Quantum spectroscopy of relativistic beams

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The spectral intensity of collisionless radiation from relativistic beams is found, with quantization of the motion of the charges taken into account. The conclusions of the theory of equilibrium and collisionless radiation from highly compressed relativistic beams are employed for an analysis of experiments with micropinches in vacuum diodes. The theory explains the observed hard x-ray quanta emitted by the electrons in quantum transitions between discrete energy levels under conditions of maximum beam compression.

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1. INTRODUCTION

Sukhorukov and the present author[1] determined the structure of relativistic beams compressed to very small dimensions by the forces of collective interaction of electrons and ions. The characteristic transverse dimensions of beams compressed due to the pinch effect turned out to be of the order of or smaller than the atomic Bohr radius. As was pointed out by Budker,[2] in beams strongly compressed by the forces of collective interaction an important role is played by the radiation connected with the finite motion of charges. This radiation bears no relation to the collisions of the charges, and it is natural to call it collisionless. A theory of the collisionless radiation of relativistic beams was developed in Ref. 3 under the assumption that the motion of the emitters is described by classical mechanics. However, the smallness of the transverse dimensions of the beam dictates the necessity of quantization of the motion of charges and calls for an investigation of its influence on the radiation.

In the present work we consider the quantum theory of relativistic beams strongly compressed by the pinch effect. As a preliminary, it is shown that the condition, used for the sake of clarity in[1], that the average velocity v_0 of the beam electrons be close to the velocity of light c, is not essential. In §2 equations determining the equilibrium structure of the beam at an arbitrary value of $\beta=v_0/c$ are given and studied. In §3 the quantization of the motion of charges in the field of the forces of collective interaction is carried out. In §4 general formulas are obtained for the intensity of spontaneous radiation of a cylindrically symmetrical plasma as a function of energy of the quanta. In §5 the spectral intensity of the radiation of charges moving in a quadratic potential is found. This case admits of an exact solution. In §6 the dependence of the spectral intensity of the radiation of the degree of degeneracy of the electrons is examined. The intensity of the radiation of strongly degenerate electrons in the region of low and high frequencies is determined. In §7 the theory developed is utilized for the explanation of the radiation accompanying the pinch effect in a vacuum diode.[4] *According to the* theory set forth, the mechanism of emission of soft x-ray quanta consists in radiation of photons in the course of the electron transitions between discrete energy states of the transverse motion in the field of collective-interaction forces. The appearance of high-energy photons solely at the very beginning of the x-ray flash indicates a rapid heating of the plasma, due to the pinch effect, to a very high temperature, followed by a slow cooling down connected with the energy losses through radiation. For a detailed comparison of theory with experiment a further development of the experimental as well as theoretical work is required. In particular, it is necessary to improve the time resolution of the emission spectrum in the experiment and also to develop a theory of the evolution of a relativistic beam in the process of the pinch effect.

2. EQUILIBRIUM OF A RELATIVISTIC BEAM AT $\beta\sim1$

In the preceding paper[1] the structure of relativistic beams was considered, assuming for clarity that the average velocity of the electrons v_0 is close to the velocity of light c. In experiments with micropinches in vacuum diodes, however, the average electron velocity is, as a rule, small in comparison with the velocity of light. To make the application of the theory to the experiment[4] possible it is necessary to consider the beam equilibrium at an arbitrary $\beta=v_0/c\sim1$.

The structure of a beam compressed by the pinch effect can be found if[1]

$$T_e\gg E_a=m_e e^4/\hbar^2=27.2\text{ eV},\tag{2.1}$$

where T_e is the temperature of the electrons in the comoving frame of reference. With the condition (2.1) satisfied, as the density is growing in the process of compression, the degeneracy of the electrons sets in earlier than the non-ideality. Thus, in studying the structure of the beam, we are allowed to regard the electrons as an ideal gas obeying Fermi statistics. The equilibrium distribution function of the electrons in the laboratory frame of reference is of the form

$$f_e=\{1+\exp[\alpha+(\mathscr{E}-v_0p_z)/T_\perp]\}^{-1},\tag{2.2}$$

where $\mathscr{E}=(m_e^2c^4+p^2c^2)^{1/2}+U_e(r)$ is the total energy, P_z is the projection of the momentum **p** on the beam axis, $T_\perp=T_e(1-\beta^2)^{1/2}$ is the effective temperature of the transverse motion of electrons in the laboratory frame of reference, and α is a scalar determined by the normalization condition.

