To obtain estimates, let us set $E=4\cdot10^{12}$ dyn/cm^2 (sapphire), $V=1$ cm^3, $\omega_e=6\cdot10^{10}$ rad/sec, $Q_e=1\cdot10^{10}$ in (12) and (13). Then

$$\left(\frac{\Delta\omega}{\omega}\right)_{min}\approx1\cdot10^{-20}(\hat{\tau})^{-1/2},\quad W_{opt}\approx2\cdot10^{3}\text{ erg/sec}.$$

These estimates show that the problem of measuring $h\approx10^{-18}$ can in principle be solved. A high frequency stability of an electromagnetic oscillator is required not only for the gravitational experiments we have described. Therefore, the limiting relations may also be helpful for planning other high-precision physics experiments.

Note added in proof (January 12, 1978). Equation (12) can be obtained on the basis of general arguments (as has been pointed out by Yu. I. Vorontsov): the relative error $\Delta\omega/\omega$ is equal to $\Delta l/l$; for continuous measurement of the coordinate of a mechanical oscillator the smallest error is $\Delta l_{min}=(\hbar/m\omega_M)^{1/2}(1/\omega_m\hat{\tau})^{1/2}$ (where m and ω_M are the mass and frequency of the oscillator). Taking into account only the fundamental mode of the mechanical vibrations of the cavity and expressing m and ω_M in terms of E and V, we readily obtain $\Delta\omega/\omega=\Delta l/l\approx(\hbar/EV\hat{\tau})^{1/2}$.

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Scale-invariant solutions in the hydrodynamic theory of multiple production

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The scale-invariant solutions in the hydrodynamic theory of multiple production are considered. The question of the behavior of a hadron system at points on its boundary with a vacuum is investigated. It is shown that the requirement of conservation of the energy and the momentum of the system indicates the necessity for the introduction of particle-like states at the periphery of the hadron liquid. These states are identified with the leading particles. The solutions to the equations of motion are found and physically analyzed.

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1. INTRODUCTION

Recently there has been a marked increase in interest in the study of the space-time picture of the processes of multiple production. In particular, in the investigation of high-energy hadron-nucleus collisions the question of the temporal evolution of the hadron systems turns out to be closely associated with the experimentally observable characteristics. In view of this, of special interest is an in-depth analysis of the hydrodynamic theory of high-energy collisions—in essence the only model that allows a detailed space-time description of the process of multiple production of particles.

One of the main achievements of the hydrodynamic approach is the fairly successful explanation of the experimental data on the structure of the secondary particles and their transverse-momentum distribution. These results, which justify the basic idea that a hydrodynamic phase exists in the course of multiple generation of particles, are insensitive to the specific choice of the equation of state of the hadron liquid and the initial conditions for its expansion. Let us recall that in the basic paper by Landau[1] the initial conditions were chosen to correspond to a homogeneous quiescent liquid in a Lorentz-contracted disk with transverse dimensions $l_\perp\approx1/\mu$ and longitudinal dimensions $l_\parallel\sim1/E_0$ (μ is the π-meson mass and E_0 is the total energy of the colliding particles in the CM system), while the equation of state was chosen in the form $p=\epsilon/3$ (p is the pressure and ϵ is the energy density). Khalatnikov[2] has obtained an exact analytic solution to the problem of the one-dimensional motion with Landau's initial conditions.[1]

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In the subsequent papers Landau's and Khalatnikov's results[1,2] were generalized: for an equation of state of the form

$$p=c_0^2\varepsilon \quad (0<c_0^2=\text{const}<1) \tag{1}$$

by Milekhin[3] and to take account of viscosity effects by Feĭnberg and Emel'yanov.[4] As to the problem of choosing the initial and boundary conditions in the hydrodynamic theory of multiple production, it remains, in our opinion, insufficiently investigated. Moreover, there exist a number of physical characteristics that depend to a considerable degree on the form of these additional conditions, which separate out the required solution of the hydrodynamic equations.

We can indicate several points that make the Landau assumption doubtful:

a) The existence of a global thermodynamic equilibrium in the entire system at the initial moment of time is assumed in his paper.[1] This assumption is quite exacting and difficult to justify. Besides, the classical description of the initial stage is at variance with the quantum-mechanical uncertainty relation.[5]

b) Estimates[6] show that in the case of high initial energies the complete stopping of two colliding protons is impossible, since the whole energy is then spent on bremsstrahlung emission.

c) In Landau's paper the presence of leading particles was not taken into account. According to current ideas,[7] it is necessary to suppose that the leading particles do not form a part of the hydrodynamic system, and that their effect on the multiple production is due only to the existence of conservation laws, i.e., the total energy of the hadron liquid is $E_l=kE_0$ $(0<k<1)$, where k is the inelasticity coefficient.

We wish to draw attention to the fact that the role of the leading particles may turn out within the framework of the hydrodynamic approach to be much more important than is customarily assumed. Although by their very meaning the leading particles are separated from the hydrodynamic system, apparently there occurs in the course of the separation energy and momentum exchange between the hadron liquid and the leading particles. In this case the boundary conditions for the expanding hadron liquid change significantly in comparison with the usual problem of expansion into a vacuum.

