Asymmetric distributions of parametrically excited waves

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An investigation is reported of asymmetric distributions of steady-state parametrically excited waves in the case when the equations of motion have spatial symmetry, for example, rotational or axial. It is shown that the symmetric and almost all the asymmetric distributions are frequently characterized by instabilities whose growth produces the lowest-symmetry distributions. These results are used to analyze the distributions of parametrically excited spin waves in ferromagnets and antiferromagnets, for which stable asymmetric distributions with several pairs of excited waves are found. Consequences of the theory and experimental data are discussed.

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INTRODUCTION

The theory of parametrically excited waves in continuous media, particularly magnetic materials and plasma, has recently received a considerable impetus (see, for example, recent reviews[1,2]). One of the most important questions is the nature of the states of motion established as a result of growth of parametric instabilities. The present paper reports an investigation of steady-state distributions of waves excited parametrically under the following conditions:

a) the initial equations of motion have some spatial symmetry in particular, they are symmetric relative to rotation or turning about some axis;

b) only the four-wave interactions of parametrically excited waves are important so that the equations of motion have the form (1.1).

We shall concentrate on parametrically excited spin waves in magnetically ordered crystals but the results will be applicable also to parametrically excited waves of other types to the extent that equations of the (1.1) type are satisfied.

If the equations of motion are invariant under certain spatial transformations, then asymptotically stable solutions, describing the steady states of motion, are either invariant under these transformations or noninvariant. The noninvariant solutions form sets such that the application of a spatial transformation which leaves the equations invariant transforms noninvariant solution into some other solution in the same set.[1)]

Symmetric solutions of the system (1) in the case of parametric excitation of spin waves in magnetic materials under rotational symmetry conditions were investigated by Zakharov, L'vov, and Starobinets.[1] The present paper is concerned with asymmetric solutions.[2)] We shall find sets of asymmetric solutions and consider instabilities whose growth lowers the symmetry of the solutions. We shall use the results to analyze the distribution of parametrically excited spin waves in ferromagnets and antiferromagnets (Sec. 3). We shall show that stable states of motion, in which two pairs of waves are excited, can exist in a parallel-pumped easy-axis ferromagnet. Study of the similar process in cubic antiferromagnets shows that symmetric distributions are always unstable and stable states of motion may be those in which one or two wave pairs are excited. We shall conclude (Sec. 4) with a discussion of the results obtained, some consequences, and experimental data.

1. PROPERTIES OF STEADY STATES OF MOTION

We shall use the reduced equations of motion which allow only for the interaction of pairs of waves (with the wave vectors k and -k) with one another and with an alternating pump field:[3)]

$$\frac{1}{2}\dot{n}_{\mathbf{k}}=n_{\mathbf{k}}\Big[-\gamma_{\mathbf{k}}+hV_{\mathbf{k}}\sin\psi_{\mathbf{k}}+\sum_{\mathbf{k}'}S_{\mathbf{kk}'}(1-\delta_{\mathbf{kk}'})(1-\delta_{-\mathbf{kk}'})\sin(\psi_{\mathbf{k}}-\psi_{\mathbf{k}'})\Big], \tag{1.1a}$$

$$\frac{1}{2}\dot{\psi}_{\mathbf{k}}=\omega_{\mathbf{k}}-\frac{\omega_0}{2}+\sum_{\mathbf{k}'}T_{\mathbf{kk}'}(2-\delta_{\mathbf{kk}'})n_{\mathbf{k}'}+hV_{\mathbf{k}}\cos\psi_{\mathbf{k}}+\sum_{\mathbf{k}'}S_{\mathbf{kk}'}(1-\delta_{\mathbf{kk}'})(1-\delta_{-\mathbf{kk}'})n_{\mathbf{k}'}\cos(\psi_{\mathbf{k}}-\psi_{\mathbf{k}'}). \tag{1.1b}$$

Here, $n_{\mathbf{k}}$ are the squares of the moduli of the amplitudes $a_{\mathbf{k}}$; $\psi_{\mathbf{k}}$ is the sum of the phases of a pair of waves; $\omega_{\mathbf{k}}$ are the frequencies and $\gamma_{\mathbf{k}}$ the damping coefficients of the waves; ω_0 and h are the frequency and amplitude of the pump field; $V_{\mathbf{k}}$ is the coefficient of the interaction of a pair with the pump field; $T_{\mathbf{kk}'}$ and $S_{\mathbf{kk}'}$ are related by simple expressions to the four-wave interaction coefficients $T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}$:

$$T_{\mathbf{kk}'}=T_{\mathbf{kkk}'\mathbf{k}'},\quad S_{\mathbf{kk}'}=T_{-\mathbf{kk}-\mathbf{k}'\mathbf{k}'},$$

which give the symmetry properties of the coefficients:

$$T_{\mathbf{kk}'}=T_{\mathbf{k}'\mathbf{k}},\quad S_{\mathbf{kk}'}=S_{-\mathbf{kk}'}=S_{\mathbf{k}'\mathbf{k}}. \tag{1.2}$$

We shall be interested in asymptotically stable solutions of the system (1.1), describing states of motion which become steady as a result of growth of a parametric instability. We can identify three classes of such solutions: stationary, conditionally periodic, and

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stochastic. The stationary solutions describe a set of waves with constant amplitudes and phases, which are defined by the equations

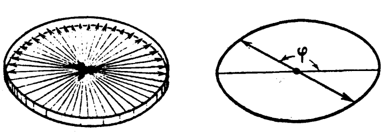
$$\dot{n}_k=0,\ \dot{\psi}_k=0. \tag{1.3}$$

The existence of stable solutions of this type may be expected for a small excess of the pump amplitude above the threshold.

