photons. For example, for two-photon absorption, from (20) we obtain

$$P(s_i \to s_j, \omega_1^-, \omega_2^-) = R(\omega_{ji}, T) \frac{4\pi^2 n_2 J(\omega_1) |M_s(\omega_1^-, \omega_2^-)|^2}{c^3 \hbar V \omega_2 m(\omega_2) \omega_1^2 \varepsilon^{1/2}(\omega_1)}, \quad \omega_{ji} = \omega_1 + \omega_2, \tag{41}$$

where M_s has the form (30).

Thus, in the approximation used above, when the heat evolved in the transitions is small compared with $\hbar\omega^0_{km}$, in the probabilities (40) of all multiphoton processes the same factor $R(\omega_{ji}, T)$ is separated out and includes the entire dependence of the probability on the temperature and on the evolved heat $l\hbar\omega_0$ accompanying the process. The second factor in (40) has the form it would have if we were considering the electronic subsystem alone, not interacting with the vibrations. It can be calculated only after the form of the electronic center and its energy spectrum and wavefunctions, and also the dispersion law $\varepsilon(\omega)$ of the light in the medium, have been specified.

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Theory of high-frequency and thermodynamic properties of iron garnets

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The high-frequency magnetic susceptibility tensor is found for iron garnets. For yttrium-iron garnet, expressions are obtained for the temperature renormalization and for the damping of both the acoustical and the optical branches of the spectrum. The temperature renormalization of the acoustic branch of the spectrum differs considerably from the case of a ferromagnet. Thus in a ferromagnet, the energy of a spin wave with a given wave vector decreases with rise of temperature; in the ferrite, it increases. Corrections to the thermodynamic potential and magnetization of the ferrite, resulting from spin-wave interaction, are also found; and it is shown that these corrections have the opposite sign to those for a ferromagnet.

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1. INTRODUCTION

The study of the high-frequency and thermodynamic properties of ferrites has been the object of a large amount of experimental and theoretical research. In a theoretical description of the observed results, a ferrite is, as a rule, treated within the framework of the single-sublattice model. Although this approach does allow one to obtain a number of results in a simple manner, nevertheless the question of the limits of its applicability remains open. This is due to the fact that a ferrite is a many-sublattice system. As is shown in the present paper, a more consistent description of an iron garnet, within the framework of a two-sublattice model, leads to some conclusions that are in direct contradiction to those that follow from the single-sublattice model. Among these must be included, in particular, the conclusion that the energy of the acoustic branch of the spectrum of spin waves with a given wave vector increases, not decreases, with rise of temperature. This result is in agreement with experiment.[1]

Precision experiments have recently been conducted in the study of the dependence of the damping of spin waves on the wave vector and on the temperature in yttrium-iron garnet (YIG).[2] These experiments showed that the conclusions obtained within the framework of the single-sublattice model,[3,4] do not describe the observed results; specifically, at temperatures 200–300 K the experimental data are systematically higher than the theoretical values.[2] As is shown in the present paper, increase of the damping coefficient oc-

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curs as a result of inclusion of the process of scattering of acoustic magnons on the optical branches of the spin-wave spectrum.

In fields at which a collinear structure of the magnet persists, exchange interaction permits processes in which an even number of magnons participate. Four-magnon processes in a two-sublattice model of a ferrite were apparently first considered in a paper of Bar'yahtar and Urushadze,[5] where their role in the establishment of thermodynamic equilibrium was investigated. In a paper of Pikin,[6] the contribution of four-magnon processes to the damping coefficient of spin waves was investigated, and expressions were obtained for the damping of the acoustic branch of the spectrum produced by scattering of acoustic magnons on each other. The question of interaction of acoustic spin waves with optical was treated qualitatively by Pikin.[6]

In the present paper, the high-frequency (hf) magnetic-susceptibility tensor of iron garnets is found. It is shown that, in contrast to the Holstein-Primakoff (HP) and Dyson-Maleev (DM) representations, in the representation of Bar'yakhtar and Yablonskiĭ[7] the calculation of the transverse components of the hf susceptibility tensor of a many-sublattice magnet reduces to the calculation of single-particle Green's functions. The amplitudes of four-magnon processes in which acoustic and optical magnons participate are calculated in the first approximation with respect to S^{-1}. For YIG, expressions are obtained for the temperature renormalization and for the damping of both the acoustical and the optical branches of the spectrum. As has already been mentioned, the temperature renormalization of the acoustic branch of the spectrum differs substantially from the case of a ferromagnet. Corrections to the thermodynamic potential and magnetization of a ferrite, resulting from spin-wave interaction, are also found, and it is shown that these corrections have the opposite sign to those for a ferromagnet.

2. THE SPIN-WAVE HAMILTONIAN

The elementary magnetic cell of YIG contains 24 Fe^{3+} ions at sites $24(d)$ and 16 Fe^{3+} ions at sites $16(a)$. Thus for exact description of such a system, it is necessary to introduce into consideration 40 sublattices. The spectrum of this system contains one acoustic and 39 optical branches[8] and in general can be found only by numerical calculation.[9] But as was shown earlier,[10] for $\mathbf{k}=0$ and $\mathbf{H}\neq 0$ one can isolate, from the system of 40 equations that describe the YIG spectrum, two frequencies that correspond to oscillations of the total spin of the 24 ions in sites $24(d)$ and of the total spin of the 16 ions in sites $16(a)$. These frequencies are at the same time also the lowest-lying,[9,10] and therefore it may be assumed that in the small-wave-vector range in a real crystal, it is primarily these branches of the spectrum that are excited. For description of the thermodynamic and high-frequency properties of YIG, we shall therefore adopt a two-sublattice model, in which one sublattice combines the 24 Fe^{3+} ions at sites $24(d)$, the other the 16 Fe^{3+} ions at sites $16(a)$. The question of the role of the other optical branches cannot be treated within the framework of such a model.

