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Alternating-current Josephson effect

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An analysis is made of the problem of nonlinear oscillations which appear in a circuit containing a Josephson tunnel junction. The nonlinear ac equation, describing steady-state spontaneous oscillations in the circuit, is solved asymptotically. Numerical methods are used to find the dependence of the oscillation period on the circuit parameters. Hysteresis effects are studied and the range of their existence is determined. The form of the current-voltage characteristics of a Josephson circuit is considered in the case of a constant voltage supplied by an external power source.

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The dc Josephson effect is produced by the passage, through a Josephson tunnel junction, of a current not exceeding a certain critical value I_c.[1-4] The voltage drop across the junction is then zero, $V=0$, and the current $I=I_c \sin\varphi$ is set up by an external voltage or current source. [The quantity $\varphi=\sin^{-1}(I/I_c)$ is known as the phase shift of the order parameter in a superconducting circuit.] If the current I exceeds I_c, the dc Josephson effect is observed: in this case a nonzero voltage drop V is established across the junction and the dependence $I(V)$ has characteristic features.[3,4]

Usually the ac Josephson effect is investigated either for a constant current I through the junction or for a constant voltage V across the junction; the majority of the experimental and theoretical investigations has been concerned specifically with these cases.[3,4] However, it is interesting to consider the problem of the ac Josephson effect in the presence of a constant external voltage U (the source of U may be a battery or a voltage generator). This case is considered in the present paper. We shall also touch upon some methodological aspects associated with the treatment of the ac Josephson effect.

We shall consider a circuit (Fig. 1) consisting of a generator of a constant voltage U, an external resistance R_0, and a tunnel junction characterized by a capacitance C and a voltage drop V. The equation for the current in the circuit is then

$$IR_0=U-V-\frac{1}{c}\frac{\partial\Phi}{\partial t},\tag{1}$$

where Φ is the magnetic flux in the circuit, and the total current is governed by its value passing through the tunnel barrier:

$$I=I_c\sin\varphi+\frac{V}{R_t}+C\frac{dV}{dt}.\tag{2}$$

Here, the first term is the Josephson current, the second is the conduction current (R_t is the resistance to the normal component of the current through the junction), and the third is the displacement current.

We shall confine our attention to the case when there is no external magnetic field and we may assume that the current through the junction is independent of the coordinates in the junction plane (this condition is satisfied, for example, by small-area contacts). The val-

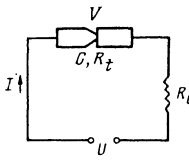

FIG. 1. Schematic diagram of a circuit with a tunnel junction.

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ue of φ is then given by[1-4]

$$\varphi=\varphi_0+\frac{e^*}{\hbar}\int_0^t V(t')dt',\quad V(t)=\frac{\hbar}{e^*}\frac{d\varphi}{dt},\tag{3}$$

where φ_0 is a constant and $e^*=2e$ is the charge of a Cooper electron pair.

Ignoring in Eq. (1) the small term $c^{-1}\partial\Phi/\partial t$ (i.e., the additional voltage in the circuit resulting from the self-induction effect), we shall rewrite Eq. (1) subject to Eqs. (2) and (3) in the form

$$\lambda\ddot{\varphi}+\dot{\varphi}+\nu\sin\varphi=1,\tag{4}$$

where $\varphi=\varphi(\tau)$, the point denotes differentiation with respect to the dimensionless time $\tau=t/t_0$, and

$$t_0=\frac{\hbar}{e^*U}\frac{R_0}{R},\quad \lambda=\frac{CR}{t_0},$$
$$\nu=\frac{R_0 I_c}{U},\quad \frac{1}{R}=\frac{1}{R_0}+\frac{1}{R_t}.\tag{5}$$