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If the temperature T_e and the Fermi energy $\epsilon_F = T_\perp|\tilde{\alpha}| = T_\perp|\alpha + m_e c^2/T_e|$ are small in comparison with $m_e c^2$:

$$T_e \ll m_e c^2, \quad |\tilde{\alpha}| \ll m_e c^2/T_\perp, \tag{2.3}$$

then the gas is nonrelativistic in the co-moving frame of reference K'. In this case the main contribution to the integrals with respect to the momenta comes from a small region near the minimum of the argument of the exponential (2.2). Denoting $T_\parallel = T_e(1-\beta^2)^{-1/2}$, we obtain for the electron density the expression

$$n_e(r) = \frac{8\pi m^* T_\perp (2m^* T_\parallel)^{1/2}}{(2\pi\hbar)^3} \int_0^\infty \frac{x^{1/2}\,dx}{1+\exp[\tilde{\alpha}+x+U_e(r)/T_\perp]}. \tag{2.4}$$

Here $U_e(r)$ is the potential of the collective-interaction forces acting on the electrons and $m^* = m_e(1-\beta^2)^{-1/2}$ is the effective mass of the transverse motion of the electrons in the laboratory frame of reference.

The equations determining the potentials $U_e(r)$ and $U_i(r)$ of the collective-interaction forces for the electrons and ions respectively, with the degeneracy of the electrons taken into account, reduce to the form

$$\frac{1}{r}\frac{d}{dr}r\frac{dU_e}{dr} = \frac{2e^2N_i}{\nu_i}\exp\left(-\frac{U_i}{T_i}\right) - \frac{T_i}{r_0^2}(1-\beta^2)\int_0^\infty \frac{x^{1/2}\,dx}{1+\exp(\tilde{\alpha}+x+U_e/T_\perp)},$$

$$\frac{1}{r}\frac{d}{dr}r\frac{dU_i}{dr} = \frac{T_i}{r_0^2}\int_0^\infty \frac{x^{1/2}\,dx}{1+\exp(\tilde{\alpha}+x+U_e/T_\perp)} - \frac{2e^2N_i}{\nu_i}\exp\left(-\frac{U_i}{T_i}\right). \tag{2.5}$$

Here N_i is the number of ions per unit length of the beam, T_i the ion temperature, ν_i the normalization integral:

$$\nu_i = \int_0^\infty \exp\left(-\frac{U_i}{T_i}\right) r\,dr,$$

$$r_0 = \frac{\pi^{1/4}}{2^{5/4}}a\left(\frac{E_a}{T_e}\right)^{1/4}\left(\frac{T_i}{T_e}\right)^{1/2}(1-\beta^2)^{1/4}, \tag{2.6}$$

and $a = \hbar^2/m_e e^2$ the Bohr radius. The scalar $\tilde{\alpha}$ is connected with the number of electrons per unit length of the beam N_e by the normalization condition

$$N_e = \frac{T_i}{2e^2 r_0^2}\int_0^\infty r\,dr\int_0^\infty \frac{x^{1/2}\,dx}{1+\exp(\tilde{\alpha}+x+U_e(r)/T_\perp)}. \tag{2.7}$$

It follows from Eqs. (2.5) and the normalization condition (2.7) that at large distances the potentials $U_e(r)$ and $U_i(r)$ increase logarithmically:

$$U_e = 2e^2[N_i - (1-\beta^2)N_e]\ln r, \quad U_i = 2e^2(N_e - N_i)\ln r, \quad r\to\infty. \tag{2.8}$$

If the number of ions N_i satisfies the inequalities

$$N_e > N_i > N_e(1-\beta^2), \tag{2.9}$$

then both the electrons and the ions are subjected at large distances to an attractive force from the other charges. Since $\beta^2 > 0$ it is clear that the inequalities (2.9) can be satisfied for any arbitrary β, and not only for $\beta\to 1$. From (2.4) it follows that at large distances both the electron density $n_e(r)$ and the ion density $n_i(r)$ obey the Boltzmann statistics:

$$n_e \sim \exp(-U_e/T_\perp), \quad n_i \sim \exp(-U_i/T_i)$$

and decrease with r according to power laws:

$$n_e \sim r^{-2K_e}, \quad n_i \sim r^{-2K_i}, \quad r\to\infty,$$
$$K_e = e^2[N_i - (1-\beta^2)N_e]/T_\perp,$$
$$K_i = e^2(N_e - N_i)/T_i. \tag{2.10}$$

For the normalization integrals ν_i and (2.7) to converge as $r\to\infty$ it is necessary that the inequalities

$$K_e,\ K_i > 1 \tag{2.11}$$

be satisfied.