When the system is located in an external magnetic field $\mathbf{H}$, its Hamiltonian has the form

$$\mathcal{H}=-\frac{1}{2}\sum_{\mathbf{n}\mathbf{n}'}J_{1\mathbf{n}\mathbf{n}'}\mathbf{S}_{1\mathbf{n}}\mathbf{S}_{1\mathbf{n}'}-\frac{1}{2}\sum_{\mathbf{m}\mathbf{m}'}J_{2\mathbf{m}\mathbf{m}'}\mathbf{S}_{2\mathbf{m}}\mathbf{S}_{2\mathbf{m}'}+\sum_{\mathbf{n}\mathbf{m}}J_{3\mathbf{n}\mathbf{m}}\mathbf{S}_{1\mathbf{n}}\mathbf{S}_{2\mathbf{m}}-\mu_0\mathbf{H}\sum_{\mathbf{n}\mathbf{m}}(g_1\mathbf{S}_{1\mathbf{n}}+g_2\mathbf{S}_{2\mathbf{m}}); \tag{1}$$

here $\mathbf{S}_{1\mathbf{n}}$ and $\mathbf{S}_{2\mathbf{m}}$ are the spin operators at sites $\mathbf{n}$ and $\mathbf{m}$; $J_{1\mathbf{n}\mathbf{n}'}$, $J_{2\mathbf{m}\mathbf{m}'}$, and $J_{3\mathbf{n}\mathbf{m}}$ are, respectively, the exchange integrals within the first, within the second, and between the sublattices; μ_0 is the Bohr magneton; and g_1 and g_2 are the g factors of the first and second sublattices. In (1) we have neglected the magnetic anisotropy energy, which in YIG is a very small quantity, and the energy of magnetic dipole interaction.

We shall investigate the system described by the Hamiltonian (1) by the method developed by Bar'yaktar *et al.*[11] For this purpose, we transform from spin to Bose operators by means of the representation of Ref. 7. At fields $H<H_c$, where H_c is the field for phase transition to a noncollinear phase, the Hamiltonian contains processes in which only an even number of magnons participate:

$$\mathcal{H}=E_0+\mathcal{H}_2+\mathcal{H}_4+\ldots$$

Here

$$E=-{}^1/_2S_1^2J_1(0)N-{}^1/_2S_2^2J_2(0)N-S_1S_2J_3(0)N-\mu_0HN(g_1S_1-g_2S_2), \tag{2}$$

$$\mathcal{H}_2=\sum_{\mathbf{k}}\{A_{1\mathbf{k}}a_{\mathbf{k}}^+a_{\mathbf{k}}+A_{2\mathbf{k}}b_{\mathbf{k}}^+b_{\mathbf{k}}+B_{\mathbf{k}}(a_{\mathbf{k}}^+b_{-\mathbf{k}}^++a_{\mathbf{k}}b_{-\mathbf{k}})\},$$

$$A_{1\mathbf{k}}=S_1\{J_1(0)-J_1(\mathbf{k})\}+S_2J_3(0)+\mu_0g_1H,$$
$$A_{2\mathbf{k}}=S_2\{J_2(0)-J_2(\mathbf{k})\}+S_1J_3(0)-\mu_0g_2H,\quad B_{\mathbf{k}}=-(S_1S_2)^{1/2}J_3(\mathbf{k}), \tag{3}$$

$J(\mathbf{k})$ is the Fourier transform of the exchange integral, S_1 and S_2 are the values of the spins at sites of the first and second sublattices, and N is the number of elementary cells in the crystal.

We transform from the operators $a_{\mathbf{k}}^+$, $a_{\mathbf{k}}$ and $b_{\mathbf{k}}^+$, $b_{\mathbf{k}}$ to the magnon creation and annihilation operators $\alpha_{\mathbf{k}}^+$, $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}^+$, $\beta_{\mathbf{k}}$ according to the formulas

$$a_{\mathbf{k}}=u_{\mathbf{k}}\alpha_{\mathbf{k}}+v_{\mathbf{k}}\beta_{-\mathbf{k}}^+,\ b_{\mathbf{k}}=u_{\mathbf{k}}\beta_{\mathbf{k}}+v_{\mathbf{k}}\alpha_{-\mathbf{k}}^+; \tag{4}$$

$$u_{\mathbf{k}}=\frac{|B_{\mathbf{k}}|}{(B_{\mathbf{k}}^2-(A_{1\mathbf{k}}-\varepsilon_{\mathbf{k}})^2)^{1/2}},\qquad v_{\mathbf{k}}=-\frac{B_{\mathbf{k}}}{|B_{\mathbf{k}}|}\frac{A_{1\mathbf{k}}-\varepsilon_{\mathbf{k}}}{(B_{\mathbf{k}}^2-(A_{1\mathbf{k}}-\varepsilon_{\mathbf{k}})^2)^{1/2}}$$

For the operator $\mathcal{H}_2$ we obtain the expression[12,13]

$$\mathcal{H}_2=\sum_{\mathbf{k}}\varepsilon_{\mathbf{k}}\alpha_{\mathbf{k}}^+\alpha_{\mathbf{k}}+\sum_{\mathbf{k}}E_{\mathbf{k}}\beta_{\mathbf{k}}^+\beta_{\mathbf{k}}+\Delta E_0; \tag{5}$$

$$\varepsilon_{\mathbf{k}}={}^1/_2(A_{1\mathbf{k}}-A_{2\mathbf{k}})+{}^1/_2[(A_{1\mathbf{k}}-A_{2\mathbf{k}})^2-4B_{\mathbf{k}}^2]^{1/2},$$
$$E_{\mathbf{k}}={}^1/_2(A_{2\mathbf{k}}-A_{1\mathbf{k}})+{}^1/_2[(A_{1\mathbf{k}}-A_{2\mathbf{k}})^2-4B_{\mathbf{k}}^2]^{1/2}, \tag{6}$$
$$\Delta E_0=\frac{1}{2}\sum_{\mathbf{k}}\{\varepsilon_{\mathbf{k}}+E_{\mathbf{k}}-A_{1\mathbf{k}}-A_{2\mathbf{k}}\}.$$

