Sound and hydrodynamic turbulence in a compressible liquid

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Canonical variables in which the potential and vortex motions are separated to a maximum degree are found for the Euler equations with small Mach numbers $M = v_T/c_s$ (v_T is the characteristic turbulent-pulsation velocity). The sound absorption decrement in a homogeneous isotropic turbulent medium calculated by the Wilde canonical diagram technique is $\Gamma_{dis} \approx M^2 v_T/L$ (L is the external scale of the turbulence pulsations). The spectral composition of sound emitted per unit tuburlence volume, $I(\omega)\sim\omega^{-7/2}$, is studied. The spectral energy density $E_s(\omega)$ of sound in equilibrium with the turbulence is calculated. It is shown that the total energy density of the "equilibrium" sound is smaller by a factor M^3 than the energy density of the turbulent pulsations $E_T \approx \rho_0 v_T^{\ 2}$. The effect of compressibility on the Kolmogorov spectrum of hydrodynamic turbulence is considered. This spectrum is of the form $J_k \approx J_k^{\ 0} + \delta J_k$, where $J_k^0 \sim k^{-11/3}$ is the Kolmogorov spectrum of an incompressible liquid and $\delta J_k \approx J_k^{\ 0} M^2 (kL)^{-2/3}$ is the compressibility correction.

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In a compressible liquid, hydrodynamic turbulence is accompanied by a number of processes that arise in the interaction of vortical and potential motions. The process of sound scattering, which has been studied in a number of researches,[1–4] is frequently the basic one. In addition, there is great interest in the processes of absorption and emission of sound by the turbulence, and also in the effect of the compressibility of the medium on the spectrum of the turbulent pulsations.

In Sec. 3 below we obtain an expression for the sound damping decrement in a homogeneous and isotropic turbulent medium:

$$\Gamma_{dis} \approx v_T M^2/L. \tag{1}$$

Here v_T is the velocity of the pulsation motion, L is the external (integrated) scale of the turbulence, $M = v_T/c_s$ is the Mach number. We note that this effect is absent in the approximation of a specified turbulence. In Sec. 3, we study the spectral composition of the sound $I_s(\omega)$ radiated by a unit turbulent volume:

$$I_s(\omega) \approx \rho_0 v_T^2 M^5 (v_T/\omega L)^{7/2}. \tag{2}$$

The integrated intensity of the radiated sound $\int I_s(\omega) d\omega$ was obtained earlier.[1,5] In acoustically opaque turbulence, the sound spectral density $E_s(\omega)$ is determined by the equilibrium between the processes of emission (2) and absorption (1) of the sound:

$$E_s(\omega) = \rho_0 v_T L M^3 (v_T/\omega L)^{7/2}. \tag{3}$$

Here the total energy density of "equilibrium" sound $E_s = \int E_s(\omega) d\omega$ is smaller by a factor of M^3 than the energy density of the turbulence pulsations $\rho_0 v_T^2$. In Sec. 5 we consider the effect of compressibility on the Kolmogorov spectrum of hydrodynamic turbulence J_k. It is shown that $J_k = J_k^0 + \delta J_k$, where $J_k^0 \sim k^{-11/3}$ is the spectrum for an incompressible liquid and

$$\delta J_k \approx J_k^0 M^2/(kL)^{2/3}. \tag{4}$$

We note that another expression was obtained previously[6] for the spectrum J_k in a compressible liquid, which does not contain the parameter M and does not therefore transform into J_k^0 as $M \to 0$.

All the enumerated questions were studied by us within the framework of a single formal scheme—the diagram technique of Wilde—for the canonical equations of motion[7] (Sec. 2). For this purpose, the Euler equations for barotropic flow of a compressible liquid in Sec. 1 are represented in Hamiltonian form (1.2), (1.3) with the aid of the Clebsch variables (1.5).[8] Then, assuming the Mach number to be small, we constructed the nonlinear canonical transformation (1.11)–(1.18) to the new variables in which the vortical motion of the liquid (a_k, a_k^*) and the potential motion (b_k, b_k^*) are separated in maximum fashion. In these variables, the Hamiltonian of the problem takes the form

$$\mathcal{H} = \mathcal{H}_s + \mathcal{H}_t + \mathcal{H}_{st}. \tag{5}$$

Here $\mathcal{H}_s$ is the Hamiltonian for sound in a quiescent liquid[8]:

$$\mathcal{H}_s = \int \omega_k b_k^* b_k \, d^3k + \frac{1}{2} \int V_{k_1|k_2k_3} (b_{k_1}^* b_{k_2} b_{k_3} + \text{c.c.}) \, \delta(k_1 - k_2 - k_3) \, d^3k_1 \, d^3k_2 \, d^3k_3, \tag{6}$$

$\mathcal{H}_t$ is the Hamiltonian for turbulent pulsations of the incompressible liquid[7]:

$$\mathcal{H}_t = \frac{1}{4} \int T_{k_1k_2|k_3k_4} a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4} \times \delta(k_1 + k_2 - k_3 - k_4) \, d^3k_1 \, d^3k_2 \, d^3k_3 \, d^3k_4, \tag{7}$$

$\mathcal{H}_{st}$ is the Hamiltonian of interaction of sound with turbulence calculated by us:

$$\mathcal{H}_{st} = \int S_{12|34} a_{k_1}^* a_{k_2} b_{k_3}^* b_{k_4} \delta(k_1 - k_2 + k_3 - k_4) \, d^3k_1 \, d^3k_2 \, d^3k_3 \, d^3k_4 + \frac{1}{4} \int W_{k,12|34} (b_k + b_{-k}^*) a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4} \delta \times (k - k_1 - k_2 + k_3 + k_4) \, d^3k \, d^3k_1 \, d^3k_2 \, d^3k_3 \, d^3k_4. \tag{8}$$

The first term describes the scattering of sound by turbulence, the second the processes of radiation and absorption of sound by the turbulence.