If the dimensionless parameters on which the beam structure depends $(\beta, T_i/T_\perp, e^2(N_e - N_i)/T_i, e^2[N_i - (1-\beta^2)N_e]/T_\perp)$ are of the order of unity, it follows from (2.5) that the characteristic transverse dimension of the beam is of the order of r_0 (2.6), that is, of the order of or smaller than the Bohr radius. In the vicinity of the axis, at $r \ll r_0$, the potentials U_e and U_i depend on r quadratically, while at large distances $r \gg r_0$ they increase logarithmically as r increases [Eq. (2.8)]. In the intermediate region $r\sim r_0$, the coordinate dependence of the potentials $U_e(r)$ and $U_i(r)$ can be found in the general case only by numerical integration of the Eqs. (2.5).[5,6] The structure of the beam has been found analytically[1] in the limiting case $\beta\to 1$ for strongly degenerate electrons under the condition $T_i \gg T_\perp$.

We note that the investigation of stationary equilibrium configurations of a cylindrically symmetrical relativistic beam does not change qualitatively also in the case of a relativistic electron gas. If the conditions (2.3) are not fulfilled, the formula (2.4) for the electron density has to be replaced by the following:

$$n_e(r) = \frac{8\pi T_e^3}{(2\pi\hbar)^3 c^3 (1-\beta^2)^{1/2}}\int_{x_0}^\infty \frac{x(x^2-x_0^2)^{1/2}\,dx}{1+\exp[\tilde{\alpha}+x+U_e(r)/T_\perp]}, \quad x_0 = \frac{m_e c^2}{T_e},$$

and a corresponding replacement of integrals must be carried out in the Eqs. (2.5).

3. QUANTIZATION OF THE MOTION OF ELECTRONS

If Eqs. (2.5) are solved and the potentials $U_e(r)$ and $U_i(r)$ found, then the vector potential $\mathbf{A} = (0, 0, A_z)$ and the scalar potential ϕ of the electromegnetic field of the collective interaction are determined from the formulas

$$\varphi = U_i/e, \quad A_z = (U_i + U_e)/\beta e. \tag{3.1}$$

The quantum-mechanical motion of an electron in the field (3.1) is described by the Dirac equation.[7] In a stationary, cylindrically symmetrical beam the time t and the coordinates z and φ are cyclic variables. The generalized momenta canonically conjugate with them—the total energy E, and the projections of the generalized momentum P_z and of the total angular momentum $\hbar_j$ onto the beam axis—are quantum numbers. The quantum number P_z corresponds to infinite motion along the beam axis and the spectrum of its values is continuous.

If the number N_e of electrons per unit length of the beam is small

$$N_e \ll m_e c^2/e^2, \tag{3.2}$$

and the transverse dimension r_0 large in comparison with the Compton wavelength

$$r_0 \gg \hbar/m_e c = 3.8\ 10^{-11}\,\text{cm}, \tag{3.3}$$

then the influence of the magnetic field of the beam current and the spin of the particles may be disregarded. In this case the problem permits separation of the variables in cylindrical coordinates. The condition (3.2) is equivalent to a situation where the Larmor radius r_H of the electrons in the magnetic field of the current is large compared to r_0 and their transverse motion reduces to oscillations in the field $U_e(r)$.

Under the conditions (2.3), (3.2) and (3.3) the wave function of the electron in the co-moving reference frame satisfies the Schrödinger equation and is of the form

$$\Psi' = \frac{R(r)}{(2\pi Z')^{1/2}} \exp\left\{-\frac{i}{\hbar}E't' + \frac{i}{\hbar}P_z'z' + il\varphi\right\}, \quad E' = \mathscr{E}' - m_e c^2.$$

The transverse coordinates r and φ remain unchanged in Lorentz transformations. The radial part R of the wave function is normalized to unity and the function itself to one particle on the length Z'. Using the formulas of the Lorentz transformations for the field (Ref. 8, §24) and expressing the energy of the transverse electron motion $\epsilon' = E' - p_z'^2/2m_e$ in the frame K' and the potential energy $U'(r)$ in terms of the respective quantities ϵ and $U_e(r)$ in the laboratory frame:

$$\varepsilon' = \varepsilon(1-\beta^2)^{-1/2}, \quad U_e' = U_e(1-\beta^2)^{-1/2},$$

we reduce the equation for the radial part R of the wave function to the form

$$\frac{1}{r}\frac{d}{dr}r\frac{dR}{dr} + \frac{2m^*}{\hbar^2}\left[\varepsilon - U_e(r) - \frac{\hbar^2 l^2}{2m^* r^2}\right]R = 0. \tag{3.4}$$