In the range of small wave vectors, $ak\ll 1$ (a is the lattice constant), and when $H\ll H_c$, we obtain the well-known results

$$\varepsilon_{\mathbf{k}}=\frac{g_1S_1-g_2S_2}{S_1-S_2}\mu_0H+D_1(ak)^2 \tag{7}$$

for the spectrum of the acoustic magnons and

$$E_{\mathbf{k}}=(S_1-S_2)J_3(0)-\frac{g_2S_1-g_1S_2}{S_1-S_2}\mu_0H+D_2(ak)^2 \tag{8}$$

for the spectrum of the optical magnons, where

$$D_1=\frac{1}{S_1-S_2}\{S_1^2I_1+S_2^2I_2+2S_1S_2I_3\}, \quad (9)$$

$$D_2=\frac{S_1S_2}{S_1-S_2}\{I_1+I_2+2I_3\}. \quad (10)$$

In the nearest-neighbor approximation, $J(0)=zI$, where z is the number of nearest neighbors.

The Hamiltonian $\mathcal{H}_4$ contains all possible processes in which four spin waves participate (35 processes in all). We shall not write the interaction amplitudes here because of their complexity.

Treatment of (1) within the framework of the HP representation leads to the following magnon-interaction Hamiltonian:

$$\mathcal{H}_4^{HP}=\sum_{1234}\{(X\alpha_1^+\alpha_2^+\alpha_3\beta_4^++\Theta_1\alpha_1^+\alpha_2^+\beta_3^+\beta_4^++\Lambda\alpha_1^+\beta_2^+\beta_3^+\beta_4+\text{h.c.})$$
$$+\Phi\alpha_1^+\alpha_2^+\alpha_3\alpha_4+\Theta_2\alpha_1^+\alpha_2\beta_3^+\beta_4+\Gamma\beta_1^+\beta_2^+\beta_3\beta_4\}, \quad (11)$$

where the amplitudes coincide with the corresponding amplitudes of the Hamiltonian $\mathcal{H}_4$ in the representation of Ref. 7. By use of the explicit form of the Hamiltonian $\mathcal{H}_4$, it is easily shown that the amplitudes of the additional processes that occur in the representation of Ref. 7 as compared with the HP representation vanish on the mass shell. Both these results—coincidence of the common amplitudes of the Hamiltonians, and vanishing of the additional amplitudes on the mass shell—are valid only in the first approximation with respect to S^{-1} (for an analogous situation for ferromagnets, see Ref. 11).

At H = 0, the interaction amplitudes satisfy symmetry conditions.[14] It has been shown[15] that fulfillment of these conditions automatically insures correct asymptotic behavior of the hf magnetic susceptibility tensor in the small-wave-vector range. The amplitudes found also satisfy the requirements of Adler's theorem[16]: that is, the interaction amplitudes that describe processes in which Goldstone particles participate vanish on the mass surface when the momentum of the latter vanishes.

3. HIGH-FREQUENCY MAGNETIC SUSCEPTIBILITY TENSOR

The components $\chi_{ij}(\mathbf{k},\omega)$ of the hf magnetic susceptibility tensor are connected with the equal-time retarded Green's functions by the relation

$$\chi_{ij}(\mathbf{k},\omega)=-\frac{\mu_0^2}{v_0}G_{ij}^{(r)}(\mathbf{k},\omega), \quad (12)$$

where $G_{ij}^{(r)}(\mathbf{k},\omega)$ is the Fourier transform of the Green's function:

$$G_{ij}^{(r)}(\mathbf{R}_m-\mathbf{R}_{m'},t)=-i\theta(t)\langle[g_1S_{1m}^i(t)$$
$$+g_2S_{2m}^i(t)\,|\,g_1S_{1m'}^j(0)+g_2S_{2m'}^j(0)]\rangle; \quad (13)$$

$S_{1m}^i(t)$ ($S_{2m}^i(t)$) is the spin operator of the m-th atom in the Heisenberg representation; $i, j=(x,y,z)$; v_0 is the volume of the elementary cell.

In the representation of Ref. 7, we get for the component $\chi_{yy}(\mathbf{k},\omega)$ of the susceptibility tensor the expression (12) with

$$G_{yy}^{(r)}(\mathbf{k},\omega)=-{}^1/_2\langle\langle(u_\mathbf{k}g_1S_1^{1/2}-v_\mathbf{k}g_2S_2^{1/2})(\alpha_\mathbf{k}^+-\alpha_{-\mathbf{k}})$$
$$+(u_\mathbf{k}g_2S_2^{1/2}-v_\mathbf{k}g_1S_1^{1/2})(\beta_\mathbf{k}^+-\beta_{-\mathbf{k}})\,|\,(u_\mathbf{k}g_1S_1^{1/2}-v_\mathbf{k}g_2S_2^{1/2})$$
$$\times(\alpha_{-\mathbf{k}}^+-\alpha_\mathbf{k})+(u_\mathbf{k}g_2S_2^{1/2}-v_\mathbf{k}g_1S_1^{1/2})(\beta_{-\mathbf{k}}^+-\beta_\mathbf{k})\rangle\rangle_\omega; \quad (14)$$

thus the problem reduces to calculation of the single-particle retarded Green's function.