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1. CANONICAL VARIABLES IN HYDRODYNAMICS. THE HAMILTONIAN

The equations of ideal hydrodynamics, which describe the barotropic flow of a compressible liquid:

$$\dot{\mathbf{v}}+(\mathbf{v}\nabla)\mathbf{v}+\nabla\delta\varepsilon/\delta\rho=0,\quad \dot{\rho}+\operatorname{div}(\rho\mathbf{v})=0, \tag{1.1}$$

permit locally the introduction of canonical variables—the Clebsch variables ρ, Φ: λ, μ,[8] the Hamilton equations for which are of the form

$$\dot{\lambda}=\delta\mathscr{H}/\delta\mu=-\operatorname{div}(\lambda\mathbf{v}),\quad \dot{\mu}=-\delta\mathscr{H}/\delta\lambda=-(\mathbf{v}\nabla)\mu; \tag{1.2}$$

$$\dot{\rho}=\frac{\delta\mathscr{H}}{\delta\Phi}=-\operatorname{div}(\rho\mathbf{v}),\quad \dot{\Phi}=-\frac{\delta\mathscr{H}}{\delta\rho}=-\frac{v^2}{2}+\frac{\lambda}{\rho}(\mathbf{v}\nabla)\mu-\frac{\delta\varepsilon}{\delta\rho}. \tag{1.3}$$

Here

$$\mathscr{H}=\int[\rho v^2/2+\varepsilon(\rho)]d^3\mathbf{r}, \tag{1.4}$$

while

$$\mathbf{v}=\nabla\Phi+\lambda\nabla\mu/\rho. \tag{1.5}$$

The case $\lambda=0$ or $\mu=\text{const}$ corresponds to potential motions of the liquid, which are described by the pair of variables ρ and Φ in correspondence with Eqs. (1.3). In the case of an incompressible liquid, $\dot{\rho}=0$ and Eqs. (1.3) reduce to the relation $\operatorname{div}\mathbf{v}=0$, which allows us to express Φ in terms of λ and μ:

$$\Phi=-\frac{1}{\Delta}\operatorname{div}\left(\frac{\lambda}{\rho}\nabla\mu\right). \tag{1.6}$$

Then, from (1.5),

$$\mathbf{v}=-\frac{1}{\Delta}\operatorname{rot}[\nabla\lambda\times\nabla\mu] \tag{1.7}$$

and Eqs. (1.3) for λ and μ describe the nonpotential motions of the incompressible liquid.

In the general case, we cannot state that the pair ρ and Φ describe the potential motions and the pair λ and μ the vortical motion. Actually, by dividing $\mathbf{v}$ into two parts

$$\mathbf{v}=\mathbf{v}_l+\mathbf{v}_t,\quad \operatorname{rot}\mathbf{v}_l=0,\quad \operatorname{div}\mathbf{v}_t=0, \tag{1.8}$$

it is easy to see that

$$\mathbf{v}_l=\nabla\tilde{\Phi},\quad \tilde{\Phi}=\Phi+\frac{1}{\Delta}\operatorname{div}\left(\frac{\lambda}{\rho}\nabla\mu\right). \tag{1.9}$$

Therefore, the initial Clebsch variables are unsuitable for a description of the turbulence of the compressible liquid: the fields (ρ,Φ) and (λ,μ) turn out to be strongly coupled even in the case in which the velocity of the pulsations is not large, i.e., at

$$\langle v_T^2\rangle/c_s^2=M^2\ll 1. \tag{1.10}$$

Formally, this is manifest by the fact that some matrix elements of the interaction Hamiltonian of these fields increase with increase in the sound velocity, $\sim c^{1/2}$.

Assuming the Mach number M to be small, we construct a canonical transformation that separates the potential and vortical motions of the liquid in the principal order in the new variables q, p; Q, P. We specify the transformation with the aid of the generating functional $F(q, Q; \Phi, M)$, which depends on the new coordinates and the old momenta[9]:

$$\rho(\mathbf{r},t)=\delta F/\delta\Phi(\mathbf{r},t),\quad p(\mathbf{r},t)=\delta F/\delta q(\mathbf{r},t),$$
$$\lambda(\mathbf{r},t)=\delta F/\delta\mu(\mathbf{r},t),\quad P(\mathbf{r},t)=\delta F/\delta Q(\mathbf{r},t). \tag{1.11}$$

Denoting by F_0 the identity-transformation functional

$$F_0=\int(\Phi_q+\mu Q)d^3\mathbf{r}, \tag{1.12}$$

we represent F in the form $F=F_0+F_1$, where $F_1=F_1(q, Q; \mu)$ is so chosen that $q=\rho$ and $p=\tilde{\Phi}$. The functional F_1 does not depend on $\tilde{\Phi}$, is bilinear in μ and Q, and is a series in powers of the variable part of the density ρ_1, $=\rho-\rho_0$:

$$F_1=\frac{1}{\rho_0}\int\left[Q(\nabla\mu)\times\frac{\nabla}{\Delta}\rho_1\right]d^3\mathbf{r}+\ldots \tag{1.13}$$