Some idea of the order of the quantities occurring in Eq. (5) can be obtained by substituting typical values $e^*U\sim\Delta\sim10^{-3}$ eV (Δ is the energy gap of the superconductor), $R_0\sim R_t\sim 1\,\Omega$, $C\sim\epsilon S/4\pi l\sim10^3$ cm ($\epsilon\sim10$ is the permittivity of the junction, $S\sim10^{-4}$ cm^2 is its area, and $l\sim10^{-7}$ cm is its thickness). Substituting these values, we find from Eq. (5) that $\lambda\sim10^2$ and that a typical oscillation period is $2\pi t_0\sim10^{-11}$ sec (this corresponds to $\sim10^{11}$ Hz and to the wavelength $\lambda_0\sim2\pi ct_0\sim0.3$ cm).[1)]

We note that an equation of the (4) type can be obtained directly from the definition of the total current (2), if we consider I as a given quantity (details of the Josephson effect under constant-current conditions can be found elsewhere[3-7]). In our formulation of the problem, we are beginning from a system of two simultaneous equations (1) and (2), but we still obtain Eq. (4) except that in our case the current I in the circuit is not constant and has to be found by solving Eq. (4). Moreover, we must bear in mind that in our case the parameters of the equation depend on two quantities R_t and R_0 and not on R_t alone, as in earlier investigations.[3-7] These differences do not alter the situation in any basic manner and we can still use the results obtained by Stewart[6] and McCumber[7] in investigating Eq. (4). To avoid repetition, we shall consider in detail only those aspects which are not discussed by Stewart[6] and McCumber[7] and, moreover, we shall discuss some methodological features which may be of general physical interest.

A qualitative study of Eq. (4) can be made by the phase picture method (see, for example, Chap. 7 in the monograph by Andronov *et al.*[9]). Figures 2 and 3 show the phase pictures of Eq. (4) drawn on cylindrical surfaces for various values of ν and λ. We can see that, in the $\nu<1$ case (Fig. 2), Eq. (4) has periodic solutions of the second kind[9] (surrounding the cylinder) and tending to the limit cycle located withing the interval

$$1-\nu<\dot{\varphi}<1+\nu,\tag{6}$$

where this cycle is reached for any initial conditions.

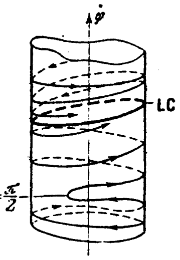

FIG. 2. Phase picture of Eq. (4) on a cylindrical surface for the $\nu<1$ case. The value of $\dot{\varphi}$ is plotted along the cylinder axis and the value of φ along the azimuth. One of the possible limit cycles, whose range of existence is defined by the inequalities of Eq. (6), is shown. For given values of ν and λ, there is only one limit cycle.

In the $\nu>1$ case (Fig. 3a), there is a periodic solution for $\nu\lambda<\nu_c\lambda_c$ ($\nu_c\lambda_c$ is some critical value) and initial conditions in the unshaded region. If $\nu>1$, but $\nu\lambda>\nu_c\lambda_c$, there are no periodic solutions and for any initial conditions (Fig. 3b) the solution tends either to a stable focus, corresponding to $\varphi=\sin^{-1}(1/\nu)$, or to an unstable saddle point, corresponding to the solution $\varphi=\pi-\sin^{-1}(1/\nu)$.

We can obtain an approximate analytic solution[2)] of Eq. (4) for small values $\nu/\lambda\ll1$. We shall seek a solution in the form

$$\varphi=\varphi_0+\nu\varphi_1.\tag{7}$$

Then, Eq. (4) can be represented by a system of two equations:

$$\lambda\ddot{\varphi}_0+\dot{\varphi}_0=1,\quad \lambda\ddot{\varphi}_1+\dot{\varphi}_1=-\sin(\varphi_0+\nu\varphi_1).\tag{8}$$