We remark that both in the HP representation and in the DM representation, the components of the hf magnetic susceptibility tensor in the case of a many-sublattice magnet will always contain, besides the single-particle, also the many-particle Green's functions. Then there arises the additional problem of calculating these functions. In the representation of Ref. 7, no difficulties of this sort arise (see Ref. 11).

In order to find $\chi_{yy}(\mathbf{k},\omega)$, we consider the matrix Green's function[15]

$$\hat{G}^{(r)}(\mathbf{k},\omega)=G_{\nu\nu'}^{(r)}(\mathbf{k},\omega)\equiv\langle\langle\alpha_{\nu\mathbf{k}}^+|\alpha_{\nu'\mathbf{k}}\rangle\rangle_\omega,$$

where $\alpha_{\nu\mathbf{k}}^+$ is the four dimensional vector $\alpha_{1\mathbf{k}}^+=\alpha_\mathbf{k}^+$, $\alpha_{2\mathbf{k}}^+=\beta_\mathbf{k}^+$, $\alpha_{3\mathbf{k}}^+=\beta_{-\mathbf{k}}$, $\alpha_{4\mathbf{k}}^+=\alpha_{-\mathbf{k}}$. As usual, we shall calculate the retarded Green's functions by analytic continuation of the temperature Green's functions $\hat{G}(\mathbf{k},i\omega_n)$. The Green's function $\hat{G}(\mathbf{k},i\omega_n)$ satisfies the equation

$$\hat{G}(p)=\hat{G}^{(0)}(p)+\hat{G}^{(0)}(p)\hat{\Sigma}(p)\hat{G}(p),$$

the solution of which has the form

$$\hat{G}(p)=[\hat{G}^{(0)-1}(p)-\hat{\Sigma}(p)]^{-1}, \quad (15)$$

where $\hat{G}^{(0)}(p)$ is a diagonal matrix whose elements are

$$G_{11}^{(0)}(p)=(i\omega_n-\varepsilon_\mathbf{k})^{-1},\quad G_{22}^{(0)}(p)=(i\omega_n-E_\mathbf{k})^{-1}$$
$$G_{33}^{(0)}(p)=G_{22}^{(0)}(-p),\quad G_{44}^{(0)}(p)=G_{11}^{(0)}(-p);$$

$\Sigma(p)$ is the matrix of irreducible self-energy parts. Here $p=(\mathbf{k},i\omega_n)$.

The mass operators $\Sigma_{\nu\nu'}(p)$ can be represented in the form of an expansion in powers of S^{-1}; to each term of the expansion corresponds a definite Feynman diagram. Substitution of the expression (15) in (14) gives the explicit form of the component $\chi_{yy}(\mathbf{k},\omega)$ of the hf susceptibility tensor of the ferrite. From symmetry considerations it is clear that $\chi_{xx}(\mathbf{k},\omega)=\chi_{yy}(\mathbf{k},\omega)$.

4. TEMPERATURE CORRECTIONS TO THE SPIN-WAVE SPECTRUM

The expression (15) determines the poles of $\chi_{yy}(\mathbf{k},\omega)$; knowing these, one can determine the spectrum of magnetic excitations in the system. It is easily verified that according to (5) and (6), we shall have in the spectrum two branches, corresponding to acoustic and to optical oscillations of the spin system. We shall consider the temperature renormalization of the acoustic branch of the spectrum. In the first approximation, the temperature shift of the frequency is described by the diagram in Fig. 1, from the mass operator $\Sigma_{11}(\mathbf{k},\omega)$. Diagram a in Fig. 1 gives the correction

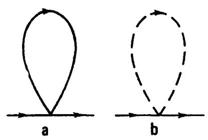

FIG. 1. In Figs. 1–5 the Green's functions of the acoustical magnons correspond to the solid lines, of the optical to the dotted.

$\Delta\omega_{1a}(T)$, equal to

$$\Delta\omega_{1a}(T)=4\sum_{\mathbf{q}}\Phi(\mathbf{k},\mathbf{q};\mathbf{k},\mathbf{q})n(\varepsilon_{\mathbf{q}}),\tag{16}$$

where $n(x)=[e^{x/T}-1]^{-1}$.

At low temperatures T, the principal contribution to (16) comes from the small-wave-vector range. We shall quote the expansion of $\Phi(\mathbf{k}, q; \mathbf{k}, q)$ in this limit, averaged over the angle between the vectors $\mathbf{k}$ and $\mathbf{q}$. Having in mind YIG, for which $I_3 \gg I_1, I_2$, we get for $g_1=g_2=g$

$$\Phi(\mathbf{k},\mathbf{q};\mathbf{k},\mathbf{q})=\frac{C_1}{N}\frac{S_1S_2}{2(S_1-S_2)^4}I_3k^2q^2.\tag{17}$$

The coefficient C_1 depends on the type of lattice: for a lattice of the NaCl type, we have $C_1=\frac{2}{3}S_1S_2$, and for a body-centered cubic lattice $C_1=\frac{1}{8}(3S_1-S_2)(3S_2-S_1)$. Using (17), we get

$$\Delta\omega_{1a}(T)=\frac{1}{(2\pi)^2}\frac{C_1}{(S_1-S_2)^3}D(ak)^2\left(\frac{T}{D}\right)^{5/2}\Gamma\left(\frac{5}{2}\right)Z_{5/2}\left(\frac{\mu H}{T}\right).\tag{18}$$

Diagram b in Fig. 1 gives the correction $\Delta\omega_{1b}(T)$, equal to

$$\Delta\omega_{1b}(T)=\sum_{\mathbf{q}}\Theta_2(\mathbf{k};\mathbf{k}|\mathbf{q};\mathbf{q})n(E_{\mathbf{q}}).\tag{19}$$