As an expansion parameter, we use

$$\xi=\frac{k_T}{k_s}\frac{\rho_1}{\rho_0}\approx\frac{\lambda_s}{L}\left(\frac{E_s}{\rho_0 c_s^2}\right)^{1/2}, \tag{1.14}$$

where k_s and k_T are the characteristic wave vectors of sound and turbulence, E_s is the energy of the acoustic motions. Substituting (1.12) and (1.13) in (1.11), and solving the resultant relation by the iteration method (in terms of the paramenter $\xi\ll 1$), we get

$$q=\rho,\quad p=\Phi+\frac{1}{\Delta}\operatorname{div}\left(\frac{\lambda}{\rho}\nabla\mu\right)=\tilde{\Phi}; \tag{1.15}$$

$$\lambda=Q+\frac{1}{\rho_0}\left(\nabla\left(\frac{\nabla}{\Delta}\rho_1\right)\right)Q+O(\xi^2),$$
$$\mu=P+\frac{1}{\rho_0}\left((\nabla P)\left(\frac{\nabla}{\Delta}\rho_1\right)\right)+O(\xi^2). \tag{1.16}$$

In the new variables

$$\mathbf{v}_l=\nabla p,\quad \rho=q,$$
$$\mathbf{v}_t=-\frac{1}{\rho_0\Delta}[\nabla\times[\nabla Q\times\nabla P]]$$
$$-\frac{1}{\rho_0^2\Delta}\left[\nabla\times\left[\nabla\times\left[\left(\frac{\nabla}{\Delta}\rho_1\right)\times[\nabla P\times\nabla Q]\right]\right]\right]+O(\xi^2). \tag{1.17}$$

Thus, the potential motions are described only by the pair q and p; the principal contribution to the vortex motion is made by the "turbulence" variables Q and P. The last term in the expression for $\mathbf{v}_t$ describes the effect of compressibility on the vortex motion.

We transform in the $\mathbf{k}$ representation in standard fashion[7,8] from the variables q_k, p_k; $Q_k\cdot P_k$ to the complex conjugate variables b_k, b_k^*; a_k, a_k^*:

$$\rho_k=(\rho_0 k/2c_s)^{1/2}(b_k+b_{-k}^*),\quad \Phi_k=i(c_s/2\rho_0 k)^{1/2}(b_k-b_k^*), \tag{1.18}$$

$$Q_k=(a_k+a_{-k}^*)/2^{1/2},\quad P_k=(a_k-a_{-k}^*)/i\cdot 2^{1/2}. \tag{1.19}$$

In these variables, the equations of hydrodynamics (1.1) take the form

$$i\dot{a}_k=\delta\mathscr{H}/\delta a_k^*,\quad i\dot{b}_k=\delta\mathscr{H}/\delta b_k^*$$

with the Hamiltonian $\mathscr{H}_2$ obtained from the substitution of (1.18) and (1.19) in (1.4):

$$\mathscr{H}=\mathscr{H}_s+\mathscr{H}_t+\mathscr{H}_{st}.$$

The acoustic Hamiltonian $\mathscr{H}_s$ has the usual form (6), in which

$$\omega_k=kc_s,\quad V_{k_1|k_2k_3}=\frac{1}{2(2\pi)^{1/2}}\left(\frac{c_s}{2\rho_0}\right)^{1/2}(k_1k_2k_3)^{1/2}$$
$$\times[(\mathbf{n}_1\mathbf{n}_2+\mathbf{n}_1\mathbf{n}_3+\mathbf{n}_2\mathbf{n}_3)+(\gamma-2)], \tag{1.20}$$

$\mathbf{n}=\mathbf{k}/k$, γ is the adiabatic exponent: $\gamma=c_p/c_v$. The Hamiltonian $\mathscr{H}_t$ is identical with the Hamiltonian (7) for the turbulence of an incompressible liquid, with

$$T_{k_1k_2|k_3k_4}=\rho_0(\varphi_{k_1k_3}\varphi_{k_2k_4}+\varphi_{k_1k_4}\varphi_{k_2k_3}),$$
$$\varphi_{kk'}=\frac{1}{2\rho_0}\frac{1}{(2\pi)^{1/2}}\left[\mathbf{k}+\mathbf{k}'-(\mathbf{k}-\mathbf{k}')\frac{k^2-k'^2}{|\mathbf{k}-\mathbf{k}'|^2}\right]. \tag{1.21}$$

In the incompressible liquid,

$$\mathbf{v}_t(\mathbf{k})=\int\varphi_{k_1k_2}a_{k_1}^*a_{k_2}\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\,d^2k_1\,d^2k_2. \tag{1.22}$$

In the interaction Hamiltonian $\mathscr{H}_{st}$ (8), the matrix elements are of the form

$$S_{k_1k_2|k_3k_4}=-\frac{1}{2}\frac{(k_3k_4)^{1/2}}{(2\pi)^{1/2}}\varphi_{k_1k_2}(\mathbf{n}_3+\mathbf{n}_4),$$
$$W_{q,k_1k_2|k_3k_4}=\frac{1}{(2\pi)^{1/2}}\left(\frac{2\rho_0 q}{c_s}\right)^{1/2} \tag{1.23}$$
$$\times[(\varphi_{k_1k_3}\mathbf{n}_q)(\varphi_{k_2k_4}\mathbf{n}_q)+(\varphi_{k_1k_4}\mathbf{n}_q)(\varphi_{k_2k_3}\mathbf{n}_q)].$$

In Eq. (8), terms of the form

$$S^{(n)}a_1a_2^*b^{n+2}\ \text{и}\ W^{(n)}a^2a^{*2}b^{n+1}$$

are not written down since they are unimportant for what follows. These terms differ from Aaa^*bb^* and $Wa^{*2}a^2b$ by the small factor $\xi^n\approx[(k_s/\rho_0c_s)^{1/2}b]^n$.