As r increases, the potential U_e in (3.4) changes from quadratic at small distances from the axis to logarithmic at large distances. The transverse motion of the charges is finite. Equation (3.4) yields a discrete set of possible energy values $\epsilon = \epsilon_{nl}$, $n = 0, 1, \ldots$, $l = 0, \pm 1, \ldots$ In the region of small energies near the potential minimum the energy spectrum is equidistant. As the energy increases, the distance between levels decreases because the potential $U_e(r)$ ceases to be quadratic. In the logarithmic region the distance between levels decreases exponentially with increasing energy.[3]

For strong-current beams $N_e \gtrsim m_e c^2/e^2$ $(v_0 \sim c)$ the spin and the radial variable in the Dirac equation are not separable. But even in this case electron states with definite values of the total energy $E = E_{P_z,j,n}$ differ from one another by one continuous index P_z and two discrete ones j and n. The discrete nature of the radial quantum number n is in general due not only to the oscillations in the field $U_e(r)$, but also to the quantization of the electron motion in the strong azimuthal magnetic field of the current.

4. QUANTUM THEORY OF COLLISIONLESS RADIATION

Condition (3.2) is equivalent to the inequality $\delta \gg r_0$, where $\delta = c(4\pi\omega\sigma)$ is the depth of penetration of a field of frequency ω into plasma, $\sigma \sim e^2 n_e/m_e\omega$ the electric conductivity, and $n_e \sim N_e/\pi r_0^2$ the electron density. This means that the radiation of each individual charge leaves the plasma without appreciable absorption by other charges, and the radiation of the beam is the sum of the radiations of individual electrons. In the opposite limiting case $\delta \ll r_0$ every photon is repeatedly absorbed and reradiated by other charges before leaving the beam. This puts the electromagnetic radiation in equilibrium with matter. To determine the intensity of radiation for $\delta \ll r_0$ the absorptivity of the beam should be calculated and Kirchhoff's law applied. In §§4 and 5 we assume the conditions (2.3), (3.2) and (3.3) as fulfilled.

The relativistically invariant formula for the transition of an electron from the initial state i to the final state f with emission of a photon with four-vector k_μ and polarization e_μ is, in the first order of perturbation theory, of the form

$$a_{if} = -ie\int A_\mu j_{fi}^\mu d^4x,$$

where $A_\mu = (2\pi/\hbar\omega V)^{1/2} e_\mu \exp(ik_\nu x^\nu)$ is the four-potential of the photon, $j_{fi}^\mu = \bar{\Psi}_f \gamma^\mu \Psi_i$ the four-vector of the current of the transition from the state i to the state f. The probability of emission of a photon into the phase-space element $d\Gamma_k = Vd^3k/(2\pi)^3$ due to the transition of an electron from the state i to the state f into the element $d\Gamma_f$ is equal, with the Pauli principle taken into account, to

$$dW_{if} = |a_{if}|^2 d\Gamma_k(1-f_f)d\Gamma_f = \frac{e^2}{(2\pi)^2}\frac{d^3k}{\hbar\omega}\left|\int d^4x\, e^{ik_\nu x^\nu} e_\mu \bar{\Psi}_f\gamma^\mu\Psi_i\right|^2 (1-f_f)d\Gamma_f. \tag{4.1}$$

Here f_f is the Fermi distribution function (2.2) of the electrons in the final state, $d\Gamma_f = (2/2\pi\hbar)Zdp_{zf}$, and Z is a normalization length.