For $\Theta_2(\mathbf{k};\mathbf{k}|\mathbf{q};\mathbf{q})$ we have

$$\Theta_2(\mathbf{k};\mathbf{k}|\mathbf{q};\mathbf{q})=-\frac{2}{N}\frac{S_1^2+S_2^2}{(S_1-S_2)^2}I_3(ak)^2\tag{20}$$

and consequently

$$\Delta\omega_{1b}(T)=-\frac{1}{(2\pi)^2}\frac{S_1^2+S_2^2}{S_1S_2(S_1-S_2)}D(ak)^2\left(\frac{T}{D}\right)^{3/2}\Gamma\left(\frac{3}{2}\right)Z_{3/2}\left(\frac{\Delta}{T}\right).\tag{21}$$

In these formulas $\Gamma(x)$ is the gamma function;

$$Z_n(x)=\sum_{m=1}^{\infty}\frac{\exp\{-xm\}}{m^n};$$

$\Delta=(S_1-S_2)J_3(0)-g\mu_0H$; the value of D is determined by formulas (9) and (10), in which we have neglected I_1 and I_2.

On taking, in the case of YIG, $S_1=24\cdot\frac{5}{2}$ and $S_2=16\cdot\frac{5}{2}$, we see that, in contrast to the case of a ferromagnet, in the two-sublattice model of a magnet the contribution of diagram a of Fig. 1 to the temperature shift of the frequency enters with a plus sign. This result is due entirely to the "many-sublattice" form of the amplitude $\Phi(12;34)$.

Diagrams a and b in Fig. 1 give contributions of opposite signs. The total temperature shift of frequency $\Delta\omega_1(T)$ will be positive. At low temperatures, as is evident from (18) and (21), $\Delta\omega_1(T)$ increases approximately as $T^{5/2}$. At $T\sim\Delta$, the contribution $\Delta\omega_{1b}(T)$ becomes important; this leads to a slowing down of the increase of $\Delta\omega_1(T)$ with temperature. According to Harris's estimates,[9] for YIG $\Delta\sim 250$ to 300 K, so that the contribution of optical magnons will become important at room temperature. From (18) and (21), the temperature dependence of the exchange constant of the acoustical branch of the spectrum is

$$\frac{D(T)}{D}=1+\frac{1}{(2\pi)^2}\left(\frac{T}{D}\right)^{3/2}\left\{\frac{3}{2}\frac{C_1}{(S_1-S_2)^3}\frac{T}{D}Z_{5/2}\left(\frac{\mu H}{T}\right)-\frac{S_1^2+S_2^2}{S_1S_2(S_1-S_2)}Z_{3/2}\left(\frac{\Delta}{T}\right)\right\}\Gamma\left(\frac{3}{2}\right),\tag{22}$$

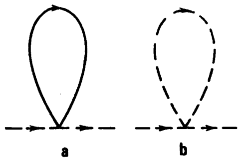

FIG. 2.

where $D=D(0)$. Such a behavior of $D(T)$ agrees with the experimentally observed dependence.[1]

We shall consider the temperature renormalization of the optical branch of the spectrum. In the first approximation, the temperature shift of the frequency is described by the diagrams in Fig. 2, from the mass operator $\Sigma_{22}(\mathbf{k},\omega)$. The analytical expressions corresponding to the diagrams are:

$$\Delta\omega_{2a}(T)=-\frac{1}{(2\pi)^2}2\frac{S_1^2+S_2^2}{(S_1-S_2)^2}I_3\left(\frac{T}{D}\right)^{5/2}\Gamma\left(\frac{5}{2}\right)Z_{5/2}\left(\frac{\mu H}{T}\right),\tag{23}$$

$$\Delta\omega_{2b}(T)=\frac{1}{(2\pi)^2}2J_3(0)\left(\frac{T}{D}\right)^{3/2}\Gamma\left(\frac{3}{2}\right)Z_{3/2}\left(\frac{\Delta}{T}\right).\tag{24}$$

The expression (23) is generated by the amplitude $\Theta_2(\mathbf{q};\mathbf{q}|\mathbf{k};\mathbf{k})$ given by (20), and the expression (24) by the amplitude

$$\Gamma(\mathbf{k},\mathbf{q};\mathbf{k},\mathbf{q})=J_3(0)/2N.\tag{25}$$

As is seen from (23) and (24), diagrams a and b in Fig. 2 give contributions of opposite signs. The total shift of frequency will be negative. At low temperatures, as follows from (23), the gap in the spectrum of the optical excitations decreases as $T^{5/2}$. At $T\sim\Delta$, the contribution $\Delta\omega_{2b}(T)$ becomes important; this leads to a slower decrease of $\Delta\omega_2(T)$ with temperature.

5. DAMPING OF ACOUSTIC MAGNONS

In the first approximation with respect to S^{-1}, the damping of acoustic magnons is determined by the imaginary parts of the diagrams in Fig. 3, from the mass operator $\Sigma_{11}(\mathbf{k},\omega)$. The heavy lines in the figure denote the fact that we have carried out temperature renormalization of the Green's functions in the random-phase approximation, which actually reduces to temperature renormalization of the spin-wave spectrum in accordance with (22) – (24). As is seen from Fig. 3, the damping $\gamma^\alpha(\mathbf{k})$ is determined by four different processes of magnon interaction, and it may accordingly be represented in the form

$$\gamma^\alpha(\mathbf{k})=\gamma_\Phi{}^\alpha(\mathbf{k})+\gamma_X{}^\alpha(\mathbf{k})+\gamma_{\Theta_2}{}^\alpha(\mathbf{k})+\gamma_\Lambda{}^\alpha(\mathbf{k}).$$