Summing up the above-discussed theoretical and experimental data on the processes of multiple production, we can draw the following conclusions. The idea that a hydrodynamic expansion phase exists prior to the appearance of real hadrons is quite fruitful and in accord with experiment. The question of the choice of the initial and boundary conditions, however, remains open. The hydrodynamic expansion apparently begins when the elements of the hadron liquid possess some nontrivial velocity distribution that arose in the nonhydrodynamic phase. The complete theoretical solution of this problem requires an analysis that takes account of the quantum effects, which, of course, falls outside the limits of the hydrodynamic model.

In the present paper we proceed in the following manner. We do not prescribe any boundary conditions; instead, we require scaling invariance. In the region of pionization, corresponding in the hydrodynamic approach to the particles produced from the hadron liquid, the scaling requirement for the inclusive spectra can be formulated in terms of a varying rapidity[7]: the rapidity distribution of the secondary hadrons should be flat with a height that does not depend on the initial energy. This property of inclusive spectra obtains in the multiperipheral and parton models (the leading particles or their excited states make a contribution to the fragmentation region).

In order to fulfill the scaling-invariance requirement, we use in our analysis a special class of solutions, recently investigated intensively by several authors,[8,9] to the equations of relativistic hydrodynamics. There, however, lies in this approach one important difficulty: the impossibility of fulfilling for these solutions in their standard formulation the laws of conservation of energy and momentum. This question is not investigated in Refs. 8 and 9. An analysis of the requirement of conservation of the total 4-momentum of the system leads to a new result: the possibility of the appearance of particle-like states at the boundary of the hadron liquid with a vacuum. We associate these states with the leading particles.

The model that is developed leads to a number of interesting physical consequences. In particular, it turns out that in the course of the expansion new elements of the liquid are produced and, consequently, the entropy of the liquid increases.

2. SCALE-INVARIANT SOLUTIONS OF THE HYDRODYNAMIC EQUATIONS

As is well known, the equations of motion of an ideal relativistic liquid can be written in the form

$$\frac{\partial T^{ik}}{\partial x^k}=0, \tag{2}$$

$$T^{ik}=(\varepsilon+p)u^iu^k-pg^{ik},$$

$$u^i=\left(\frac{1}{(1-v^2)^{1/2}},\ \frac{\mathbf{v}}{(1-v^2)^{1/2}}\right). \tag{3}$$

g^{ik} is the diagonal tensor $(1,-1,-1,-1)$. The liquid being studied by us occupies in coordinate space a cylinder whose axis (the x axis) coincides with the collision axis for the primary particles. As time goes on, the length of the cylinder increases, but we shall assume that the transverse cross section, which is a circle of radius $1/\mu$, does not change $(u^2=u^3=0)$. We shall discuss the validity of this one-dimensional approximation later.

From the requirement that the chemical potential of the system be equal to zero, i.e., that $\epsilon-Ts+p=0$ (T is the temperature and s is the entropy density), and the second law of thermodynamics, $d\epsilon=Tds$, we can find with the aid of (1) all the thermodynamic relations:

$$\varepsilon=\lambda\left(\frac{T}{T^*}\right)^{(1+c_0^2)/c_0^2}, \quad s=\frac{\lambda(1+c_0^2)}{T^*}\left(\frac{T}{T^*}\right)^{1/c_0^2}, \tag{4}$$

where λ is the constant of integration, $T^*\approx\mu$ is the critical temperature corresponding to the breakup of the liquid into the individual hadrons.

In the one-dimensional approximation, the equations (2) can be rewritten in the following form:

$$\frac{\partial(su^0)}{\partial t}+\frac{\partial(su^1)}{\partial x}=0, \tag{5}$$

$$\frac{\partial(Tu^1)}{\partial t}+\frac{\partial(Tu^0)}{\partial x}=0. \tag{6}$$

To separate out a single solution of Eqs. (5) and (6), we must specify additional conditions. As these conditions, Chiu *et al.*[8] have introduced the requirement of non-dependence of the motion of the liquid on the choice of the Lorentz frame of reference (i.e., a frame independence symmetry), which they regard as some new principle in high-energy hadron collisions that leads to the fulfillment of the scaling requirement.

In our arrangement there exist the vector $x^\mu=(t,x)$ and the scalar $\tau=(x^\mu x_\mu)^{1/2}=(t^2-x^2)^{1/2}$. The requirement of Lorentz invariance of the form of the solution to Eqs. (5) and (6) will be fulfilled if all the vector quantities are proportional to x^μ and all the scalar quantities depend on τ. Thus, $u^\mu\sim x^\mu$ (from the normalization condition $u^\mu u_\mu=1$ we find $u^\mu=x^\mu/\tau$, or $v=x/t$) and $s=s(\tau)$, $T=T(\tau)$, etc. The substitution of $v=x/t$ and $T=T(\tau)$ into Eqs. (5) and (6) transforms (6) into an identity, while (5) is reduced to the equation

$$\frac{ds}{d\tau}+\frac{s}{\tau}=0. \tag{7}$$

Let us write the solution to Eq. (7), $s(\tau)=\text{const}.\tau^{-1}$, in the form