The solution of the first of the two equations in the system (8) is

$$\varphi_0=C_1e^{-\tau/\lambda}+\tau+C_2.\tag{9}$$

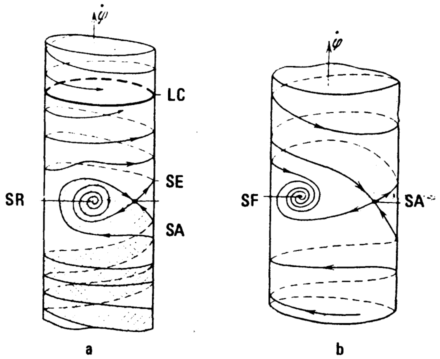

FIG. 3. Phase picture of Eq. (4) in the $\nu>1$ case: a) $\nu\lambda<\nu_c\lambda_c$; b) $\nu\lambda>\nu_c\lambda_c$. Here, $\nu_c\lambda_c$ is some critical value (see also Fig. 6). In the case shown in Fig. 3a there are two asymptotic solutions, one of which corresponds to a limit cycle (LC) and the other to a stable focus (SF). In the case shown in Fig. 3b there is only one asymptotic solution corresponding to a stable focus; SA is a saddle point and SE denotes a separatrix.

The second equation in the system (8) can be solved by variation of the constants, which gives

$$\varphi_1 = -\frac{1}{\lambda}\int\limits_0^\tau e^{-t/\lambda}dt\int\limits_0^t e^{\xi/\lambda}\sin\varphi(\xi)\,d\xi. \qquad (10)$$

Combining Eqs. (7), (9), and (10), we obtain

$$\varphi(\tau) = C_1e^{-\tau/\lambda}+\tau+C_2-\frac{\nu}{\lambda}\int\limits_0 e^{-t/\lambda}dt\int\limits_0 e^{\xi/\lambda}\sin\varphi(\xi)\,d\xi. \qquad (11)$$

We shall apply the method of successive approximations, replacing φ on the right-hand side of Eq. (11) with Eq. (6):

$$\varphi = C_1e^{-\tau/\lambda}+\tau+C_2-\frac{\nu}{\lambda}\int\limits_0^\tau e^{-t/\lambda}dt\int\limits_0^t e^{\xi/\lambda}\sin(C_1e^{-\xi/\lambda}+\xi+C_2)\,d\xi. \qquad (12)$$

For the initial conditions $\varphi(0)=0$ and $\dot{\varphi}(0)=0$, we have

$$C_1=\lambda,\quad C_2=-C_1=-\lambda.$$

Integrating twice by parts, we find from Eq. (12) that

$$\dot{\varphi} = 1-e^{-\tau/\lambda}-\frac{\nu}{1+\lambda^2}\sin\varkappa+\frac{\nu\lambda}{1+\lambda^2}\cos\varkappa-\frac{\nu\lambda}{1+\lambda^2}e^{-\tau/\lambda}$$
$$-\frac{\nu}{1+\lambda^2}e^{-\tau/\lambda}\int\limits_0^\tau\cos\varkappa\,dt-\frac{\nu\lambda}{1+\lambda^2}e^{-\tau/\lambda}\int\limits_0^\tau\sin\varkappa\,dt, \qquad (13)$$

where

$$\varkappa = \lambda e^{-t/\lambda}-\lambda+t.$$

The expression (13) allows us to find the solution $\varphi(\tau)$ for any time.

The solution (13) applies if $\nu<1$, $\nu/\lambda\ll 1$ and it discribes oscillations which occur when the circuit is subjected to a constant voltage $U>U_1=I_cR_0$. We shall be interested in the steady-state forced oscillation regime in the limit $\tau\to\infty$. In this case, we find from Eq. (13) that φ has the asymptotic solution

$$\varphi(\tau) = \tau+\frac{\nu\lambda}{1+\lambda^2}\sin\tau+\frac{\nu}{1+\lambda^2}\cos\tau+\text{const}. \qquad (14)$$

The spontaneous oscillation regime described by Eq. (14) is established in the circuit irrespective of the

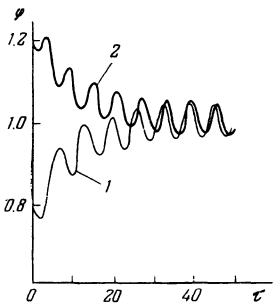

FIG. 4. Example of a numerical solution of Eq. (4) for $\lambda=10$ and $\nu=0.5$, illustrating that the asymptotic solution is independent of the initial conditions. Curve 1 corresponds to the initial values $\varphi_0=1$ and $\dot{\varphi}_0=0.8$ and curve 2 to the values $\varphi_0=2$ and $\dot{\varphi}_0=1.2$.