We consider $\gamma_\Phi{}^\alpha(\mathbf{k})$:

$$\gamma_\Phi{}^\alpha(\mathbf{k})=8\pi\sum_{234}|\Phi(\mathbf{k}2;34)|^2\frac{[n(\varepsilon_2)+1]n(\varepsilon_3)n(\varepsilon_4)}{n(\varepsilon_{\mathbf{k}})}\delta(\varepsilon_{\mathbf{k}}+\varepsilon_2-\varepsilon_3-\varepsilon_4).$$

In the small-wave-vector range, the following expan-

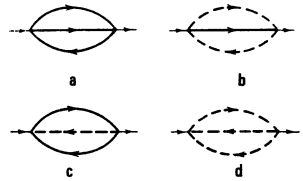

FIG. 3.

sion is valid for the amplitude $\Phi(12;34)$:

$$\Phi(12;34)=-\frac{1}{2N}\frac{S_1S_2}{(S_1-S_2)^2}I_3a^2(\mathbf{k}_1\mathbf{k}_2+\mathbf{k}_4\mathbf{k}_3)\Delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4).$$

By using the same method of calculating integrals that was used by Wang,[4] we get for $\varepsilon_{\mathbf{k}}\ll T$ and $g_1=g_2$:

$$\gamma_{\Phi}^{\alpha}(\mathbf{k})=\frac{1}{(2\pi)^3}\frac{(ak)^2\varepsilon_{\mathbf{k}}T^2}{12(S_1-S_2)^2}\frac{D^2(0)}{D^4(T)}F\left(\frac{T}{\omega_{\mathbf{k}}},R\right), \tag{26}$$

$$F\left(\frac{T}{\omega_{\mathbf{k}}},R\right)=\frac{1}{2}\ln^2\frac{T}{\omega_{\mathbf{k}}}+\Big\{-\ln(1+R)$$
$$+2\left[\frac{5}{6}-R+R^{3/2}\operatorname{arctg}(R^{-1/2})\right]\Big\}\left\{1+\ln\frac{T}{\omega_{\mathbf{k}}}\right\}+3G(R). \tag{27}$$

Here $\omega_{\mathbf{k}}=D(T)(ak)^2$, and R is the ratio of the magnetic-field energy to the exchange energy, $R=\mu H/\omega_{\mathbf{k}}$; $D(T)$ is determined by the expression (22), and the function $G(R)$ was found by Wang.[4]

The expressions (26) and (27) go over, in the case of the single-sublattice model of a magnet and with replacement of $D(T)$ by $D(0)$, to formulas (10) and (11) of Ref. 4.[1)]

The damping produced by the process Φ was considered also in Ref. 6. But the expression (26) has a more general character than does that found in Ref. 6. This is so because (26) is correct for arbitrary values of the parameter R, whereas Pikin's result[6] was found only in the two limiting cases $R\gg 1$ and $R\ll 1$. Furthermore, Pikin[6] did not consider temperature renormalization of the spin-wave spectrum.

We now consider $\gamma_{\Theta 2}^{\alpha}(\mathbf{k})$:

$$\gamma_{\Theta 2}^{\alpha}(\mathbf{k})=\pi\sum_{234}|\Theta_2(\mathbf{k};2|3;4)|^2\frac{n(\varepsilon_2)[n(E_3)+1]\,n(E_4)}{n(\varepsilon_{\mathbf{k}})}\delta(\varepsilon_{\mathbf{k}}+E_3-\varepsilon_2-E_4).$$

Using the expression for $\Theta_2(\mathbf{k};2|3;4)$ in the small-momentum range,

$$\Theta_2(\mathbf{k};2|3;4)$$
$$=\frac{1}{N}\frac{2I_3}{(S_1-S_2)^2}a^2\{(S_1^2+S_2^2)\mathbf{k}\mathbf{k}_2-S_1S_2(\mathbf{k}\mathbf{k}_3+\mathbf{k}_2\mathbf{k}_4)\}\Delta(\mathbf{k}+\mathbf{k}_3-\mathbf{k}_2-\mathbf{k}_4),$$

we get for $\varepsilon_{\mathbf{k}}\ll T$ and $g_1=g_2$:

$$\gamma_{\Theta 2}^{\alpha}(\mathbf{k})=\frac{1}{(2\pi)^3}\frac{(ak)^2\varepsilon_{\mathbf{k}}T^2}{12(S_1-S_2)^2}\frac{D^2(0)}{D^4(T)}\left\{F_1\left(\frac{T}{\omega_{\mathbf{k}}},R,\frac{\Delta(T)}{T}\right)+\frac{(S_1-S_2)^4}{4S_1^2S_2^2}F_2\left(\frac{\Delta(T)}{T}\right)\right\}; \tag{28}$$

$$F_1\left(\frac{T}{\omega_{\mathbf{k}}},R,\frac{\Delta(T)}{T}\right)=\Big\{\ln\frac{T}{\omega_{\mathbf{k}}}-\ln(1+R)$$
$$+2\left[\frac{5}{6}-R+R^{3/2}\operatorname{arctg}(R^{-1/2})+\frac{\Delta(T)}{4T}\right.$$
$$\left.+\frac{1}{2}\ln\left[2\operatorname{sh}\left(\frac{\Delta(T)}{2T}\right)\right]\right]\Big\}\left\{\frac{\Delta(T)}{2T}-\ln\left[2\operatorname{sh}\left(\frac{\Delta(T)}{2T}\right)\right]\right\}$$
$$+\frac{e^{\Delta(T)/T}}{e^{\Delta(T)/T}-1}Z_2\left(\frac{\Delta(T)}{T}\right)-\sum_{n,m=1}^{\infty}\frac{e^{-\Delta(T)m/T}}{n(n+m)}; \tag{29}$$

$$F_2\left(\frac{\Delta(T)}{T}\right)=\frac{1}{e^{\Delta(T)/T}-1}\left\{\zeta(2)-(1+e^{\Delta(T)/T})Z_2\left(\frac{\Delta(T)}{T}\right)\right\}$$
$$-\left\{\frac{\Delta(T)}{2T}-\ln\left[2\operatorname{sh}\left(\frac{\Delta(T)}{2T}\right)\right]\right\}\ln\left[2\operatorname{sh}\left(\frac{\Delta(T)}{2T}\right)\right]+\sum_{n,m=1}^{\infty}\frac{e^{-\Delta(T)m/T}}{n(n+m)}. \tag{30}$$