The nominally periodic solutions describe sets of waves with conditionally periodic variations of the amplitudes and phases. These solutions can be found by substituting, in the equations of motion, the values of $a_k(t)$ in the form of Fourier-series expansions in which the expansion coefficients and frequencies are unknown. This produces a system of algebraic equations for the frequencies and expansion coefficients [a special case of such a system is (1.3)], which describes the conditionally periodic solutions. The existence of stable solutions of this class may also be expected for a small excess of the pump amplitude above the threshold.

We may find that among these stationary and conditionally periodic solutions there are none which are asymptotically stable. Then, clearly, the solutions are stochastic. The existence of stochastic solutions may be expected for a considerable excess of the pump amplitude above the threshold. It should be noted that the existence of such states of motion in dynamic systems of the general kind demonstrated by Ruelle and Takens[4] and discovered, in particular, for systems of interacting waves.[5,6]

As pointed out earlier, if the equations are spatially symmetric, we can have symmetric or asymmetric asymptotically stable solutions. Let us consider a simple example. We shall assume that there is a preferred axis, for example, one coinciding with the direction of the pump field, and the coefficients describing the interaction of waves with one another and the pump field are independent of the angle φ associated with this axis. We shall assume that there is symmetry of rotations about the axis. There may exist a solution which is invariant under these rotations. For example, the solution may be such that the ends of the wave vectors fill a ring or other geometric figure invariant relative to rotations and the amplitudes of all the waves are equal (Fig. 1a). There may also be solutions in each of which only the amplitudes of one pair of waves do not vanish (these are represented schematically in Fig. 1b). The orientation of the wave vectors relative to φ is arbitrary, i.e., there is an infinite number of asymmetric states of motion differing only in respect of the orientations of the wave vectors relative to φ.



a b

FIG. 1. In addition to the symmetric states of motion in which waves have equal amplitudes and wave vectors filling a ring (a), there may be asymmetric states in each of which one pair of waves is excited (b).

The problem is to find and investigate the stability of asymmetric steady-state solutions of the system (1.1).

2. SYMMETRIC AND ASYMMETRIC STATES OF MOTION

We shall assume that the problem has symmetry relative to rotations through angles which are a multiple of $\Delta\varphi=\pi/R$ (R is an integer) and the rotation takes place about a certain axis $\mathbf{H}_0$. Considering hV_k/γ_k as a function of the polar and azimuthal angles (θ_k and φ_k) linked with the $\mathbf{H}_0$ axis, we shall use $\Omega_0=\{\theta_k^0,\ \varphi_k^0\}$ to denote the set of angles at which the ratio hV_k/γ_k has maxima. The waves whose wave vectors belong to the set Ω_0 are excited first and there is a symmetric solution[1,3] such that

$$n_k\equiv n=hV_k(2R\bar{S}_k)^{-1}\cos\psi_k,\qquad \bar{S}_k=(2R)^{-1}\sum_{k'}S_{kk'},\quad k'\in\Omega_0, \tag{2.1a}$$

$$\sin\psi_k\equiv\sin\psi=\gamma_k/hV_k \tag{2.1b}$$

for θ_k and φ_k belonging to the selected set of angles Ω_0, and

$$n_k=0$$

for all other orientations of the wave vectors.

The value of $|\mathbf{k}|$ in Eq. (2.1a) is found from the condition of stability of the solution (2.1a) relative to variations of the amplitudes whose stationary values vanish (in the terminology of Zakharov *et al.*,[1] this represents an "external instability"), which gives the relationship

$$\omega_k-\omega_0/2+(4R\bar{T}_k-T_{kk}-2S_{kk})n=0,\qquad \bar{T}_k=(2R)^{-1}\sum_{k'}T_{kk'},\quad k'\in\Omega_0. \tag{2.2}$$