2. STATISTICAL DESCRIPTION OF NONLINEAR TURBULENT AND ACOUSTIC FIELDS

1. Diagram Technique. For the statistical description of nonlinear fields a_k and b_k, we use the canonical diagram technique of Wilde, which is similar to that analyzed in Ref. 7. This technique is analogous to the Keldysh diagram technique and is adapted for the description of classical (nonquantum) systems far from thermodynamic equilibrium. In contrast with Ref. 7, we consider a system of two coupled fields. We introduce the graphic notation for the Green's function (which has the meaning of a linear response to the force F, f):

$$\delta^4(q-q')G_q=\langle\delta b_q/\delta F_{q'}\rangle,\quad \delta^4(q-q')g_q=\langle\delta a_q/\delta f_{q'}\rangle; \tag{2.1}$$

$G_q\sim$ - - - - →, $g_q\sim$ ——→

and the pair correctors:

$$\delta^4(q-q')N_q=\langle b_q^*b_{q'}\rangle,\quad \delta^4(q-q')n_q=\langle a_q^*a_{q'}\rangle, \tag{2.2}$$
$$q=(\mathbf{k},\omega),\quad \delta^4(q)=\delta(\omega)\delta^3(\mathbf{k});$$

$N_q\sim$ ∿∿∿∿→, $n_q\sim$ ꝏꝏꝏ→

These quantities satisfy the Dyson equations

$$G_q=(\omega-\omega_k-\Sigma_q)^{-1},\quad g_q=(\omega-\sigma_q)^{-1}, \tag{2.3}$$

$$N_q=|G_q|^2\Phi_q,\quad n_q=|g_q|^2\varphi_q, \tag{2.4}$$

where Σ_q, σ_q; Φ_q, ϕ_q are the sums of the corresponding irreducible diagrams.

We write down the Dyson equation for $N_{k\omega}$ in the form

$$L_{k\omega}=\Sigma_{k\omega}''N_{k\omega}-\Phi_{k\omega}G_{k\omega}''=0. \tag{2.5}$$

In the case in which the interaction is weak, only diagrams of second order in the vertices need be retained in the series for Σ_q'' and Φ_q, and Eq. (2.5) can be integrated with respect to ω. Then the condition $L_k=\int d\omega L_{k\omega}=0$ will coincide with the stationary kinetic equations for the waves:

$$0=-\Gamma_kN_k+\pi\Phi_k, \tag{2.6}$$

where

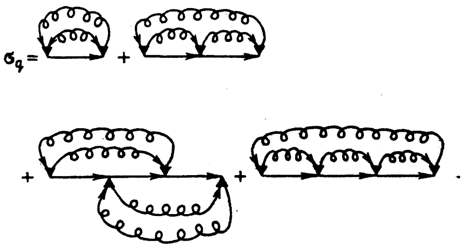
$$\Gamma_k=\Sigma_{k,\omega_k}'',\quad \Phi_k=\Phi_{k,\omega_k}.$$

It is seen from Eq. (2.6) that physically vertex Γ_k—the "damping decrement"—has the meaning of the departure term and Φ_k the meaning of the approach term in the kinetic equation. The kinetic-equation approximation in a number of cases is insufficient and it is necessary to substitute the already partially summed quantities Γ_k and Φ_k in Eq. (2.6). We shall use a similar equation for the hydrodynamic pulsations of the velocity:

$$l_{k\omega}=(n_{k\omega}\sigma_{k\omega}''-\varphi_{k\omega}g_{k\omega}''). \tag{2.7}$$

In the equations $L_{k\omega}=0$, $l_{k\omega}=0$, a cancellation of the longwave divergences responsible for transport takes place. The reason for this is that the transport of the elementary excitation (vortices, sound waves) by a homogeneous velocity field does not lead to a redistribution of the energy in k space.

2. Procedure for the isolation of the dynamic interaction. The diagrams for the turbulence of an incompressible liquid (containing only the vertices T) was analyzed in Ref. 10, where it was shown that at $kL>1$, the odd sequence of diagrams for σ_q is so constructed that the external momentum $\mathbf{k}$, ω is carried through a "backbone" of Green's functions $g_{k\omega}$ oriented from the input to the output. Integrals over the interior lines diverge in the range of small k and the principal contribution to them is made at $k'\sim L^{-1}$ and $\omega'\sim v_T/L$. The contribution of the region $\mathbf{k}'$ is of the order of the external momentum $\mathbf{k}$, and the contributions of the remaining diagrams to the Kolmogorov spectrum are small in the parameter $(kL)^{-1/2}$. The integrals in the higher-order sequence of diagrams can be calculated approximately by assuming that the arguments in the Green's functions in the "backbone" are equal to the external momentum $\mathbf{k}$, ω. This can be drawn graphically as follows:



(2.8)

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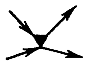
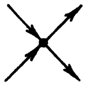


$\sim T_{k_1 k_2 | k_3 k_4} \delta(\mathbf{k}_2 - \mathbf{k}_4) \delta(\omega_2 - \omega_4)$

and differs from the usual vertex:



$\sim T_{k_1 k_2 | k_3 k_4} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4).$

The diagram series (2.8) is summed in Ref. 10, yielding for $g_{k\omega}$ an expression

$$g_{k\omega} = \left\langle \frac{1}{\omega - \mathbf{kv} + i0} \right\rangle_{\mathbf{v}}, \tag{2.9}$$

in which $\langle\ \rangle_{\mathbf{v}}$ denotes averaging over the ensemble of the turbulent velocity field $\mathbf{v}$ at the arbitrary point $\mathbf{r}, t$. For example, for a Gaussian velocity field,

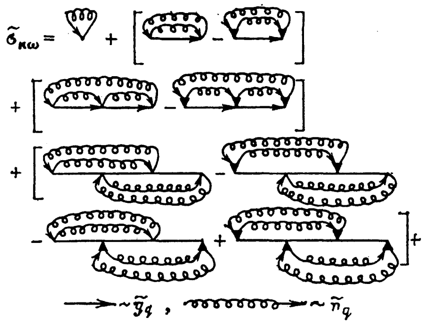
$$\langle f(\mathbf{v}) \rangle_{\mathbf{v}} = \frac{1}{\pi^{3/2} v_T^3} \int f(\mathbf{v}) \exp(-v^2/2v_T^2)\, d^3\mathbf{v}. \tag{2.10}$$

The expression (2.9) and the diagram series (2.8) have a simple physical meaning—they describe the transport of vortices with characteristic dimension k_T^{-1} as a whole by vortices with energy-content scale $L \gg k_T^{-1}$; the gradients of the velocity of the energy-containing vortices are neglected and, consequently, the vortices of scale k_T^{-1} are not deformed.