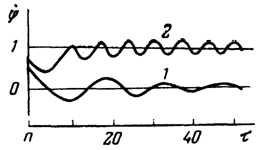

FIG. 5. Numerical solution of Eq. (4) plotted for $\lambda=10$ and $\nu=2$, illustrating the existence of two asymptotic (in the limit $\tau\to\infty$) solutions. Solution 1 corresponds to the initial conditions $\varphi_0=1$ and $\dot{\varphi}_0=0.5$ (resulting in a steady regime), whereas solution 2 corresponds to the initial conditions $\varphi_0=1$ and $\dot{\varphi}_0=0.7$ (resulting in stable spontaneous oscillations).

initial conditions[3)] at the moment $t=0$ (see also Fig. 2).

This is illustrated in Fig. 4, which shows the solution of Eq. (4) obtained on a computer for $\lambda=10$ and $\nu=0.5$. We can easily see that the curves in Fig. 4 are described satisfactorily by analytic solutions of the (13) and (14) type. Naturally, the asymptotic solution (14) is independent of the initial values of the phase φ_0 and voltage $\dot{\varphi}_0$ (this is also clear from the curves in Fig. 4). We can see that variation of the asymptotic solutions by some constant amount relative to one another from the abscissa.

If $\nu>1$ (i.e., for $U<U_1=I_cR_0$), we always have an asymptotic solution of the type

$$\varphi(\tau)\,|_{\tau\to\infty}=\varphi_\infty=\arcsin^{-1}(1/\nu),\qquad \nu>1 \qquad (15)$$

(this solution corresponds to a stable focus in Figs. 3a and 3b). Then, Eq. (4) describes the process of approach of the circuit current to its steady-state value

$$I\,|_{\tau\to\infty}=I_c\sin\varphi_\infty,$$

and

$$V\,|_{\tau\to\infty}\to 0$$

(see curve 1 in Fig. 5). The current reaches a value smaller than I_c and the asymptotic solution (15) represents the usual dc Josephson effect. However, if $\nu>1$, we can have not only the asymptotic solution (15), but also the asymptotic oscillatory solution (14) (corresponding to the limit cycle in Fig. 3a). The latter solution is obtained by specifying some finite initial voltage in the system $\dot{\varphi}_0>\dot{\varphi}_0$ and then the solution of Eq. (4) reaches asymptotically the oscillatory regime of the (14) type in spite of the fact that the battery voltage is less than the critical value $U<U_1$ and the circuit is under conditions corresponding to the dc Josephson effect. This is illustrated by curves 1 and 2 in Fig. 5, found by numerical integration of Eq. (4) for various values of $\dot{\varphi}_0$, and also by the results in Fig. 3a. (The oscillatory regime appears in Fig. 3a for initial conditons outside the shaded region.)

Figure 6 shows the dependence of $\dot{\varphi}_c$ on ν and λ. For a given value of $\dot{\varphi}_0>\dot{\varphi}_c$, the solution of Eq. (4) becomes of the spontaneous oscillation type, whereas for $\dot{\varphi}_0<\dot{\varphi}_c$, the solution tends to the constant (15), i.e., it corresponds to the dc Josephson effect. The dashed curve in Fig. 6 joins the end points of the branches $\dot{\varphi}_c$. In

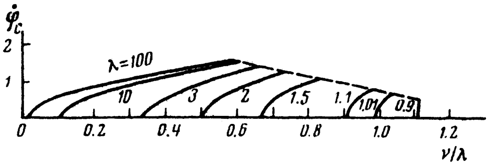

FIG. 6. Critical initial values of $\dot{\varphi}_c$ plotted as a function of the ratio ν/λ (continuous curves). The numbers alongside the curves give the values of the parameter λ. If the initial derivative $\dot{\varphi}_0$ lies above the corresponding curve, i.e., if $\dot{\varphi}_0 > \dot{\varphi}_c$, the solution of Eq. (4) gives spontaneous oscillations. However, if $\dot{\varphi}_0 < \dot{\varphi}_c$, the solution is the constant $\varphi = \varphi_\infty = \sin^{-1}(1/\nu)$ (steady-state case).