$$s=s^*\tau^*/\tau. \tag{8}$$

The quantity τ is the proper time of the element moving with velocity $v=x/t$. With the aid of (4) we have

$$\varepsilon=\varepsilon^*\left(\frac{\tau^*}{\tau}\right)^{1+c_0^2}, \quad T=T^*\left(\frac{\tau^*}{\tau}\right)^{c_0^2}, \tag{9}$$

from which it can be seen that the quantity τ^* plays the role of the proper time of the elements of the liquid at the moment of disintegration into the final particles, while ϵ^* and s^* are the critical values at the moment of disintegration. It is also easy to see that all the thermodynamic quantities, p, ϵ, s, and T, at fixed t have their minimum values at the center, $x=0$, of the system and attain their maximum values at the boundary points $x_1(t)$ (the left boundary) and $x_2(t)$ (the right boundary). As a consequence, in the CM system the slowest particles ($v\approx0$ at $x\approx0$) are the first, and the fastest ones the last, to appear at the borders of the system.

Let us emphasize once again that the functions

$$v=x/t, \quad s=\text{const}\,\tau^{-1} \tag{10}$$

are a solution to Eqs. (5) and (6) inside the liquid for any c_0^2. The difference between our model and the Landau model[1] lies not in the change in the equation of state, but in the new initial conditions. It is convenient to prescribe them on the curve $(t^2-x^2)^{1/2}=\tau_0$ in (x,t)-space in the form $v=x/t$, $s(\tau_0)=s_0=\text{const}$. It is clear that the liquid occupies a finite region in x space, in view of the finiteness of the energy of the system.

The solutions thus constructed lead to scaling-invariant behavior of the inclusive spectra of the secondary hadrons. The secondary particles appear when the elements of the liquid attain the critical temperature T^*. The transverse momenta, $p_\perp$, are then determined by the thermal motion with temperature T^*, while the longitudinal momenta, $p_\parallel$, are determined by the purely hydrodynamic motion (in the case of high initial energies the thermal motion in the longitudinal direction does not play a significant role).

Let us introduce the variables

$$\alpha=\text{arc th}\,v, \quad \theta=\ln(T/T^*). \tag{11}$$

Upon the appearance of the secondary particles

$$\alpha=\text{arcth}\,v=\text{arcth}\frac{p_\parallel}{p_0}=\frac{1}{2}\ln\frac{p_0+p_\parallel}{p_0-p_\parallel}, \tag{12}$$

i.e., α plays the role of rapidity. From (9)–(11) we find

$$x=\tau^*\exp[-\theta/c_0^2]\,\text{sh}\,\alpha, \quad t=\tau^*\exp[-\theta/c_0^2]\,\text{ch}\,\alpha. \tag{13}$$

At the moment of disintegration the number of secondary particles is proportional to the total entropy, S, of the system. The rapidity distribution of the entropy can be found by going over from x,t to θ,α with the aid of the formulas (13):

$$dS=\frac{\pi}{\mu^2}(su^0dx-su^1dt)=\frac{\pi}{\mu^2}s^*\tau^*d\alpha, \tag{14}$$

where π/μ^2 is the area of the transverse cross section of the liquid. We find on the disintegration curve

$$(t^2-x^2)^{1/2}=\tau^*$$

the rapidity distribution of the secondary hadrons:

$$\frac{dN}{d\alpha}\sim\frac{dS}{d\alpha}=\frac{\pi}{\mu^2}s^*\tau^*, \quad \alpha_{min}<\alpha<\alpha_{max}. \tag{15}$$

The formula (15) shows that the rapidity distribution is flat. The scaling requirement will be fulfilled if τ^* does not depend on the initial energy, E_0. To obtain agreement with the experimental data, we must have that, in order of magnitude, $\tau^*\sim1/\mu$.

For brevity, we shall call the solution (10) a scaling solution. Many of its attractive features have been considered in a number of papers.[8,9] The authors of these papers did not, however, investigate the extremely important problem connected with the violation of the energy-momentum conservation laws for the scaling solution. The point is that the solution (10) possesses an important characteristic: all the thermodynamic quantities are discontinuous at the boundary of the liquid with a vacuum. It is clear that for a physical system with an energy density that does not vanish at the boundary the conservation of the total energy and total momentum follows from the local conservation laws (2) only when certain boundary conditions are fulfilled. In this respect the situation is analogous to the bag model.[10]

3. THE LAWS OF CONSERVATION OF ENERGY AND MOMENTUM FOR THE SCALING SOLUTION

A necessary and sufficient condition for the conservation of the total energy and total momentum is the vanishing of the energy-momentum flux density across the lateral surface of the tube described by the liquid in 4-space:

$$T^{\mu\nu}n_\nu^{(k)}=0, \quad \mu,\nu=0,1, \quad k=1,2, \tag{16}$$

where

$$n_\nu^{(k)}=\frac{1}{(1-x^2)^{1/2}}(-\dot{x}_k(t),1)$$

is the normal, directed outwards from the system, to the lateral face of the tube and $k=1,2$ correspond to the left and right boundaries; the point denotes differentiation with respect to time (in (16) we have dropped the indices $\mu,\nu=2,3$, which are unessential in our analysis). Let us recall that the necessity of the conditions (16) is connected with the requirement of conservation of the 4-momentum P^μ in any coordinate system. In the CM system $\mathbf{P}=0$ and is conserved by virtue of the symmetry of the problem. This symmetry is, however, broken by the Lorentz transformations, which inevitably leads to the violation of the conservation laws when the conditions (16) are not fulfilled.