In addition to the symmetric solutions [Eqs. (2.1) and (2.2)], there can be asymmetric stationary solutions. We shall draw attention to one property of the relationship (2.1a) which governs the stationary amplitudes: it includes the sum over all the wave vectors in the set Ω_0, i.e., in the final analysis it includes the quantity $\bar{S}_k$, independent of the angles. Consequently, in addition to Eqs. (2.1) and (2.2), there may be solutions of lower symmetry. We shall show when this is possible. Let us assume that R is not a prime and r is a factor which is an integer so that $R=mr$. Then, the set Ω_0 can be split into m subsets of the wave vectors $\Omega_r^{(s)}$, where $s=1, 2, \dots, m$, in each of which there are $2r$ values of the angles φ, the differences between which are multiples of $m\Delta\varphi$. It may then happen that the average value of the coefficient $S_{kk'}$ over a subset $\Omega_r^{(s)}$ is equal to its average value over the set Ω_0, i.e.,

$$(2r)^{-1}\sum_{\mathbf{k}'} S_{\mathbf{kk}'}=(2R)^{-1}\sum_{\mathbf{k}''} S_{\mathbf{kk}''}=\bar{S}_{\mathbf{k}}, \quad \mathbf{k}'\in\Omega_r^{(s)}, \quad \mathbf{k}''\in\Omega_0. \tag{2.3}$$

We can easily show that then, in addition to the symmetric solution of Eqs. (2.1) and (2.2), there are m asymmetric solutions and in every one of these the only wave amplitudes that do not vanish are those whose wave vectors belong to one of the subsets $\Omega_r^{(s)}$. These solutions are of the following form:

$$\begin{gathered} n_{\mathbf{k}}=n_r=-hV_{\mathbf{k}}(2r\bar{S}_{\mathbf{k}})^{-1}\cos\psi_{\mathbf{k}}, \\ \sin\psi_{\mathbf{k}}=\sin\psi=\gamma_{\mathbf{k}}/hV_{\mathbf{k}}, \quad \mathbf{k}\in\Omega_r^{(s)}, \\ \omega_{\mathbf{k}}-\omega_0/2+(4r\bar{T}_{\mathbf{k}}-T_{\mathbf{kk}}-2S_{\mathbf{kk}})n_r=0, \end{gathered} \tag{2.4}$$

and $n_{\mathbf{k}}=0$ if $\mathbf{k}$ does not belong to $\Omega_r^{(s)}$.

Combining an arbitrary number of the wave-vector subsets $\Omega_r^{(s)}$ we obtain subsets whose average coefficients $S_{\mathbf{kk}'}$ are identical with $\bar{S}_{\mathbf{k}}$ on the basis of Eq. (2.3). We shall use $\Omega_{lr}^{(i)}$ to denote the i-th combination of this kind consisting of $l(1\leq l\leq m)$ subsets $\Omega_r^{(s)}$. By analogy with Eq. (2.4), we can construct stationary solutions of the system (1.1) in each of which the only nonvanishing and equal wave amplitudes are those with the wave vectors belonging to $\Omega_{lr}^{(i)}$. Then, the stationary value $\psi_{\mathbf{k}}=\psi$, $n_{\mathbf{k}}=n_{lr}$, and also the value of $|\mathbf{k}|$ are given by the relationships (2.4) where r is replaced with lr. We can easily see that for $l=m$ [i.e., when r is replaced with R in Eq. (2.4)], the solution (2.4) is identical with the symmetric solution of Eqs. (2.1) and (2.2).

We shall now investigate the stability of the above solutions in the presence of variations of amplitudes and phases of the excited waves (in the terminology of Zakharov *et al.*,[1] this corresponds to an "internal instability").

We shall first consider the stability of the resultant solutions in the presence of homogeneous perturbations of amplitudes and phases. Let

$$n_{\mathbf{k}}=n_{lr}, \quad \psi_{\mathbf{k}}=\psi, \quad \mathbf{k}\in\Omega_{lr}^{(i)} \tag{2.5}$$

be one of the stationary solutions and assume that perturbations of all the amplitudes and phases of the excited waves are equal, so that

$$n_{\mathbf{k}}=n_{lr}+\alpha_{lr}, \quad \psi_{\mathbf{k}}=\psi+\varphi_{lr}, \quad \mathbf{k}\in\Omega_{lr}^{(i)}. \tag{2.6}$$

We shall substitute Eq. (2.6) in the equations of motion (1.1) linearizing them with respect to α_{lr}, φ_{lr} and we shall seek solutions of the resultant equations in the form $\varphi_{lr}\propto\exp(\lambda^{(lr)}t)$. The characteristic exponents are given by the expression

$$\lambda_{1,2}^{(lr)}=-\gamma_{\mathbf{k}}\pm\{\gamma_{\mathbf{k}}^2-2\bar{S}_{\mathbf{k}}[2lr(2\bar{T}_{\mathbf{k}}+\bar{S}_{\mathbf{k}})-T_{\mathbf{kk}}-2S_{\mathbf{kk}}]lrn_{lr}^2\}^{1/2}, \tag{2.7}$$

from which we obtain the criterion of stability of the stationary solution (2.5) in the presence of homogeneous perturbations:

$$\bar{S}_{\mathbf{k}}(4lr\bar{T}_{\mathbf{k}}+2lr\bar{S}_{\mathbf{k}}-T_{\mathbf{kk}}-2S_{\mathbf{kk}})>0 \tag{2.8}$$