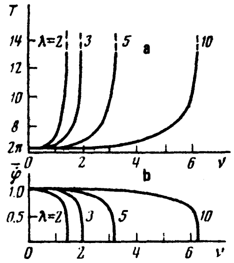

FIG. 8. a) Dependences of the oscillation period on the parameter ν plotted for different values of λ (these values are given alongside each curve). b) Dependences of the average voltage across the junction $\langle\dot{\varphi}\rangle = 2\pi/T$ on ν plotted for different values of λ (taken from Stewart's[6] and McCumber's[7] papers).

the range of parameters ν/λ to the right of the end point of a given branch we have only the steady-state asymptotic solution (the end points on the dashed curve correspond to the critical values of the parameter $\nu_c\lambda_c$ in Fig. 3).

Thus, in the $U < U_1$ case (i.e., for $\nu > 1$), depending on the initial conditions, we can have either the regime (15) corresponding to the dc Josephson effect (in this case the voltage across the junction is $V = 0$) or the stable oscillatory regime of the (14) type, when the voltage across the junction is $V \neq 0$. As long as the ratio ν/λ remains small ($\nu/\lambda \ll 1$), the spontaneous oscillations which develop in the system for $\nu > 1$ are described well by the simple harmonic expression (14), but on increase of ν/λ these oscillations become nonsinusoidal (Fig. 7). Figure 8 shows the dependences of the oscillations period on the quantities ν and λ. We can see that there are certain critical values ν_c and λ_c, at which the oscillation period tends to infinity. These values ν_c and λ_c lie on the dashed curve shown in Fig. 6.

We shall be interested mainly in the oscillations described by the harmonic law (14) for the quantity $\varphi - \tau$. In accordance with this law, the following quantities vary with time in the investigated circuit (Fig. 1): the quantity $\varphi(\tau)$, the voltage across the junction $\dot{\varphi} = V/U$ (the charge in the tunnel-junction capacitor $Q = CV$ varies proportionally to this voltage), the displacement current $\sim\ddot{\varphi}$, the superconducting current through the junction $I_s/I_c = \sin\varphi$, and the total current in the circuit I [Eq. (2)]. Figure 9 shows schematically the time dependences of all these quantities.

We can easily calculate also the total current $\langle I\rangle$ averaged over one oscillation period. The solution (14) can be conveniently represented in the form

$$\varphi(\tau) = \tau + \frac{\nu}{(1+\lambda^2)^{1/2}}\sin(\tau+\psi), \quad \mathrm{tg}\,\psi = \frac{1}{\lambda}.$$

Then, bearing in mind that

$$\langle\sin\varphi\rangle = J_1\left(\frac{\nu}{(1+\lambda^2)^{1/2}}\right)\sin\psi, \quad \langle V\rangle = U\frac{R_t}{R_0+R_t}, \qquad (16)$$
$$\left\langle\frac{dV}{dt}\right\rangle = 0, \quad \langle f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\tau)\,d\tau,$$

where $J_1(z)$ is a Bessel function, we obtain

$$\langle I\rangle = \frac{I_c}{(1+\lambda^2)^{1/2}} J_1\left(\frac{\nu}{(1+\lambda^2)^{1/2}}\right) + \frac{U}{R_t+R_0}. \qquad (17)$$

Using the expansion $J_1(z) = \frac{1}{2}z$ for $z \ll 1$, we find that if $\lambda \gg 1$, then the following relationships apply:

$$\langle I\rangle = \langle I_s\rangle + \langle I_n\rangle, \quad \langle I_s\rangle = I_c\frac{\nu}{2\lambda^2} \ll I_c, \quad \langle I_n\rangle = \frac{U}{R_t+R_0}. \qquad (18)$$

It follows that the average current across the barrier

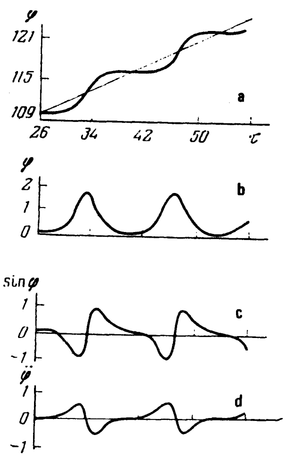

FIG. 7. Example of stable nonlinear spontaneous oscillations in the $\lambda = 10$ and $\nu = 6.2$ case: a) phase φ; b) voltage $\dot{\varphi}$; c) current $\sin\varphi$; d) displacement current $\ddot{\varphi}$.