In order to isolate the much weaker dynamic interaction of vortices of a single scale from the background of the kinematic diagrams (2.8), it was suggested in Ref. 10 to seek the Green's function in the form

$$g_{k\omega} = \langle \tilde{g}_{k,\, \omega - \mathbf{k}\mathbf{v}} \rangle_{\mathbf{v}}, \quad \tilde{g}_{k\omega} = (\omega - \tilde{\sigma}_{k\omega})^{-1}. \tag{2.11}$$

The diagram series for $\tilde{\sigma}_{k\omega}$ contains diagrams of the series for $\sigma_{k\omega}$, from which diagrams of (2.8), which describe the transport, are calculated. We give the first diagrams for $\sigma_{k\omega}$[10]:

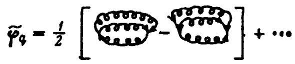


(2.12)

The following equation for $\tilde{n}_q$ was obtained in Ref. 10:

$$\tilde{n}_q = |\tilde{g}_q| \tilde{\Phi}_q. \tag{2.13}$$

The series for φ_q contains the subtractions



In the following, we shall use in a number of cases, in place of (2.5) and (2.7) similar equations in a diagram technique that does not include transport:

$$\tilde{L}_{k\omega} = \tilde{\Sigma}_{k\omega}'' \tilde{N}_{k\omega} - \tilde{\Phi}_{k\omega} \tilde{G}_{k\omega}'', \quad \tilde{l}_{k\omega} = \tilde{\sigma}_{k\omega}'' \tilde{n}_{k\omega} - \tilde{\varphi}_{k\omega} \tilde{g}_{k\omega}''. \tag{2.14}$$

The following relation exists between the quantities $L_{k\omega}$ and $\tilde{L}_{k\omega}$, $l_{k\omega}$ and $\tilde{l}_{k\omega}$[10]:

$$L_{k\omega} = \langle \tilde{L}_{k,\, \omega - \mathbf{k}\mathbf{v}} \rangle_{\mathbf{v}}, \quad l_{k\omega} = \langle \tilde{l}_{k,\, \omega - \mathbf{k}\mathbf{v}} \rangle_{\mathbf{v}}. \tag{2.15}$$

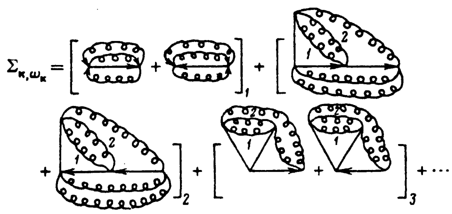
In conclusion of this section, it must be remarked that the integrals in the higher-order diagrams of the series for $\tilde{\sigma}$ and $\tilde{\varphi}$ unfortunately diverge. Starting from the hypothesis of Kolmogorov that the "dynamic" interaction of the vortices is local, we shall assume that these divergences should cancel out and the functions $\tilde{\sigma}_{k\omega}$ and $\tilde{\varphi}_{k\omega}$ can be estimated from the first diagrams which do not contain divergences. It was shown in Ref. 10 that in such a case the equation has a solution in the form of the Kolmogorov spectrum:

$$\tilde{n}_{k\omega} = \rho k^5 f\left(\frac{\omega L}{v_T (kL)^{2/3}}\right), \quad \tilde{g}_{k\omega} = \frac{1}{\omega} g\left(\frac{\omega L}{v_T (kL)^{2/3}}\right). \tag{2.16}$$

3. DAMPING OF SOUND IN A TURBULENT MEDIUM

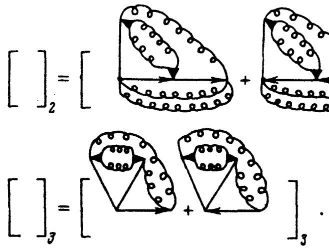
In a turbulent medium, the contribution to the damping decrement Γ_k arises both as a result of the direct absorption of the sound energy by the turbulent pulsations and from processes of sound scattering. We study the first of these mechanisms.

1. *Sound absorption.* We calculate the contribution to $\Sigma_{k,\omega k}$ made by sound absorption in processes described by the W vertices in (8). Here it suffices to take into account diagrams of the order W^2; diagrams containing W^4 and $S^n W^2$ are small in comparison with M. Thus,

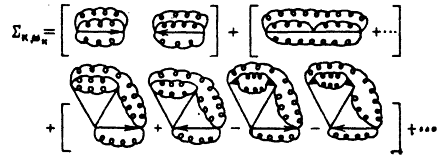


(3.1)

In the calculation of $\mathrm{Im}\, \Sigma_{k\omega}$ a strong cancellation takes place in the diagrams that differ in the arrow directions. As a result, the sums of each of the two diagrams in the brackets in (3.1) turn out to be of the order of $(kv_T)(kL)^{-2/3} M^{3/2}$. Moreover, the sum of the entire diagram series (3.1) is still less than this quantity in terms of the parameter $(kL/M)^{-1/3}$. Actually, the principal contribution to the integration over those internal lines n_q which are not joined to an input or output [n_{q_1}, n_{q_2} in diagrams (3.1)] is made by the energy-containing region: k_1, $k_2 \sim L^{-1}$, ω_1, $\omega_2 \sim v_T L^{-1}$. In this region, we can take in place of the diagram $[\ ]_2$ and $[\ ]_3$