At room temperature and for $\varepsilon_{\mathbf{k}}\sim 1$ K we find that in YIG the contribution $\gamma_{\Theta 2}^{\alpha}(\mathbf{k})$ amounts to 20% of the contribution $\gamma_{\Phi}^{\alpha}(\mathbf{k})$. But if $\varepsilon_{\mathbf{k}}\sim 10$ K, this contribution becomes 50%. On the other hand, at low temperatures, $T\ll\Delta$, the contribution of this process is exponentially small. Analysis of the processes of magnon interaction described by the amplitudes $X(12;3|4)$ and $\Lambda(1|23;4)$ shows that in the case $\varepsilon_{\mathbf{k}}\ll T$ and $T\lesssim\Delta$, their contribution is orders of magnitude smaller than $\gamma_{\Theta 2}^{\alpha}(\mathbf{k})$.

We shall give the expressions for the damping of the acoustic branch of the spectrum in the case $\varepsilon_{\mathbf{k}}\gg T$.

If $T\ll\omega_{\mathbf{k}}\ll T_c$,

$$\gamma_{\Phi}^{\alpha}(k)=\frac{1}{(2\pi)^3}\frac{D(ak)^3}{24(S_1-S_2)^2}\left(\frac{T}{D}\right)^{5/2}Z_{5/2}\left(\frac{\mu H}{T}\right)\Gamma\left(\frac{5}{2}\right). \tag{31}$$

If $\omega_{\mathbf{k}}\ll T\ll\mu H\ll T_c$

$$\gamma_{\Phi}^{\alpha}(\mathbf{k})=\frac{1}{(2\pi)^3}\frac{D(ak)^2}{12(S_1-S_2)^2}\left(\frac{T}{D}\right)^3e^{-\mu H/T}. \tag{32}$$

The contribution of optical magnons in this temperature range is exponentially small.

As is seen from formulas (26) and (28), the damping coefficient $\gamma^{\alpha}(\mathbf{k})$ vanishes when $\mathbf{k}=0$. In Pikin's paper[6] it was shown that allowance for inequality of the g factors leads to a finite value of $\gamma^{\alpha}(0)$.

By calculating the amplitude $\Phi(12;34)$ for $g_1\neq g_2$, we find the damping coefficient of uniform precession resulting from this process when $\varepsilon_0\ll T$:

$$\gamma_{\Phi}^{\alpha}(0)=\frac{1}{(2\pi)^3}\left\{\frac{S_1S_2(S_1+S_2)}{12D(S_1-S_2)^4}\right\}^2(g_1-g_2)^2\mu_0^2H^2\varepsilon_0\left(\frac{T}{D}\right)^2\ln\frac{T}{\varepsilon_0}; \tag{33}$$

this expression is similar to that given by Pikin.[6]

The presence of an optical branch in the spectrum leads to additional damping of spin waves with $\mathbf{k}=0$. The process described by the amplitude $\Theta_2(1;2|3;4)$ gives a damping coefficient

$$\gamma_{\Theta 2}^{\alpha}(0)=\frac{1}{(2\pi)^3}\left\{\frac{S_1+S_2}{4D(S_1-S_2)^2}\right\}^2(g_1-g_2)^2\mu_0^2H^2\varepsilon_0\frac{T}{D}F\left(\frac{\varepsilon_0}{T},\frac{E_0}{T}\right); \tag{34}$$

$$F\left(\frac{\varepsilon_0}{T},\frac{E_0}{T}\right)=\frac{1}{\exp(E_0/T)-1}\left\{\ln\frac{T}{\varepsilon_0}+1+\frac{E_0}{T}-\ln\left[2\operatorname{sh}\left(\frac{E_0}{2T}\right)\right]\right\}$$
$$+\ln\left[2\operatorname{sh}\left(\frac{E_0}{2T}\right)\right]-\frac{E_0}{2T}. \tag{35}$$

6. DAMPING OF OPTICAL MAGNONS

The damping coefficient $\gamma^{\beta}(\mathbf{k})$ of optical magnons is determined by the imaginary part of the diagrams shown in Fig. 4, from the mass operator $\Sigma_{22}(\mathbf{k},\omega)$. Just as in the preceding case, we represent it in the form

$$\gamma^{\beta}(\mathbf{k})=\gamma_{\Theta 2}^{\beta}(\mathbf{k})+\gamma_{\Gamma}^{\beta}(\mathbf{k})+\gamma_{X}^{\beta}(\mathbf{k})+\gamma_{\Lambda}^{\beta}(\mathbf{k});$$

$$\gamma_{\Theta 2}^{\beta}(\mathbf{k})=\pi\sum_{124}|\Theta_2(1;2|\mathbf{k};4)|^2\frac{[n(\varepsilon_1)+1]\,n(\varepsilon_2)\,n(E_4)}{n(E_{\mathbf{k}})}\delta(E_{\mathbf{k}}+\varepsilon_1-\varepsilon_2-E_4); \tag{36}$$

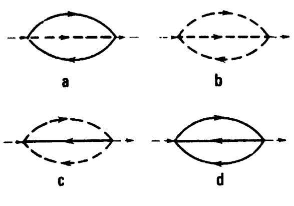

FIG. 4.