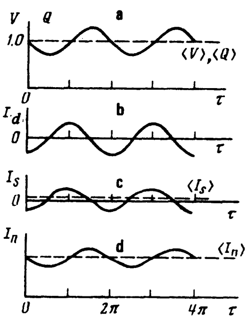

FIG. 9. Schematic time dependences of the voltage ($V \propto \dot{\varphi}$) and charge in the tunnel capacitor ($Q = CV$), displacement current ($I_d = CdV/dt$), superconducting current ($I_s \propto \sin\varphi$), and normal current ($I_n = V/R_t$). The dashed lines are the average values of $\langle I_n\rangle$ and $\langle I_s\rangle$ ($\langle I_s\rangle \ll \langle I_n\rangle$).

is governed mainly by the normal component, whereas the superconducting current oscillates at an amplitude $\pm I_c$ but the average of this current is small: $\langle I_s \rangle \ll I_c$. An increase in U results in an increase of the oscillation frequency $2\pi/t_0$, in accordance with Eqs. (5) and (14), and the oscillation amplitude decreases.

An increase in U alters the voltage $\langle V \rangle$ across the junction and also the normal current (18). It is known that when the voltage across the junction attains the value $\langle V \rangle = 2\Delta$, where Δ is the energy gap of the superconductor, the single-particle current rises strongly.[2-4] This means that the tunnel resistance R_t in Eq. (18) is generally a nonlinear function[4)] of the voltage across the junction $\langle V \rangle$. The experimentally determined dependences $\langle V \rangle(U)$ and $\langle I \rangle(U)$ can be used, in principle, to obtain information on the behavior of the nonlinear function $R_t(\langle V \rangle)$ or $R_t(U)$. For example, the relationship

$$\langle V \rangle = UR_t/(R_t+R_0), \quad \langle I \rangle = U/(R_t+R_0)$$

yields the dependences

$$R_t(U) = \langle V \rangle (U)/[U - \langle V \rangle (U)], \quad R_t(\langle V \rangle) = \langle V \rangle / I(\langle V \rangle).$$

The last dependence can be obtained naturally also from the usual measurements in the case of a constant voltage $\langle V \rangle$ across the junction.

In discussing some qualitative aspects of our approach we shall model the dependence $R_r(\langle V \rangle)$ or $R_t(U)$ by a step function, as shown in Fig. 10. The tunnel resistance is absent right up to the point U_1 at which the transition takes place to the ac regime. The kink at the point U_2 corresponds to the appearance of the single-particle current; the section c represents the region where the current-voltage characteristic has its conventional form. The dashed curve describes the region of possible hysteresis of the resistance. (In fact, the curves may be more complex.)

Figure 11 shows schematically the dependences $\langle I \rangle(U)$ and $\langle V \rangle(U)$ in the case of a constant voltage supplied by an external source; these dependences are found using the approximation for the function $R_t(U)$ shown in Fig. 10. These dependences are nonmonotonic, in accordance with the nonmonotonic nature of $R_t(U)$ (Fig. 10). The dependence of $\langle I \rangle$ on $\langle V \rangle$ corresponding to the conventional current-voltage characteristic is shown in Fig. 12. The sudden fall of the current from $I = I_c$ to $I = I_1$ (Figs. 11 and 12) is due to the appearance of a tunnel resistance R_t on transition to the ac regime. The values of $I_1 = U_1/(R_t + R_0)$ and $V_1 = U_1 R_t/(R_t + R_0)$ are then governed by the total resistance of the system and they can be varied by a suitable choice of the load resistance R_0. In particular, if the load resistance if high ($R_0 \gg R_t$),

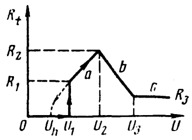

FIG. 10. Schematic model of the nonlinear dependence $R_t(U)$ used to plot the current-voltage characteristics in Fig. 11.