We shall now consider perturbations of the following kind which affect one of the states of motion in which lr pairs of waves described by Eq. (2.5) are excited: we shall assume that only the amplitudes of the waves of two of the subsets $\Omega_r^{(s)}$ occurring in $\Omega_{lr}^{(i)}$ are perturbed, for example the amplitudes of the subsets $\Omega_r^{(1)}$ and $\Omega_r^{(2)}$ so that

$$n_{\mathbf{k}}=n_{lr}+\alpha_r^{(1,2)}, \quad \psi_{\mathbf{k}}=\psi+\varphi_r^{(1,2)}, \quad \mathbf{k}\in\Omega_r^{(1,2)}\subset\Omega_{lr}^{(i)}.$$

Let us assume that $\alpha_-=\alpha_r^{(1)}-\alpha_r^{(2)}$ and $\varphi_-=\varphi_r^{(1)}-\varphi_r^{(2)}$. It is the growth of instabilities in the presence of perturbations of this kind that reduces the symmetry of the solution and, in particular, establishes the state of motion described by the relationships (2.4). A standard investigation of the stability of the solutions (2.5) in the presence of perturbations α_- and φ_- readily yields the following expression for the characteristic exponents:

$$\lambda_{1,2}^{(-)}=-\gamma_{\mathbf{k}}\pm[\gamma_{\mathbf{k}}^2+2\bar{S}_{\mathbf{k}}(T_{\mathbf{kk}}+2S_{\mathbf{kk}})r(l-2)n_{lr}^2]^{1/2}, \tag{2.9}$$

and the stability criterion

$$\bar{S}_{\mathbf{k}}(T_{\mathbf{kk}}+2S_{\mathbf{kk}})<0. \tag{2.10}$$

It is clear from Eq. (2.9) that for $l=2$ one of the characteristic exponents vanishes and, consequently, this case requires additional study.

For $l=2$ the equations for $\alpha_r^{(1)}$, $\alpha_r^{(2)}$, and φ_- can be described with high accuracy by

$$\begin{gathered} {}^1/_2(\dot{\alpha}_r^{(1)}-\dot{\alpha}_r^{(2)})=\gamma_{\mathbf{k}}(\cos\varphi_--1)(\alpha_r^{(1)}-\alpha_r^{(2)})-8lr\bar{S}_k(\cos{}^1/_2\varphi_--1)n_{lr}^2 \\ {}^1/_2\dot{\varphi}_-=-\gamma_{\mathbf{k}}\varphi_--(T_{\mathbf{kk}}+2S_{\mathbf{kk}})(\alpha_r^{(1)}-\alpha_r^{(2)}), \end{gathered} \tag{2.11}$$

which shows that the symmetric solution is unstable if

$$\bar{S}_{\mathbf{k}}(T_{\mathbf{kk}}+2S_{\mathbf{kk}})>0. \tag{2.12}$$

The instability increment is then proportional to the magnitude of the perturbation.

A comparison of Eqs. (2.10) and (2.12) shows that the stability criterion of interest to us is independent of l. Thus, if the condition (2.12) is satisfied, only an asymmetric distribution of parametrically excited waves (2.4) may be realized in this system. It is necessary to investigate also the stability of this state in the presence of perturbations of the amplitudes of the waves which occur in this state.

The symmetry relative to rotation through finite angles is the special case of the rotation symmetry and it reduces to the latter in the limit $\Delta\varphi\to 0$. The above stability criteria are applicable in the limit if we find discrete sets of angles $\Omega_r^{(s)}$ such that the average values of the coefficients $S_{\mathbf{kk}'}$ are identical with the values averaged over all the angles φ. If the rotations are symmetrical, $S_{\mathbf{kk}'}$ depend only on the difference between the azimuthal angles $S_{\mathbf{kk}'}=S(\varphi_{\mathbf{k}}-\varphi_{\mathbf{k}'})$ and can be represented by the expansion

$$S(\varphi)=\sum_m S_m e^{im\varphi}. \tag{2.13}$$

In view of the parity of $S(\varphi)$ only the even values of m occur in Eq. (2.13).

Usually the coefficients $S(\varphi)$ are polynomials of finite amplitude consisting of trigonometric functions of the angle φ and Eq. (2.13) contains a finite number $(2N+1)$ of such terms. We can show (see Appendix) that in this case the sets $\Omega_r^{(s)}$ contain a finite number of the angles φ:

$$\Omega_r^{(s)}=\left\{\varphi_s,\varphi_s+\frac{\pi}{r},\ldots,\varphi_s+\frac{(2r-1)\pi}{r}\right\},\quad 0\leqslant\varphi_s<2\pi,$$

where $2r=2^{l+1}$ and l is an integer such that

$$\ln(N/2)/\ln 2\leq l\leqslant \ln N/\ln 2.$$

A useful example is the case when $S(\varphi)$ and $T(\varphi)$ are independent of φ. Since the Fourier expansion contains only one term, it follows that $r=1$ and, consequently, $\Omega_r^{(s)}$ contains only two angles φ_s and $\varphi_s+\pi$. The criteria of stability of the symmetric solution obtained from Eqs. (2.8) and (2.10) have the form

$$S(2T+S)>0,\quad S(T+2S)<0. \tag{2.14}$$

These conditions are incompatible. If the first is disobeyed, then all the asymmetric solutions appear simultaneously with the symmetric unstable solution. The system then has no stable stationary solutions and it is necessary to consider conditionally periodic or stochastic solutions. If the first condition is obeyed but not the second, then the stable state of motion is the one in which only one pair of waves is excited.