They describe the transport of the absorbing volume by vortices of scale L. It can be summed by the method described in Sec. 2. Then the Green's function G takes the form (2.9) and we obtain for $\tilde{\Sigma}_q$ a diagram series which must be calculated with the Kolmogorov functions (2.16) that do not contain the transport:



(3.2)

The analytic expression for the first pair of diagrams has the form

$$\Sigma_q'' = \int |W_{\mathbf{k},\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3\mathbf{k}_4}|^2 n_{\mathbf{k}_1\omega_1} n_{\mathbf{k}_2\omega_2} n_{\mathbf{k}_3\omega_3} g''_{\mathbf{k}_4\omega_4} \times[\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_4)\delta(kc_s+\omega_1-\omega_2+\omega_3-\omega_4) -\delta(-\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_4)\delta(-kc_s+\omega_1-\omega_2+\omega_3-\omega_4)] \times d^3\mathbf{k}_1\, d\omega_1\, d^3\mathbf{k}_2\, d\omega_2\, d^3\mathbf{k}_3\, d\omega_3\, d^3\mathbf{k}_4\, d\omega_4. \quad (3.3)$$

The basic contribution to this expression is made by integration over the region of scales $k_i \cong k_T$, where the characteristic Kolmogorov frequency $\omega_T \approx v/L(k_T L)^{2/3}$ is of order kc_s. We assume that k_T falls in the inertial interval $L^{-1} < k_T < L^{-1}\mathrm{Re}^{3/4}$, or

$$ML^{-1} < k_s < ML^{-1}\,\mathrm{Re}^{1/2}. \quad (3.4)$$

With account of this circumstance, it is easy to estimate (3.3):

$$\Gamma_{\mathbf{k}} = -\mathrm{Im}\,\Sigma_{\mathbf{k},\omega_\mathbf{k}} \approx v_T M^2/L. \quad (3.5)$$

We obtain this result qualitatively by considering the change in energy of turbulent pulsations with a characteristic frequency of motion:

$$\omega_T \approx \frac{v_T}{L}(k_T L)^{2/3} \approx \omega_s = k_s c_s.$$

In the field of a sound wave of intensity E_s, the density of the liquid oscillates with amplitude $\delta\rho_s = \rho_0(E_s/\rho_0 c_s^2)^{1/2}$ which produces changes in the velocity v by an amount $\delta v_T \approx v_T \delta\rho_s/\rho_0$ within the time of a single period ω_s^{-1}. The corresponding change in the energy of the vortex motion has the form

$$\delta E_T \approx E_T\left[\frac{\delta\rho_s}{\rho_0} + \frac{\delta\rho_s^2}{\rho_0^2} + \dots\right],$$

whence

$$-\frac{dE_s}{dt} = \frac{d}{dt}\overline{\delta E_T} \approx \omega_s \overline{\delta E_T} \approx \omega_s E_T\left(\frac{\delta\rho_s}{\rho_0}\right)^2 \approx \omega_s \frac{E_T}{\rho_0 c_s^2} E_s.$$

Then, substituting $E_T \approx \rho_0 v_T^2 (k_T L)^{-2/3}$ and expressing k_T in terms of k_s, we get, with the help of the relation $\omega_T \approx k_s c_s$,

$$dE_s/dt \approx -v_T M^2 E_s/L,$$

which corresponds to the estimate (3.5).

We compare the sound absorption by turbulence (3.5) with its damping due to viscosity and heat conduction $\Gamma_0 \approx \nu k_s^2$.[11] At small k_s, the damping (3.5) predominates and is comparable with Γ_0 at $k_s = k_0$, where

$$k_0^2 \approx \frac{v_T M^2}{\nu L} = \frac{M^2}{L^2}\frac{v_T L}{\nu} = \frac{M^2}{L^2}\mathrm{Re}. \quad (3.6)$$

Thus, the damping due to turbulence predominates over the whole range (3.4).

We can now formulate the question as to the acoustic transparency of a turbulent layer of thickness Λ. If $f_s < L^{-1}$, then the sound propagates along a straight line and the layer of thickness $\Lambda_{tr} = LM^{-3}$ turns out to be opaque because of the direct absorption of the sound in the processes (3.2). At $L^{-1} < k_s < l_0^{-1}$, it is necessary to take into account processes of elastic sound scattering, which lead to random walk of the phonons in the turbulent medium.[4]

4. STUDY OF THE SOUND OF TURBULENCE

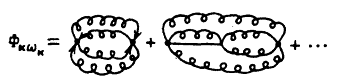
1. *Study of "transparent turbulence."* We use the kinetic equations for sound (2.6), to which we add the phenomenological component $\dot{N}_\mathbf{k}^0$, which describes the losses due to radiation of sound from the turbulent volume:

$$\tfrac{1}{2}\dot{N}_\mathbf{k}^0 = -\Gamma_\mathbf{k} N_\mathbf{k} + \pi\Phi_{\mathbf{k},\omega_\mathbf{k}}. \quad (4.1)$$

If the volume occupied by the turbulence is so small that it is acoustically transparent, then the losses due to damping Γ_k can be neglected. Then we get for the energy flux density (in a unit interval of frequency per unit time per unit volume)

$$I_\mathbf{k}(\omega) = 4\pi k^3 \dot{N}_\mathbf{k} = 4\pi^2 k^3 \Phi_{\mathbf{k},\omega_\mathbf{k}}. \quad (4.2)$$

The principal sequence of diagrams for Φ is proportional to W^2 and does not contain sound lines:



It is not difficult to understand that this sequence is summed to the fourth correlator of the turbulent velocity:

$$\Phi_{\mathbf{k},\omega_\mathbf{k}} = \frac{2\rho_0 k}{(2\pi)^3 c_s}\int I^{zzzz}_{1,2,3,4}\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_3)\delta(\mathbf{k}+\mathbf{k}_2+\mathbf{k}_4)\delta(\omega_\mathbf{k}+\omega_1+\omega_3)\cdot\delta(\omega_\mathbf{k}+\omega_2+\omega_4)\, d^3\mathbf{k}_1\, d\omega_1\, d^3\mathbf{k}_2\, d\omega_2\, d^3\mathbf{k}_3\, d\omega_3\, d^3\mathbf{k}_4\, d\omega_4. \quad (4.3)$$

The z axis is oriented along k and

$$\langle v^z_{\mathbf{k}_1\omega_1} v^z_{\mathbf{k}_2\omega_2} v^z_{\mathbf{k}_3\omega_3} v^z_{\mathbf{k}_4\omega_4}\rangle = I^{zzzz}_{1,2,3,4}\,\delta^4(q_1+q_2+q_3+q_4). \quad (4.4)$$

Estimate of this expression on the Kolmogorov spectrum of isotropic turbulence gives

$$\Phi_{k_s \omega_k} \approx \rho_0 v_T^2 L^3 M^2 (M/kL)^{13/3}. \tag{4.5}$$

In correspondence with (4.2), this expression determines the spectral content of the sound radiated acoustically by the transparent turbulence:

$$I_s(\omega) = \rho_0 v_T^2 M^5 (M/kL)^{7/3}. \tag{4.6}$$

We note that the principal contribution to the emission of sound at a frequency ω_s is made by vortices of scale l/k_T, for which the characteristic angular frequency is $\omega_T \approx v_T L^{-1} (k_T L)^{2/3}$ is of the order of ω_s. The most intensively radiated is sound of vortices of the energy content scale L at frequency v_T/L.

We obtain the estimate (4.6) qualitatively. In the decay of the vortex of scale k_T^{-1} in time ω_T^{-1}, density pulsations

$$\delta\rho_T \approx \rho_0 \frac{v_T^2(k_T)}{c_s^2} \approx \rho_0 \frac{M^2}{(k_T L)^{2/3}}$$

develop which produce changes in the volume of the vortex V at a rate

$$\dot V = \frac{dV}{dt} \approx V \frac{\delta\rho_T}{\rho_0} \omega_T \approx \frac{v_T}{L} \frac{M^2}{k_T^3}.$$

The pulsating volume causes radiation of sound with intensity

$$I \approx \rho_0 \omega_s^2 \dot V^2 / c_s^2$$

(see Ref. 11). Substituting there the estimate for $\dot V$, multiplying the result by the number of vortices per unit volume, $1/k_T^3$, and expressing k_T in terms of k_s according to the formula $\omega_T \simeq k_s c_s$, we obtain the formula (4.6) for $dI/d\omega \approx I/\omega = I(\omega)$.

The total flow is determined by the integral of the expression (5.6) with respect to ω:

$$I \approx \rho_0 v_T^3 M^5 / L. \tag{4.7}$$

The estimate (4.7) for the integrated intensity was obtained previously by other methods.[1,5]

2. *The sound spectrum in a non-transparent medium.* It is determined by Eq. (4.1), in which it is necessary to neglect losses due to radiation:

$$N_k = \pi \Phi_{k, \omega_k} / \Gamma_k. \tag{4.8}$$

This spectrum is determined by the equilibrium between the process of sound radiation (4.5) and the reverse process of its absorption (3.5). Scattering processes lead only to isotropization and, of course, do not affect the spectral composition of $I_s(\omega)$ (4.6). Thus,

$$N_k \approx \rho_0 v_T L^4 (M/kL)^{13/3}. \tag{4.9}$$

This expression is valid in the interval (3.4) for k_s at which k_T falls in the inertial interval. The sound energy density per unit frequency interval is of the form

$$E_s(\omega_k) = 4\pi k_s^2 N_k \approx \rho_0 v_T L M^3 (M/kL)^{7/3}. \tag{4.10}$$

The total "equilibrium" sound energy density

$$E_s = \int E_s(\omega)\, d\omega \approx \rho_0 v_T^2 M^3 \tag{4.11}$$

is smaller by a factor of M^3 than the energy density of the turbulent pulsations.

5. EFFECT OF COMPRESSIBILITY ON THE KOLMOGOROV SPECTRUM OF HYDRODYNAMIC TURBULENCE

It is known that the pressure fluctuations in a turbulent medium are proportional to ρv_T^2. Consequently, the relative changes in the density of the medium is

$$\delta\rho_T/\rho_0 \approx v_T^2/c_s^2 = M^2,$$

and the effect of compressibility on the hydrodynamic turbulence is small in terms of this parameter. Therefore, in contrast with Ref. 6, in which the spectrum was found independent of the Mach number M for the hydrodynamic turbulence in a compressible liquid J_k, we shall seek J_k in the form

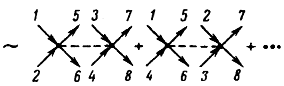
$$J_k = J_k^0 (1 + M^2 f(k)). \tag{5.1}$$

Here $J_k^0 \sim^{-11/3}$ is the Kolmogorov spectrum of the incompressible liquid.