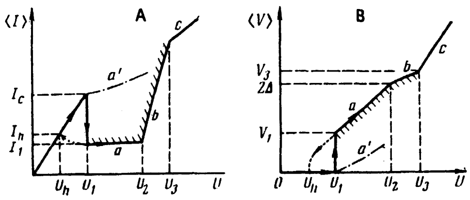

FIG. 11. A) Time-average value $\langle I \rangle$ of the current through a junction plotted as a function of the generator voltage U (schematic representation). The section $0 < U < U_1$ corresponds to the dc Josephson effect. The fall of the current at $U = U_1$ corresponds to the appearance of the tunnel resistance R_1 (see Fig. 10). The kink at the point U_2 simulates a rapid rise of the single-particle current for $\langle V \rangle > 2\Delta$. Thermally unpaired electrons contribute the single-particle current in the shaded region. In the limit $T \to 0$, this region decreases and the characteristic tends to the line a, b, due to nonthermal pair dissociation processes. The dashed curve in the range $U_h < U < U_1$ corresponds to ac states of the type given by Eq. (14). Hysteresis of the current is possible (this is indicated by arrows). The point (I_h, U_h) corresponds to the end points of the curves $\dot{\varphi}_c$ (dashed curve in Fig. 6). For $U < U_h$, only the dc Josephson effect is possible. B) Time-average value $\langle V \rangle$ of the voltage across the junction plotted as a function of the generator voltage U (schematically). The dc Josephson effect ($\langle V \rangle = 0$) in the range $0 < U < U_1$. For $U > U_1$, the conditions are unstable ($\langle V \rangle \neq 0$, curve a). On reduction of U, the ac regime and the nonzero voltage (dashed curve) may be retained right down to $U = U_h$ (voltage hysteresis).

the current and voltage jumps may not be observed and the current-voltage characteristics may have the form represented by the dependences a' shown as chain curves in Figs. 11 and 12. (In view of the damping of the high-frequency oscillations along the circuit, the influence of the resistance R_0 should be much more marked if this resistance is in the immediate vicinity of the Josephson junction: see Fig. 1.) The temperature-dependent parts of the characteristics in Figs. 11 and 12 are identified by shading. In these regions there are, at $T \neq 0$, thermally unpaired electrons which contribute to the quasiparticle current. Cooling reduces the number of electrons exponentially but in the limit $T \to 0$ the normal current in the voltage range $U_1 < U < U_2$ (section a in Fig. 11) still remains finite. It follows that, in this range of voltages, electron pairs are dissociated by a nonthermal mechanism and the distribution of electrons between the states is of nonequilibrium type. This disequilibrium is created and

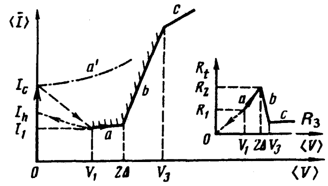

FIG. 12. Conventional current-voltage characteristic $\langle I \rangle(\langle V \rangle)$ plotted schematically on the basis of Figs. 10 and 11. The inset on the right gives the dependence $R_t(\langle V \rangle)$ corresponding to Fig. 10.

maintained by an external source of the voltage U so that an ac current I passes through the junction and there is a voltage drop V across it.[5)]

The dashed curves in Fig. 11 (ending with arrows) indicate the possibility of hysteresis effects associated with the existence of spontaneous oscillation states of the (14) type in the range $U<U_1$ [these oscillations may differ from the simple sinusoidal form predicted by Eq. (14)]. In principle, such states may appear if they are approached from the side of finite voltages $\langle V\rangle \neq 0$, as shown in Fig. 11. The possibility of hysteresis effects follows also from an analysis of the phase picture (Fig. 3a) and from numerical solutions of Eq. (4) shown in Figs. 5–8. The existence of hysteresis solutions has been pointed out earlier.[6,7] The possibility of establishing an ac nonequilibrium state in the range of voltages $U<U_1$ characteristic of the dc Josephson effect is interesting and deserves a more detailed experimental investigation. (Hysteresis effects are discussed also in Solymar's monograph,[4] where the literature of the subject is given.)