The substitution of (3), (9), and (10) into (16) leads, after elementary calculations, to the solution

$$c_0^2=0, \quad \dot{x}_k(t)=x_k(t)/t=v(t,x_k(t)),$$

i.e., the fulfillment of the conservation laws is possible only in the idealized case of a liquid with zero pressure (the boundaries then move with a velocity equal to the velocity of the liquid at the boundaries). When $c_0^2=0$, the temperature of any element of the liquid is constant, and does not depend on the energy density. This particular case does not, in essence, bear any relation to hydrodynamics, but describes the free (uniform and rectilinear) motion of an ensemble of individual particles.

In a real physical system, the pressure is not equal to zero, so that the scaling solution leads to the violation of the conservation laws (i.e., (16) is not fulfilled). This circumstance is directly connected with the fact that the scaling solution (9), (10) satisfies the equations of motion (2) inside the liquid, but breaks down at the boundary points. In order to investigate this question, let us separate out the discontinuities of the thermodynamic quantities in the energy-momentum tensor (3) in their explicit form with the aid of the θ step functions. Then we find in the case when the equations of hydrodynamics are satisfied inside the liquid that

$$\frac{\partial}{\partial x^\nu}[T^{\mu\nu}\theta(x-x_1(t))\theta(x_2(t)-x)] = \sum_{k=1,2}(-1)^k(\dot{x}_k(t)T^{\mu 0}-T^{\mu 1})\delta(x-x_k(t)). \tag{17}$$

It is not difficult to see that the requirement that (17) be equal to zero (i.e., the satisfaction of the equations (2) at the boundary points) is equivalent to the conditions (16). For the solution (9), (10) these conditions are not fulfilled when $c_0^2\neq 0$.

In order to give a physical meaning to the scaling solution, we shall assume that the energy-momentum tensor, (3), of the liquid is not the total tensor of the hadron system. Let us, in accordance with (17), assume that the total tensor of the hadron system differs from the tensor, (3), of the ideal liquid at points on the boundary with a vacuum:

$$\mathcal{T}^{\mu\nu}=T^{\mu\nu}\theta(x-x_1(t))\theta(x_2(t)-x)+\sum_{k=1,2}t^{\mu\nu}\delta(x-x_k(t)). \tag{18}$$

Let us determine the form of the auxiliary tensor $t^{\mu\nu}$ from the requirement that the local energy-momentum conservation laws be fulfilled in the whole space:

$$\frac{\partial\mathcal{T}^{\mu\nu}}{\partial x^\nu}=\sum_{k=1,2}\Big\{\Big[(-1)^k(\dot{x}_k(t)T^{\mu 0}-T^{\mu 1})+\frac{\partial t^{\mu\nu}}{\partial x^\nu}\Big]\delta(x-x_k(t)) +[\dot{x}_k(t)t^{\mu 0}-t^{\mu 1}]\delta'(x-x_k(t))\Big\}, \quad \delta'(y)=\frac{d}{dy}\delta(y). \tag{19}$$

Taking into account the rules for handling the derivatives of generalized functions, we arrive at the following system of equations:

$$\Big[(-1)^k(\dot{x}_k(t)T^{\mu 0}-T^{\mu 1})+\frac{\partial t^{\mu 0}}{\partial t}-\dot{x}_k(t)\frac{\partial t^{\mu 1}}{\partial x}\Big]_{x=x_k(t)}=0, \tag{20}$$

$$[\dot{x}_k(t)t^{\mu 0}-t^{\mu 1}]_{x=x_k(t)}=0. \tag{21}$$

If the tensor $t^{\mu\nu}$ is symmetric at $x=x_k(t)$, then it follows from (21) that

$$t^{\mu\nu}=\omega\dot{x}^\mu\dot{x}^\nu, \tag{22}$$

where we have introduced the notation $\omega=t^{00}(t,x_k(t))$..

4. FORMULATION OF THE MODEL

In the model developed by us, the hadron system's total energy-momentum tensor, integrated over the transverse cross section, can be given in the form

$$\mathcal{T}^{\mu\nu}=\frac{\pi}{\mu^2}[(\varepsilon+p)u^\mu u^\nu-pg^{\mu\nu}]\theta(x-x_1(t))\theta(x_2(t)-x) +\sum_{k=1,2}\frac{m_k(t)\dot{x}^\mu\dot{x}^\nu}{(1-\dot{x}_k^2)^{1/2}}\delta(x-x_k(t)), \tag{23}$$

where

$$m_k(t)=\pi\mu^{-2}\omega(t, x_k(t))(1-\dot{x}_k^2)^{1/2}.$$

It is clear that the auxiliary tensor $t^{\mu\nu}\delta(x-x_k(t))$ is the energy-momentum tensor of a particle of mass $m_k(t)$ and energy