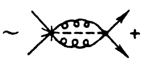
Compressibility leads to the appearance of additional vertices of interaction for the canonical variables a_k that describe the vortex motion. The simplest of such vertices arises in second order perturbation theory in the interaction W [see the Hamiltonian (8)]:

$$T^{(1)}_{1234|5678} = \int [W_{k,12|56} W_{k,34|78} \delta(-k + k_1 + k_2 - k_3 - k_4) + \ldots] (G_{k,\omega_k} + G^*_{-k,\omega_k})\, d^3k. \tag{5.2}$$

Terms not written down in (5.2) were obtained by permutations within the groups of indices $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, $\mathbf{k}_4$ and $\mathbf{k}_5$, $\mathbf{k}_6$, $\mathbf{k}_7$, $\mathbf{k}_8$. We represent (5.2) graphically:

$T^{(1)}_{1234|5678} \sim$ 

In hydrodynamic diagrams, the interaction $T^{(1)}$ leads to the appearance of a correction to the fourth-order vertex:

$\delta T_{12|34} \sim$  $+ \cdots$

Using (5.2) and substituting the explicit expression (1.23) for W, we obtain after partial summation in order of magnitude,

$$\delta T_{12|34} \approx T_{12|34} \int \frac{k}{c_s} J_{k'\Omega'} G_{k\omega} \delta(k_1 - k_2 + k' - k) \times \delta(\Omega_1 - \Omega_2 + \Omega' - \omega)\, d^3k\, d^3k'\, d\Omega'\, d\omega. \tag{5.3}$$

We divide $\delta T_{12|34}$ into two parts in the following manner:

$$\delta T_{12|34} = \delta T^{(1)}_{12|34} + \delta T^{(2)}_{12|34}; \tag{5.4}$$

$$\delta T^{(1)}_{12|34} \approx T_{12|34} \int \frac{k}{c_s} J_{k'\Omega'} G_{k\omega} \delta(k_1 - k_2 - k) \delta(\Omega_1 - \Omega_2 - \omega)\, d^3k\, d^3k'\, d\Omega'\, d\omega \approx T_{12|34} M^2,$$

$$\delta T^{(2)}_{12|34} \approx T_{12|34} \int \frac{k}{c_s} J_{k'\Omega'} G_{k\omega} [\delta(k_1 - k_2 + k' - k) \delta(\Omega_1 - \Omega_2 + \Omega' - \omega) - \delta(k_1 - k_2 - k) \delta(\Omega_1 - \Omega_2 - \omega)]\, d^3k\, d^3k'\, d\Omega'\, d\omega \approx T_{12|34} M^2 (k_1 L)^{-2/3}. \tag{5.5}$$

These estimates are made under the assumption that $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, $\mathbf{k}_4$ are of the same order of magnitude. We note that in the expression (5.4), the chief contribution is made by the region of integration $k' \sim L^{-1}$, and in (5.5), by the region $k' \sim k_1$.

The turbulence spectrum is determined from Eq. (2.14) for $l_{\mathbf{k},\omega}$. The increment to the vertex $\delta T^{(1)}_{12|34} \approx M^2 T_{12|34}$ has the same index of homogeneity as the vertex $T_{12|34}$ for the incompressible liquid. Therefore, it does not lead to a change in the Kolmogorov exponent. The increment

$$\delta T^{(2)}_{12|34} \approx T_{12|34} M^2 (kL)^{-2/3}$$

leads to the appearance in the equation for $l_{\mathbf{k}\,\omega}$ of additional terms with the parameter of smallness $M^2(kL)^{2/3}$. They should be cancelled by the terms which arise because of the correction (5.1) to the spectrum J_k. Thus $f(\mathbf{k}) \approx (kL)^{-2/3}$,

$$J_k = J_k^0 + \delta J_k, \quad \delta J_k \approx J_k^0 M^2/(kL)^{2/3}. \tag{5.6}$$

This result has a simple meaning; in correspondence with the Kolmogorov hypothesis that the spectrum interaction is local, the value of J_k in the inertial interval of the scales ($kL > 1$) cannot depend on the velocity of motion v_T in the scale L, i.e., on $M^2 = v_T^2/c_s^2$. Only the fluctuations of density due to motions of the liquid $v(k)$ of the same scale $1/k$ are important. Taking it into account that in the inertial interval $v(k) \approx v_T(kL)^{-1/2}$, we obtain

$$\frac{\delta\rho_T(k)}{\rho_0} \approx \frac{v^2(k)}{c_s^2} \approx M^2(kL)^{-2/3}.$$

It can be shown that the relative correction to the spectrum ($\delta J_k/J_k$) is of the order of the square of the Mach number calculated from the circular velocity of motion of vortices of scale $1/k$.

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Auxiliary boundary conditions in the theory of additional light waves and excitons in bounded crystals

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Frenkel excitons [Phys. Rev. **37**, 17, 1276 (1931)] and corresponding optical excitons (optical waves, polaritons) are considered for a semiinfinite crystal of any symmetry but possessing an inversion center. It is shown that the solutions for a semiinfinite crystal can be constructed as linear combinations of solutions for an infinite crystal only in the case of normal incidence or for selected wave polarizations, and if the interactions of close (A) or distant (B) cells in a crystal predominate. The coefficients of such linear combinations are found from auxiliary boundary conditions whose justification is the main task of the present paper. It is shown that these auxiliary boundary conditions, proved by the present author for the case of degenerate excitons and restricted crystal symmetries [S. I. Pekar, Sov. Phys. JETP **6**, 785, (1958)], are always valid in case A, but in case B they apply only for normal incidence and certain polarizations of light. Auxiliary boundary conditions of a new type are obtained for all other cases and for dipole-forbidden radiative transitions. The results obtained agree, in the appropriate special cases, with microcalculations based on the simplest model of a crystal.

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1. INTRODUCTION

According to the conventional theory of birefringence, two orthogonal waves of the same frequency and with the same direction of propagation may travel in a crystal. Their amplitudes can be described uniquely in terms of the amplitude of a wave incident on a crystal from vacuum, using the well-known conditions of continuity of the tangential projections of an electron and magnetic fields. However, in the vicinity of the ex-

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