We shall conclude by considering the physical aspects of the appearance of the ac Josephson effect, which seems to us to describe adequately Eqs. (1)–(4) and fit the phenomenological approach adopted in the present paper. An external power source supplying a constant voltage $U<U_1=R_0I_c$ produces just the superconducting dc Josephson current $I_s=I_c\sin\varphi_\infty \leq I_c$ through the junction; the voltage across the junction then vanishes, $V=0$, and the normal current is $I_n=0$. On increase of the external voltage to $U>U_1$, the superconducting component fails to ensure a current $I>I_c$ and an additional normal current I_n should appear in the circuit (Fig. 1); this results in a partial conversion of the superconducting to the normal current, produces a nonequilibrium state, and a finite tunnel resistance R_t. At the same time the tunnel capacitor C becomes charged and an electric field appears in the junction. This field accelerates electrons and Cooper pairs travel faster than normal excitations (in the absence of collisions of the superconducting system with the lattice). The resultant excess charge in the capacitor (Fig. 9a) produces a reverse displacement current (Fig. 9b) and, in view of the current (2), undamped nonlinear oscillations are established in the circuit (Figs. 9c and 9d) at a nonzero voltage $\langle V\rangle$ when the junction charge is $Q=C\langle V\rangle$.

Spontaneous oscillations which appear in the circuit are described by the nonlinear equation (4); Eqs. (5) and (14) then show that the amplitude of these oscillations depends on the parameter λ proportional to the tunnel junction capacitance C.[6)] From the phenomenological point of view, a Josephson tunnel junction then acts as an oscillator in the current circuit shown in Fig. 1 (it then resembles a conventional rf tube or some other nonlinear element). In some cases this macroscopic approach may be useful because, in addition to the usual quantum-mechanical terminology based on the concept of the difference between the phases of wave functions,[3,4] it allows us to employ the very illuminating representations of the classical nonlinear theory of oscillations in describing some of the phenomena associated with the ac Josephson effect.

[1)]Equation (4) corresponds to the quasisteady approximation in the theory of alternating currents.[8] Since the characteristic frequencies of natural oscillations in the circuit under consideration are high and the corresponding wavelength is short, this equation clearly cannot provide an accurate description of oscillations in the circuit. In a more rigorous formulation of the problem one would have to allow for the delay and damping of the waves traveling along the circuit. However, we shall not introduce these complications because we are interested in the general qualitative pattern of the effect described by Eq. (4).

[2)]We take this opportunity to thank S. E. Konshteĭn for a valuable discussion of this topic.

[3)]More exactly, the initial conditions govern only the constant in Eq. (14).

[4)]In our phenomenological approach the quantity R_t occurs as some given parameter. We can find it by applying the microscopic theory and considering the kinetics of the processes responsible for the tunnel resistance R_t (the microscopic approach can be found, for example, in the monograph by Kulik and Yanson[3]).

[5)]Disequilibrium discussed here is, in a sense, analogous to the disequilibrium which appears on passage of a static current across a boundary between a superconductor and a normal metal.[10,11] In such a case we also have conversion of the superconducting to the normal current, i.e., the pair dissociation process is of nonthermal origin.

[6)]Here, it is appropriate to mention the fact that the nonlinear equation (4) has also an undamped spontaneous oscillation solution for $\lambda=0$ (i.e., in the absence of the capacitor with $C=0$); an analytic solution of this case can be found in the de Gennes monograph.[12] In the absence of the capacitor electric field ($C=0$) the oscillation regime is entirely due to the nonlinear dependence of the current of the voltage, represented by the term $\nu\sin\varphi$ in Eq. (4). In general, these two factors (i.e., the capacitor field and the nonlinear term) play some role in the development of nonlinear spontaneous oscillations.

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