3. ASYMMETRIC DISTRIBUTIONS OF PARAMETRICALLY EXCITED SPIN WAVES IN FERROMAGNETS AND ANTIFERROMAGNETS

The coefficient $S(\varphi)$ for a cubic ferromagnet was calculated by L'vov *et al.*[7] They showed that the expansion of this coefficient as a Fourier series consists of only zeroth and second harmonics, given by the following expressions

$$S_0=2\pi g^2\left(\frac{\omega_M}{\omega_0}\right)^2\left\{\left[\left(\frac{\omega_0}{\omega_M}\right)^2+1\right]^{1/2}+N_{z0}-1\right\},$$
$$S_{\pm2}=2\pi g^2\left[(\tilde N_{\pm2}-1)u_\pm{}^2+\frac{\omega_M}{2\omega_0}u_\pm\right], \tag{3.1}$$

where

$$\omega_M=4\pi gM,\quad N_{z0}=N_z+\beta\omega_a/\omega_M,$$
$$\tilde N_{\pm2}=\frac{1}{\omega_M}[\omega_{ex}(ak)^2+\beta\omega_a],\quad \beta=\begin{cases}-8, & \mathbf{M}\parallel\langle111\rangle\\ 9, & \mathbf{M}\parallel\langle100\rangle\end{cases},$$
$$u_\pm={}^1/_2\{[(\omega_M/\omega_0)^2+1]\mp1\},$$

N_z is the demagnetization factor; ω_a/g is the crystallographic anistropy field; ω_{ex} is the exchange frequency; a is the lattice constant.

Typical numerical values of these parameters are as follows: $\omega_0=10^{10}$ sec^{-1}, $\omega_M=5\times10^{-9}$ sec^{-1}, $\omega_a=0.25\times10^9$ sec^{-1}, $\omega_{ex}=kT$, $N_z=1/3$.

Since $S(\varphi)$ contains only the zeroth and second harmonics, it follows that $r=2$ and, consequently, $\Omega_r^{(s)}$ contains two mutually perpendicular pairs of vectors forming a cross and the corresponding asymmetric solution is a quartet of waves of equal amplitudes.

Thus, in addition to the symmetric (independent of φ) distribution of parametrically excited spin waves, there are also asymmetric distributions containing only two pairs of waves. We shall find which of these solutions is realized by applying the stability criteria (2.8) and (2.10). Moreover, in addition to $S(\varphi)$, we have to know also $\overline{T}$ and $T(0)$. As far as the coefficients $T(\varphi)$ are concerned, only the expressions for T_0 and $T_{\pm2}$ are given in the earlier treatments[1,4]:

$$T_0=S_0+2\pi g^2(N_{z0}-1),$$
$$T_{\pm2}=2\pi g^2\left(\frac{\omega_M}{2\omega_0}\right)^2\left\{\tilde N_{\pm2}-1+\left[\left(\frac{\omega_0}{\omega_M}\right)^2+1\right]^{1/2}\right\}, \tag{3.2}$$

whereas the expression for the coefficients $T_{\pm1}$ are not given; however, the knowledge of T_0 and $T_{\pm2}$ is sufficient for our purpose. The definitions of the coefficients $S(\varphi)$ and $T(\varphi)$ lead to the relationship $S(0)=T(\pi)$, which allows us to find the sum of the coefficients T_1+T_{-1} and to reconstruct $T(0)$. In fact,

$$S(0)=S_0+S_2+S_{-2},\quad T(\pi)=T_0-T_1-T_{-1}+T_2+T_{-2},$$

so that

$$T_1+T_{-1}=T_0+T_2+T_{-2}-S_0-S_2-S_{-2}$$

and, consequently,

$$T(0)=2T_0+2T_2+2T_{-2}-S_0-S_2-S_{-2}. \tag{3.3}$$

The following stability criteria of the symmetric (independent of φ) distribution are obtained from Eqs. (2.8), (2.10), and (3.3):

$$S_0(2T_0+S_0)>0, \tag{3.4}$$

$$S_0(2T_0+4T_2+S_0+S_2+S_{-2})<0. \tag{3.5}$$

It follows from Eqs. (3.1) and (3.2) that the stability condition (3.4) is disobeyed for the ⟨111⟩ orientation and, consequently, there are no stable symmetric or asymmetric stationary solutions and we may expect conditionally periodic or stochastic oscillations in a system of parametrically excited spin waves. This has been established earlier[1,7] and it accounts for the experimentally observed appearance of strong self-modulation of parametrically excited spin waves along the ⟨111⟩ orientations.