$$w_k(t)=\pi\mu^{-2}\omega(t, x_k(t)).$$

As follows from (23), in the interval $x_1(t)<x<x_2(t)$ the local energy-momentum conservation law

$$\frac{\partial\mathcal{T}^{\mu\nu}}{\partial x^\nu}=0,$$

leads to the equations, (2), of relativistic hydrodynamics. As their solution, we choose (10). At the boundary points $x_1(t), x_2(t)$ we arrive at the equations (20), which we rewrite as follows:

$$(-1)^k(\dot{x}_k(t)T^{\mu0}-T^{\mu1})\frac{\pi}{\mu^2}+\frac{d}{dt}\frac{m_k(t)\dot{x}_k^\mu}{(1-\dot{x}_k^2)^{1/2}}=0,$$
$$x_k^\mu=(t, x_k(t)),\quad k=1, 2. \tag{24}$$

The formula (24) can be represented in a form from which the relativistic invariance of the model clearly follows:

$$\frac{\pi}{\mu^2}T^{\mu\nu}n_\nu^{(k)}=\frac{d}{d\tau}\left(m_k\frac{dx_k^\mu}{d\tau}\right),$$

where $n_\nu^{(k)}$ is defined in (16) and $d\tau=dt(1-\dot{x}_k^2)^{1/2}$. It is not difficult to verify that the conservation of the total energy and total momentum of the system follows from the Eqs. (24), i.e., that

$$\frac{d}{dt}\int\mathcal{T}^{\mu0}dx=0.$$

Thus, the obtained equations imply that, in the physically correct formulation of the scaling solution, we should explicitly take into account the exchange of energy and momentum between the liquid and the boundary, which is a particle-like object. It seems natural to identify these objects with the leading particles that appear in hadron-hadron collisions. Such a model is quite close to the quark-gluon picture of hadron interactions[11]: the valence quarks of the colliding hadrons fly through each other and interact in the process of separation through exchange of gluons, being converted in the final phase into leading particles. They correspond in our model to the particle-like states at the periphery of the "gluon liquid."

Let us explicitly write out Eq. (24) in the CM system for one of the boundaries (for definiteness, the right-hand boundary $x_2(t)\equiv z(t)$):

$$\frac{\pi}{\mu^2}\{[(1+c_0^2)\varepsilon(u^0)^2-c_0^2\varepsilon]\dot{z}(t)-(1+c_0^2)\varepsilon u^0u^1\}+\frac{dw(t)}{dt}=0, \tag{25}$$

$$\frac{\pi}{\mu^2}\{[(1+c_0^2)\varepsilon u^0u^1]\dot{z}(t)-[(1+c_0^2)\varepsilon(u^1)^2+c_0^2\varepsilon]\}+\frac{d}{dt}[w(t)\dot{z}(t)]=0\cdot \tag{26}$$

Here

$$\varepsilon=\varepsilon^*(\tau^*)^{1+c_0^2}(t^2-z^2(t))^{-(1+c_0^2)/2},$$
$$u^0=t(t^2-z^2(t))^{-1/2},\quad u^1=z(t)(t^2-z^2(t))^{-1/2},$$

$w(t)$ is the energy of a particle-like state at a boundary of the liquid with a vacuum. The Eqs. (25) and (26) constitute a system of ordinary nonlinear differential equations for the two unknown functions $z(t)$ and $w(t)$. In spite of their quite complicated character, it is possible to find an exact analytic solution to these equations.

It can be verified by a direct calculation that the first integrals of (25), (26) have the form

$$az(t)(t^2-z^2(t))^{-(1+c_0^2)/2}+w(t)=E, \tag{27}$$

$$at(t^2-z^2(t))^{-(1+c_0^2)/2}+w(t)\dot{z}(t)=P, \tag{28}$$

where $a\equiv\pi\mu^{-2}\epsilon^*(\tau^*)^{1+c_0^2}$, while E and P are constants of integration.

Equation (27) expresses the law of conservation of energy for the right half of the system. The first term on the left-hand side of (27) is, as is easy to verify, the energy of the liquid

$$E_l=\int_0^{z(t)}T^{00}dx,$$

while the second term is the energy of the particle. In the case of symmetric separation $E=E_0/2$.

The constant of integration P does not have as simple a physical meaning as E has. The momentum, Π, of the right half of the system increases in time because of the action of the force

$$f=\frac{\pi}{\mu^2}p(t,0)=\frac{\pi}{\mu^2}c_0^2\varepsilon^*(\tau^*)^{1+c_0^2}t^{-1-c_0^2}$$

on the right half of the system at the point $x=0$, i.e., $d\Pi/dt=f(t)$. Then we can write

$$\Pi(t)=\int f(t)dt=P'-at^{-c_0^2}, \tag{29}$$

where $P'=$const.

A direct computation yields

$$\Pi=\int_0^{z(t)}T^{10}dx+w(t)\dot{z}(t)=at(t^2-z^2(t))^{-(1+c_0^2)/2}-at^{-c_0^2}+w(t)\dot{z}(t). \tag{30}$$

From a comparison of (28), (29), and (30), it is clear that $P=P'$.