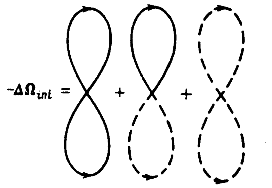

FIG. 5.

$$\gamma_\Gamma^\beta(\mathbf{k}) = 8\pi\sum_{234}|\Gamma(\mathbf{k}2;34)|^2\frac{[n(E_2)+1]\,n(E_3)\,n(E_4)}{n(E_\mathbf{k})}\delta(E_\mathbf{k}+E_2-E_3-E_4). \tag{37}$$

When $D(ak)^2 \ll T$, $H \ll H_c$, and $g_1 = g_2$,

$$\gamma_{\varepsilon 2}^\beta(\mathbf{k}) = \frac{1}{(2\pi)^3}\frac{D(S_1^2+S_2^2)^2}{16S_1^2S_2^2(S_1-S_2)^2}\left(\frac{T}{D}\right)^4 I\left(\frac{\Delta}{T}\right) \tag{38}$$

$$I\left(\frac{\Delta}{T}\right) = (1-e^{-\Delta/T})\int_0^\infty \frac{x^2e^x\,dx}{(e^x-1)(e^{x-\Delta/T}-1)}\ln\frac{1-e^{-x}}{1-e^{-\Delta/T}},$$

$$I\left(\frac{\Delta}{T}\right) = 2\sum_{\substack{n=0\\m=1}}^{\infty}\frac{1}{m(n+m)^3}, \quad \Delta \gg T,$$

$$\gamma_\Gamma^\beta(\mathbf{k}) = \frac{1}{(2\pi)^3}\frac{J_1^2(0)}{4D}\left(\frac{T}{D}\right)^2(1-e^{-\Delta/T})F_\Gamma\left(\frac{\Delta}{T}\right), \tag{39}$$

$$F_\Gamma\left(\frac{\Delta}{T}\right) = Z_2\left(\frac{\Delta}{T}\right) + \left\{\frac{\Delta}{2T} - \ln\left[2\,\mathrm{sh}\left(\frac{\Delta}{2T}\right)\right]\right\}^2.$$

As is evident, in contrast to the damping of acoustic magnons, the damping coefficient $\gamma^\beta(0)$ of optical magnons does not vanish, even when $g_1 = g_2$.

We shall not give the expressions for $\gamma_\chi^\beta(\mathbf{k})$ and $\gamma_\Lambda^\beta(\mathbf{k})$, since, as in the case of damping of acoustic magnons, their contribution is negligibly small.

7. THERMODYNAMIC POTENTIAL AND MAGNETIZATION OF FERRITES

The thermodynamic potential of the ferrite can be represented in the form

$$\Omega = E_0 + \Delta E_0 + \Omega_0 + \Delta\Omega_{int}, \tag{40}$$

where the thermodynamic potential of noninteracting particles is

$$\Omega_0 = T\sum_\mathbf{k}\{\ln[1+n(\varepsilon_\mathbf{k})] + \ln[1+n(E_\mathbf{k})]\}, \tag{41}$$

and where $\Delta\Omega_{int}$ is a correction resulting from interaction of magnons. In the leading approximation with respect to S^{-1}, $\Delta\Omega_{int}$ is determined by the diagrams shown in Fig. 5. The corresponding analytic expression has the form

$$\frac{\Delta\Omega_{int}}{NT} = \frac{1}{(2\pi)^4}\left\{\frac{9}{2}\frac{C_1}{(S_1-S_2)^3}\left(\frac{T}{D}\right)^2 Z_{5/2}^2\left(\frac{\mu H}{T}\right) - 6\frac{S_1^2+S_2^2}{S_1S_2(S_1-S_2)}\frac{T}{D}Z_{3/2}\left(\frac{\Delta}{T}\right)Z_{5/2}\left(\frac{\mu H}{T}\right) + 2\frac{S_1-S_2}{S_1S_2}Z_{3/2}^2\left(\frac{\Delta}{T}\right)\right\}\left(\frac{T}{D}\right)^2\Gamma^2\left(\frac{3}{2}\right). \tag{42}$$

Knowing $\Omega(T,H)$, one can find the magnetization of the ferrite, $M(T,H) = -N^{-1}\partial\Omega/\partial H$.

We shall give the expression for the correction to the magnetization resulting from interaction of magnons:

$$\Delta M_{int} = \frac{\mu}{(2\pi)^4}\left\{\frac{9}{2}\frac{C_1}{(S_1-S_2)^3}\left(\frac{T}{D}\right)^2 Z_{5/2}\left(\frac{\mu H}{T}\right)Z_{3/2}\left(\frac{\mu H}{T}\right) - 6\frac{S_1^2+S_2^2}{S_1S_2(S_1-S_2)}\frac{T}{D}\left[Z_{1/2}\left(\frac{\Delta}{T}\right)Z_{5/2}\left(\frac{\mu H}{T}\right) + Z_{3/2}\left(\frac{\Delta}{T}\right)Z_{3/2}\left(\frac{\mu H}{T}\right)\right] + 4\frac{S_1-S_2}{S_1S_2}Z_{1/2}\left(\frac{\Delta}{T}\right)Z_{3/2}\left(\frac{\mu H}{T}\right)\right\}\left(\frac{T}{D}\right)^2\Gamma^2\left(\frac{3}{2}\right). \tag{43}$$

As is seen from (42) and (43), the principal contribution to $\Delta\Omega_{int}$ and ΔM_{int} at $T \ll \Delta$ is produced by interaction of acoustic magnons with each other. We see that the functional dependence of the corrections on T and H is the same as in a ferromagnet,[17] but they enter with the opposite sign; this is essentially due to the ferrimagnetic structure of the magnon-interaction amplitudes.

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[1)]In Wang's paper[4] the numerical coefficient in formula (10) is twice as large, since the author finds $\gamma(k)$ from the kinetic equation.

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