The stability criterion (3.4) is satisfied by the ⟨100⟩ orientation but it follows from Eqs. (3.1) and (3.4) that the condition (3.5) is not obeyed and, therefore, the symmetric distribution is unstable and only the asymmetric distribution in the form of a quartet of waves can be stable.

We shall now consider the internal stability of a quartet of waves, i.e., the stability in the presence of perturbations of amplitudes of each pair of waves. We shall assign the indices 1 and 2 to the pairs of waves with mutually perpendicular wave vectors and we shall use $\alpha_{1,2}=n_{1,2}-n_r$, and $\varphi_{1,2}=\varphi_{1,2}-\varphi$. An investigation of the equations of motion which follow from the system (1.1) for $\alpha_- = \alpha_1 - \alpha_2$ and $\varphi_- = \varphi_1 - \varphi_2$:

$$\tfrac{1}{2}\dot{\alpha}_- = -8sn_r^2\varphi_-, \quad s = S_0 - S(0),$$
$$\tfrac{1}{2}\dot{\varphi}_- = -\gamma_k\varphi_- - [2T(\pi/2) + 2T(-\pi/2) + 2S(\pi/2) - 2T(\pi) - T(0)]\alpha_-, \tag{3.6}$$

shows that the two pairs are stable if

$$s[2T(\pi/2) + 2T(-\pi/2) + 2S(\pi/2) - 2T(\pi) - T(0)] < 0. \tag{3.7}$$

Since

$$s = -S_2 - S_{-2}, \quad T(\pi/2) + T(-\pi/2) = 2T_0 - 2T_2 - 2T_{-2},$$
$$S(\pi/2) = S_0 - S_2 - S_{-2}, \quad T(\pi) = T_0 - T_1 - T_{-1} + T_2 + T_{-2},$$

it follows that the condition (3.7) can be rewritten in the form

$$(S_2 + S_{-2})(6T_0 - 8T_2 + S_0 - 3S_2 - 3S_{-2}) > 0. \tag{3.8}$$

It is useful to identify the ranges of the wave vectors in which the condition (3.8) is satisfied. It is clear from Eqs. (3.1) and (3.2) that only $\tilde{N}_{z2}$ depends on k, and that S_{-2} changes most on increase of k. Using the approximate values of the parameters given above, we readily find from Eqs. (3.8), (3.1), and (3.2) that the cross is unstable for $0.75 \lesssim \tilde{N}_{z2} \lesssim 1.5$. Noting that for $k = 0$ we have $\tilde{N}_{z2} \approx 0.45$, we find that the cross is unstable for small and large wave vectors, with the exception of the interval just given. Experimental evidence shows indeed self-modulation of a system of parametrically excited spin waves in approximately this range (see Zakharov *et al.*[1]). The review of Zakharov *et al.*[1] gives the results of a numerical integration of a system of parametrically excited spin waves concentrated in two rays in the wave space (i.e., of a system corresponding to a wave quartet) in the presence of an instability similar to that defined by the criterion (3.8). It is reported there that the growth of this instability results in oscillations of the amplitudes of parametrically excited spin waves, i.e., it produces the conditionally periodic (or stochastic) regime.

We shall now consider the steady states of motion of parametrically excited spin waves in cubic antiferromagnets. It is shown by L'vov and Shirokov[8] that the coefficients $S_{\mathbf{kk'}}$, $T_{\mathbf{kk'}}$, and $\gamma_{\mathbf{k}}$ and quadratic polynomials in $X_{\mathbf{k}}$ and $X_{\mathbf{k'}}$, where $X_{\mathbf{k}} = \sin^2\theta_{\mathbf{k}} \sin 2\varphi_{\mathbf{k}}$, $\theta_{\mathbf{k}}$ and $\varphi_{\mathbf{k}}$ are the polar and azimuthal angles in a spherical system of coordinates oriented along the magnetic field H which lies in the easy plane of the antiferromagnet. The steady states of motion found by L'vov and Shirokov[8] represent symmetric distributions of $n_{\mathbf{k}}$ along the lines $X_{\mathbf{k}} = X_c$ and $X_{\mathbf{k}} = \pm X_c$, where X_c is some value which depends on the pump intensity and tends to zero when this intensity approaches the threshold value. In view of these properties of the coefficients $S_{\mathbf{kk'}}$, $T_{\mathbf{kk'}}$, and $\gamma_{\mathbf{k}}$, they are constant on the $X_{\mathbf{k}} = \pm X_c$ lines and equal to, respectively, $S_{\mathbf{kk'}} = S(\pm X_c)$ and $T_{\mathbf{kk'}} = T(\pm X_c)$. The application of the results of Sec. 2 shows that the symmetric distributions are unstable. Stable states of motion are those for which only one pair of waves is excited along the $X_{\mathbf{k}} = \pm X_c$ lines.