Eliminating $w(t)$ from (27) and (28), we obtain

$$\dot{z}(t)[E-az(t)(t^2-z^2(t))^{-(1+c_0^2)/2}]-[P-at(t^2-z^2(t))^{-(1+c_0^2)/2}]=0. \tag{31}$$

Equation (31) is an equation in total differentials. Its solution for $c_0^2<1$ is

$$Ez(t)-Pt+\frac{a}{1-c_0^2}(t^2-z^2(t))^{(1-c_0^2)/2}=Q=\text{const}, \tag{32}$$

while for $c_0^2=1$

$$Ez(t)-Pt+a\ln\frac{(t^2-z^2(t))^{1/2}}{\tau^*}=Q=\text{const}. \tag{33}$$

Using (32) and (33), we obtain an equation characterizing the variation of the mass of a boundary particle with the quantity $\tau=(t^2-z^2(t))^{1/2}$. For $c_0^2<1$,

$$m^2(\tau)=M^2+a^2\frac{1+c_0^2}{1-c_0^2}\tau^{-2c_0^2}-2aQ\tau^{-(1+c_0^2)} \tag{34}$$

and for $c_0^2=1$

$$m^2(\tau)=M^2+a^2[\ln(\tau/\tau^*)^2-1]\tau^{-2}-2aQ\tau^{-2}, \tag{35}$$

where $M^2\equiv E^2-P^2$.

By the use of the variable τ the particle trajectory is found in the explicit form

$$z(\tau)=\frac{E}{M^2}\left(A+\frac{P}{E}(A^2+M^2\tau^2)^{1/2}\right),$$

where

$$A=\begin{cases} Q-\dfrac{a}{1-c_0^2}\tau^{1-c_0^2}, & c_0^2<1,\\ Q-a\ln(\tau/\tau^*), & c_0^2=1.\end{cases}$$

Using the obtained relations, we can also easily find the dependence $w(t)$.

Let us investigate the obtained solutions. In the $t\to\infty$ asymptotic limit we find from (32) that $\dot z(t)\to P/E$, with, as can be seen from (29), the total momentum of the right-hand part of the system $\Pi(t)\to P$. In view of this, we require on the basis of physical arguments that $0<P\leqslant E$. An investigation of Eq. (32) shows that the function $z(t)$ can be determined uniquely provided $|\dot z(t)|<1$. And if at the initial moment $z(t_0)\geqslant 0$ and $\dot z(t_0)\geqslant 0$, then these inequalities and the condition $\dot z(t)<1$ are maintained at all $t>t_0$. We have directly from (25) and (26) that

$$\ddot z(t)=\frac{\pi(1-\dot z^2(t))^{1/2}\varepsilon}{\mu^2 m(t)}\left[(1+c_0^2)(u^0)^2\left(\dot z(t)-\frac{z(t)}{t}\right)^2+c_0^2(1-\dot z^2(t))\right]>0, \tag{36}$$

from which can be seen the nature of the convexity of the $z=z(t)$ curves.

The behavior of the solutions in the physical region is shown in Fig. 1 for different values of Q.

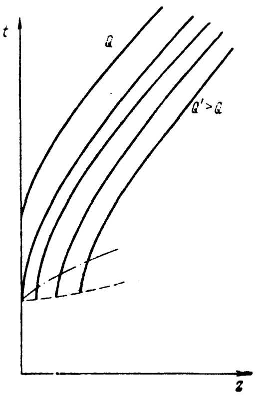

FIG. 1. The trajectories $z(t)$ in the region $z(t)\geqslant 0$, $\dot z(t)\geqslant 0$. As Q grows at fixed E and P, the curves shift to the right. The dashed line represents the curve

$$z=[t^2-(at/P)^{2/(1+c_0^2)}]^{1/2},$$

on which $\dot z=0$ for different Q. Above the curve

$$Ez-Pt+a(t^2-z^2)^{(1-c_0^2)/2}=0,$$

which is depicted in the figure by the dash-dot line, $\dot z\geqslant z/t$.

An important characteristic of the model under consideration is the growth of the entropy of the liquid in the course of the expansion. For the right half of the liquid, with allowance for the local conservation law (5) inside the liquid, we have

$$\frac{dS}{dt}=\frac{d}{dt}\int_0^{z(t)} su^0dx=su^0(\dot z(t)-v)|_{x=z(t)}. \tag{37}$$

Using (36), we can easily show by *reductio ad absurdum* that, if at the initial moment t_0

$$\dot z(t_0)>v(t_0,z(t_0))=z(t_0)/t_0,$$

then this inequality is maintained during the whole separation time, i.e., the entropy of the liquid increases, or $dS/dt>0$. Notice that in the $t\to\infty$ asymptotic limit we always have $\dot z(t)>v(t,z(t))$ (see Fig. 1).