4. DISCUSSION

The nature of a steady-state distribution of parametrically excited waves can be determined quite simply if direct methods for the observation of waves are available. There are relatively simple techniques for observing plasma and elastic oscillations and in their case the conclusions of the theory can easily be checked whenever the process in question is described by equations of the (1.1) type. In the case of spin waves, direct observations may be provided by the diffraction of light on parametrically excited spin waves (see, for example, the work of Bakai and Sergeeva[9]). The main experimental difficulty in the diffraction of light on spin waves is the smallness of the photon—magnon interaction but the process is sensitive to the nature of the distribution of parametrically excited spin waves. Light is scattered resonantly on spin waves whose wave vectors $\mathbf{k}$ make an angle $\theta_{\mathbf{qk}}$ with the wave vector of the incident light $\mathbf{q}$, so that

$$\cos\theta_{\mathbf{qk}} = k/2q. \tag{4.1}$$

Since the wave vectors of parametrically excited spin waves all lie in the same plane, the condition (4.1) defines two resonance orientations of the wave vectors of pairs. The sum of the squares of the amplitudes of parametrically excited spin waves depends weakly on their number r [see Eq. (2.4)] and this is why the amplitudes n_r as well as the intensity of scattered light decrease on increase of r. However, if only a few pairs are excited in the steady regime, we have a situation favorable for the investigation of parametrically excited spin waves by the diffraction of light. The circumstance that the orientations of pairs of parametrically excited spin waves are not fixed (only the differences between the polar angles of the wave vectors are fixed) makes it difficult to select the orientation of the wave vector of the incident light. This difficulty can be avoided by disturbing the symmetry of the problem by external means so as to fix the directions of the wave vectors of parametrically excited spin waves.

There are several indirect manifestations of the nature of distribution of parametrically excited spin waves but these can be interpreted in two ways because measurements usually yield integrated quantities, whose behavior is not distinguished in any way for different types of distribution of parametrically excited spin waves. We shall now consider some of them.

According to the current ideas,[10,11] the hard parametric excitation of spin waves is associated with the saturation of the wave absorption channels. This is manifested by the presence of a negative nonlinear correction to the damping coefficient:

$$\gamma_{\mathbf{k}} = \gamma_{\mathbf{k}}^0 - \sum_{\mathbf{k'}} \gamma_{\mathbf{kk'}} n_{\mathbf{k'}},$$

where $\gamma_{\mathbf{k}}^0$ is the linear damping coefficient.

If the wave dissipation channels with different orientations of the wave vectors are different, we may assume that $\gamma_{\mathbf{kk'}} \sim \delta_{\mathbf{kk'}}$ and

$$\gamma_k\approx\gamma_k^0-\gamma_{kk}n_r\approx\gamma_k^0-\gamma_{kk}\frac{[(hV_k)^2-\gamma_k^2]^{1/2}}{2r\bar{S}_k}. \qquad (4.2)$$

It is clear from Eq. (4.2) that for high values of r the nonlinear correction and the hard excitation effect are not significant so that the very discovery of this effect in ferrites and antiferromagnets[10,12-15] may be regarded as supporting the asymmetry of the steady-state distributions of parametrically excited spin waves. In the hard excitation of such waves there is a considerable time interval between the moment of application of a microwave field and a jump in the absorption of this field (indicating wave excitation), which may be regarded as the time needed for the transformation of a largely symmetric distribution of spin waves created from noise fluctuations into an asymmetric distribution. The increments representing the rate of this process are small, as indicated by Eq. (2.9). It is worth noting that a preliminary excitation of an asymmetric "priming" wave of finite amplitude reduces strongly this time interval.[14]

Another important phenomenon closely associated with the nature of the distribution of parametrically excited spin waves is the appearance of collective oscillations. It is clear from Eq. (2.5) that the frequency of homogeneous collective oscillations $\Omega_0=\mathrm{Im}\,\lambda$ depends very weakly on the number of pairs of excited waves. For example, the ratio of the frequencies of collective oscillations in symmetric and asymmetric distributions, the latter being characterized by the excitation of just one pair,[4)] is equal to $\Omega_{0s}/\Omega_{0A}=\sqrt{2}$, as easily deduced from Eq. (2.7). In an experimental check of this quantitative difference it is necessary to determine independently the coefficient $\bar{S}_k$.

Self-modulation in a system of parametrically excited spin waves corresponds to a transition from stationary to conditionally periodic and stochastic states of motion. The stability criterion of a symmetric distribution[1] is

$$S_{\pm2}(2T_{\pm2}+S_{\pm2})>0, \qquad (4.3)$$

differs considerably from the corresponding criterion of an asymmetric (3.8) distribution of waves in a ferromagnet and be checked experimentally. As pointed out in the preceding section, the available experimental results are not in conflict with the conclusions which follow from Eq. (3.8) but they are insufficient for a detailed comparison with Eqs. (4.3) and (3.8).