It can be seen from (37) that the growth of the entropy follows from the fact that the boundary moves faster than the boundary elements of the liquid, i.e., new streamlines appear at the boundary (see Fig. 2). This implies the production of new liquid elements at the boundary in the course of the expansion (let us recall that the chemical potential of the liquid has been taken to be equal to zero). The energy expended on the production of the new liquid particles is borrowed both from the boundary particles and from the interior region occupied by the liquid. The nature of the energy exchange between a boundary particle and the liquid depends on the specific choice of the constants of integration and on the quantity c_0^2, different signs of $\dot w(t)$ being possible at different stages of the expansion. The indicated entropy-growth effect is connected with the decrease of the mass of the boundary particle, a result which can be seen from the following relation

$$\frac{dm}{dt}=\frac{\pi}{\mu^2}(1+c_0^2)\varepsilon(u^0)^2(1-vz(t))(1-\dot z^2(t))^{1/2}(v-\dot z(t)),$$

i.e.,

$$\operatorname{sign}\left(\frac{dm}{dt}\right)=-\operatorname{sign}\left(\frac{dS}{dt}\right),$$

which can be derived with the use of (36).

On the disintegration hypersurface $(t^2-x^2)^{1/2}=\tau^*$, with with allowance for (13), we have

$$E_l=kE=\frac{\pi}{\mu^2}\varepsilon^*z(\tau^*)=\frac{\pi}{\mu^2}\varepsilon^*\tau^*\operatorname{sh}\alpha_{max},$$

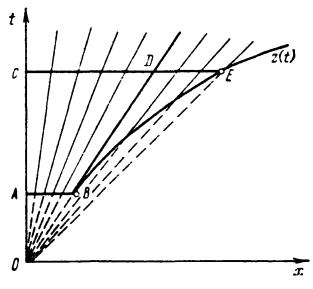

FIG. 2. The straight lines $x=ct$ (c = const) are the lines of flow of the liquid. The segment AB corresponds to the hadron system at the moment of time t_0, the segment CE at the moment $t>t_0$. The segment DE contains the liquid elements produced during the period (t_0, t).

whence for $E \to \infty$ we have

$$\alpha_{max} \approx \ln\left(\frac{2k\mu^2}{\pi\varepsilon^*\tau^*}E\right).$$

According to (15), we then have a logarithmic growth of the multiplicity of the secondary particles from the disintegration of the hadron liquid:

$$N \sim S = \frac{\pi}{\mu^2} s^*\tau^* \ln\left(\frac{2k\mu^2}{\pi\varepsilon^*\tau^*}E\right).$$

Let us write down the relations on the disintegration hypersurface that allow us to obtain the quantities that are measurable in experiment. For $\tau = \tau^*$ we have the following expression for the inelasticity coefficient k, which we define as the fraction of the energy possessed by the liquid:

$$k = \frac{E_l}{E} = \frac{\pi\varepsilon^*}{\mu^2 M^2}\left[B + \frac{P}{E}(B^2 + M^2\tau^{*2})^{1/2}\right], \tag{38}$$

and for the leading-particle mass

$$m^2(\tau^*) = M^2 - \frac{2\pi\varepsilon^*}{\mu^2}\left(B + \frac{\pi\varepsilon^*\tau^{*2}}{2\mu^2}\right), \tag{39}$$

where

$$B = \begin{cases} Q - \dfrac{\pi\varepsilon^*\tau^{*2}}{(1-c_0^2)\mu^2}, & c_0^2 < 1, \\ Q, & c_0^2 = 1. \end{cases} \tag{40}$$

The presence of the three constants of integration, E, P, and Q, reflects the arbitrariness in the choice of the initial values for $z(t_0)$, $\dot{z}(t_0)$, and $w(t_0)$ at the moment of time t_0.

Because of the Lorentz contraction of the longitudinal dimensions of the colliding hadrons, it is natural to assume that the initial dimensions of the hadron system decrease with increasing E. Let us, therefore, set $z(t_0) \sim E^{-n}$ $(n > 0)$. In the Landau model[1] $n = 1$, whereas allowance for the quantum effects[5] yields $n = 1/3$. For an arbitrary $n > 0$, we obtain, under the assumption that the mass of the leading cluster is either finite or increases with E not faster than E^γ $(\gamma < 1)$, the estimate $P/E \to 1$ for $E \to \infty$.

Thus, the relations (38) and (39) express k and m in terms of two parameters: Q and M^2. Notice that for fixed P, E, and Q, and for $0 < c_0^2 < 1$, the mass, m, of the leading cluster increases, while the coefficient k decreases, with increasing c_0^2. Using the experimental values for k and m ($k \approx 0.5$; m is of the order of one or several GeV), we can easily solve (38) and (39) for Q and M^2 and thereby fix the parameters of the model under consideration.

5. CONCLUSION

The physical picture of hadron-hadron collisions at high energies looks as follows in the model under consideration. Excited particle-like states arise during collisions of initial particles at the boundaries of the hadron system. In the process of separation they generate a trail of hadron liquid behind them, so that new liquid elements, which are drawn into the motion, appear during the entire time of expansion. The entropy of the liquid increases, which is directly connected with the decrease of the boundary-particle mass (in introducing an appropriate definition for the entropy of the boundary particles, we could have considered the entropy of the entire expanding hadron system to be constant).