In the case of a discrete distribution of parametrically excited spin waves the processes of the interaction of these waves with nonparametric waves, ignored in the system (1.1), become important. The latter waves include, in particular, the processes of coalescence discussed above and resulting in a change in γ_k, the processes of decay investigated earlier[16,17]

$$2\omega_k\to\omega_{k+x}+\omega_{k-x},\quad \omega_k+\omega_{-k}\to\omega_{k+x}+\omega_{-k-x},$$

as well as the decay involving acoustic waves. All of them alter the distributions of parametrically excited spin waves and of nonparametric waves.

It is necessary to point out one other feature of the asymmetric distributions of parametrically excited spin waves found above. In addition to a discrete set of excited waves, there is a continuum of waves whose damping is compensated by the action of the pump field

$$\gamma_k=\left|hV_k+\sum_{k'}S_{kk'}n_{k'}\exp(i\psi_k-i\psi_{k'})\right|,$$

and which satisfy the resonance condition

$$\omega_k\approx\omega_0/2+\sum_{k'}T_{kk'}n_{k'}.$$

The amplitudes of these waves are negligible only if we ignore thermal fluctuations. At finite temperatures the level of fluctuations of these amplitudes is very high and it differs considerably from the equilibrium fluctuations.

APPENDIX

Let us assume that $S(\varphi)$ is a periodic function (period 2π) and

$$S(\varphi)=\sum_k S_k e^{ik\varphi}. \qquad (A.1)$$

We can easily see that the Fourier-series expansion of the function

$$S_1(\varphi)={}^1/_2[S(\varphi)+S(\varphi+\pi)] \qquad (A.2)$$

contains only the terms with even numbers, and the average value $S_1(\varphi)$ is identical with the average $S(\varphi)$, i.e., it is equal to S_0. Similarly, in the case of the function

$$S_r(\varphi)=\frac{1}{2r}\sum_{n=1}^{2r}S\left(\varphi+\frac{n\pi}{r}\right), \qquad (A.3)$$

where $r=2^l$, and l is a positive integer, the only nonvanishing Fourier coefficients are those with numbers which are multiples of 2^{l+1} and the average value of the function $S_r(\varphi)$ is S_0.

Therefore, if the only nonvanishing coefficients in Eq. (A.1) are those with numbers k not exceeding the modulus of N, the average value $S(\varphi)$ in the interval $[0,2\pi]$ is identical with the average value of the sequence

$$\Omega_r^{(s)}=\left\{\varphi_s,\varphi_s+\frac{\pi}{r},\ldots,\varphi_s+\frac{(2r-1)\pi}{r}\right\},\quad 0\leqslant\varphi_s<2\pi, \qquad (A.4)$$

where

$$r=2^{l(N)},\quad \ln(N/2)/\ln 2<l(N)\leqslant\ln N/\ln 2.$$

It follows that the sequences (A.4) are the required sets $\Omega_r^{(s)}$. We can see that $N<2r\leqslant 2N$.

1)In other words, the invariant solutions form spaces which transform in accordance with the representation of the group of given spatial transformations.

2)We shall call the solutions asymmetric if they have symmetry lower than that of the equation in question.

3)The derivation and analysis of the equations can be found in the work of Zakharov *et al.*[1,3]

4)This case corresponds to $T_{kk'}=S_{kk'}=\bar{S}_k$ and it occurs, for example, in the case of a cubic antiferromagnet with the easy-plane anisotropy.

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Splitting of ENDOR lines by a strong microwave field

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The first experiments on the splitting of ENDOR lines in a solid by a strong microwave field are reported. The experiment was performed on *F* centers in KCl. The splitting was registered at ENDOR sum lines from the Cl^{35} and Cl^{37} nuclei of coordination sphere II at $T = 300$ K. The dependence of the effect on the microwave and RF field powers, on the detuning of the stationary magnetic field, and on the temperature is investigated. The maximum microwave field intensity attained in the experiment reached 0.14 Oe and corresponded to a splitting of 380 kHz. A theoretical explanation is offered for the magnitude of the splitting, for the dependence of H_1, and for intensity ratio of the split component, as well as for the independence of the splitting of the magnetic-field detuning.

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INTRODUCTION

The study of various types of double magnetic resonances in strong alternating fields that lead to coherent motion of the spins is of considerable interest, since it permits observation of qualitatively new effects that yield additional information on the investigated objects. Up to now, these resonances were investigated mainly in liquids and were used most extensively for nuclear-nuclear double resonance.[1,2] In electron-nuclear double resonance (ENDOR), the manifestation of strong alternating fields (which can be in either the microwave or the RF band) has been investigated relatively little,[2-6] and notice should be taken of the contradictory character of a number of these studies.[5,6] The phenomena connected with a strong microwave field in solutions of organic compounds were considered by Freed *et al.*[3] However, both the observed effects and the theoretical analysis for liquids, as is well known, have features that greatly encumber their interpretation; it is of therefore of interest to investigate such phenomena in solids, where the ENDOR had been most extensively used.

We have registered, for the first time ever, effects connected with the manifestation of a strong microwave field in ENDOR of solids, studied these effects, and deduced a theoretical interpretation that explains the most significant aspects of the phenomenon.

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