We have thus far discussed that part of the hadron matter which moves in the CM system to the right. The analysis of the left half is completely analogous. Notice, however, that the division of the energy and the momentum between the liquid and a boundary particle may not be the same for the two halves, depending on the choice of the constants of integration for the left- and right-hand parts of the system. Thus, the model assumes that the symmetry in the final-particle distribution, arising when the distribution is averaged over a large number of collisions, can be broken in each individual event.

Notice also that the proper expansion time τ^* does not depend on the initial energy. The finiteness of τ^* ($\tau^* \sim 1/\mu$) justifies the use of the one-dimensional approximation. Indeed, the one-dimensional approximation for a liquid element moving with velocity v is valid as long as the signal from the lateral face (its velocity is equal to c_0) has not reached the center of the system, i.e., as long as

$$t(1-v^2)^{1/2} \lesssim 1/\mu c_0^2.$$

Since $v = x/t$, this is equivalent to the condition that $\tau \lesssim 1/\mu c_0$, which is fulfilled in our estimates: $\tau \leqslant \tau^* \sim 1/\mu$.

The space-time picture developed in the present paper for the process of multiple production can, apparently, be verified in the study of high-energy hadron-nucleus collisions. Let us mention in this connection the well-known Gottfried model,[12] whose hydrodynamic basis consists in precisely the use of solutions of the type (10). From the standpoint of our analysis, that model should be reexamined.

In conclusion, we should like to note one interesting interconnection between the multiperipheral and hydrodynamic approaches. The model considered by us yields the same results as the multiperipheral model. The basic premises and their justification are, however, not the same in the two models. If models of the hydrodynamic type meet with difficulties in the analysis of the initial phase of the process because of the neglect of the quantum effects, the multiperipheral model, in its conventional formulation, apparently contains inherent inconsistencies, which manifest themselves in the study of the space-time picture of the appearance of the real hadrons. As has been shown by Feĭberg in a recent paper,[13] the space volumes in which the particles of the multiperipheral chain are produced turn out to be quite small, so that the final phase of the process should contain motion of the hydrodynamic type. A synthesis of the two approaches is called for.

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Translated by A. K. Agyei

Solution of the Fokker-Planck equation for a laminar medium

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An exact analytic solution (Green's function) of the Fokker-Planck equation is found for the case in which the mean square of the multiple-scattering angle is a function of a longitudinal coordinate.

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The Fokker-Planck kinetic equation, which describes the spatial and angular distribution of particles subject to multiple scattering in the small-angle approximation, is of the well-known form

$$\frac{\partial w}{\partial z}+\theta\frac{\partial w}{\partial \mathbf{r}}=q\Delta_\theta w, \tag{1}$$

where $\mathbf{r}$ is the radius vector in the transverse direction, z is the coordinate of longitudinal displacement, $\boldsymbol{\theta}$ is a two-dimensional angular vector fixing the projection of the vector of the velocity's direction on a plane normal to the z axis, and q denotes one fourth of the mean square of the multiple scattering angle in unit path length along the longitudinal coordinate.

In applying this equation to actual physical problems it is necessary, as a rule, to take into account a dependence of q on the spatial coordinates. This is the situation, for example, when what is required is an estimate of the spatial and angular distribution of a beam of fast particles as they pass through an inhomogeneous medium. Examples are analysis of the passage of cosmic rays through the atmosphere, calculation of effectiveness of targets and of shielding arrangements in accelerators, investigation of the properties of charged-particle detectors, estimates of the effect of the spread in position and direction of travel on transition and Cherenkov radiation; there are many other cases. Moreover, if the particles are subject to continuous energy loss, one can take the effect of a dependence of q on the energy of a particle by introducing an appropriate dependence of the scattering properties of the medium on the coordinate. The case of dependence of q on the longitudinal coordinate z is of particular interest, since because the transverse displacements are small the dependence of the scattering properties of the medium on $\mathbf{r}$ are usually of little importance.

Therefore the aim of this paper is to find the distribution function in closed form (in terms of quadratures) in its dependence on the function $q(z)$.

We shall look for a solution of Eq. (1) in the form

$$w(\mathbf{r},\theta,z)=\frac{s}{4\pi^2}\exp(-p_1\mathbf{r}^2+p_2\mathbf{r}\theta-p_3\theta^2), \tag{2}$$

where p_1, p_2, p_3, and s are functions of the coordinate z.

Substitution of Eq. (2) in Eq. (1) gives the following system of equations:

$$\frac{1}{s}\frac{ds}{dz}=-4p_3q,\quad \frac{dp_1}{dz}=-p_2^2q,$$
$$\frac{dp_2}{dz}=2p_1-4p_2p_3q,\quad \frac{dp_3}{dz}=p_2-4p_3^2q. \tag{3}$$

Normalizing the function $w(\mathbf{r},\boldsymbol{\theta},z)$ to unity,

$$\iint w(\mathbf{r},\theta,z)\,d\mathbf{r}\,d\theta=1, \tag{4}$$

we find that $s=4p_1p_3-p_2^2$ and that the first equation is a consequence of the others. Successive use of the second, third, and fourth equations in combination with