The separation of the integration variables into factors $d^4x = d^2x_\perp dzdt$ is relativistically invariant. The dependence of the transition current j_{fi}^μ on t and z also has the form of a relativistically invariant factor

$$\exp\{-i(\mathscr{E}_i - \mathscr{E}_f)t/\hbar + i(p_{zi} - p_{zf})z/\hbar\}.$$

Taking, as usual, integrals of the form

$$\left|\int dt \exp\left\{\frac{i}{\hbar}(\mathscr{E}_i - \mathscr{E}_f - \hbar\omega)t\right\}\right|^2$$

to mean $2\pi\hbar T\delta(\mathscr{E}_i - \mathscr{E}_f - \hbar\omega)$, where T is the duration of the emission process, we obtain for the probability (4.1) the expression

$$dW_{if} = e^2\hbar^2 TZ\delta(\mathscr{E}_i - \mathscr{E}_f - \hbar\omega)\delta(p_{zi} - p_{zf} - \hbar k_z)A\frac{d^3k}{\hbar\omega}(1-f_f)d\Gamma_f,$$

where

$$A = \left|\int d^2x_\perp\, e^{ik_\perp x_\perp} e_\mu \bar{\psi}_f\gamma^\mu\psi_i\right|^2 \tag{4.2}$$

is a relativistically invariant quantity. This means that if it is expressed in some frame K', its expression in the frame K is

$$A^{(K)}(\mathscr{E}_i, \mathscr{E}_f, p_{zi}, p_{zf}, \dots) = A^{(K')}(\mathscr{E}_i', \mathscr{E}_f', p_{zi}', p_{zf}', \dots),$$

where the primed quantities are expressed in terms of the unprimed quantities by the formulas of the Lorentz transformation.

The scalar A is easiest to find in the simplest way in the co-moving frame of reference K'. Under the condition (2.3) the radiation is described in the frame K' by means of the formulas of the nonrelativistic dipole approximation.[7] In this approximation the integrals (4.2) reduce to matrix elements of the velocity operator computed with the help of nonrelativistic wave functions of the transverse motion. According to the rule of operator differentiation[9] the velocity matrix element $\mathbf{v}_{fi}'$ is conveniently expressed in terms of the acceleration matrix element $\mathbf{W}_{fi}'$:

$$|\mathbf{v}_{fi}'|^2 = |\mathbf{W}_{fi}'|^2 \hbar^2/(\mathscr{E}_i' - \mathscr{E}_f')^2.$$

Taking the four-vector of photon polarization in a three-dimensional transverse gauge and summing over the photon polarizations:

$$\sum_\lambda e_\mu^*(\lambda) e_\nu(\lambda) = \delta_{ik} - n_i' n_k',$$

where $\mathbf{n}'$ is the unit vector in the direction of $\mathbf{k}'$, we obtain

$$\sum_\lambda A^{(K')} = \frac{\hbar^2}{(\mathscr{E}_i' - \mathscr{E}_f')^2} (|W_{fi}'|^2 - |(\mathbf{n}'\mathbf{W}_{fi}')|^2).$$

We change now to the laboratory frame. Recognizing that U_e' does not explicitly depend on φ we have

$$\mathbf{W}' = -\frac{1}{m_e}\frac{\partial U_e'}{\partial \mathbf{r}} = -\frac{1}{m_e}\frac{dU_e'}{dr}\frac{\mathbf{r}}{r},$$

and since the transverse coordinates are not changed by Lorentz transformations, we obtain

$$\mathbf{W}' = \mathbf{W}/(1-\beta^2), \quad \mathbf{W} = -\frac{1}{m^*}\frac{dU_e}{dr}\frac{\mathbf{r}}{r}.$$

After separation of the variables z and t the wave functions still contain the normalization length Z', which is connected with the normalization length Z in the laboratory frame by the Lorentz contraction formula $Z = Z'(1-\beta^2)^{1/2}$. If a coordinate system is now chosen in such a way that the direction $\mathbf{n}'$ of photon propagation lies in the xz-plane, we obtain, in virtue of azimuthal symmetry

$$|(\mathbf{n}'\mathbf{W}_{fi}')|^2 = {}^1/_2 \sin^2\theta' |\mathbf{W}_{fi}'|^2,$$

where θ' is the angle between the propagation direction of the radiation and the beam axis in the frame K'.

Taking into consideration the law of conservation of energy $\mathscr{E}_i' - \mathscr{E}_f' = \hbar\omega'$, the Doppler effect $\omega' = \omega(1-\beta\cos\theta)(1-\beta^2)^{-1/2}$, and the aberration of light $\sin\theta' = \sin\theta(1-\beta^2)^{1/2}(1-\beta\cos\theta)^{-1}$, we obtain

$$\sum_\lambda A^{(K)} = \frac{|\mathbf{W}_{fi}|^2}{\omega^2(1-\beta\cos\theta)^2}\left\{1 - \frac{1}{2}\frac{(1-\beta^2)\sin^2\theta}{(1-\beta\cos\theta)^2}\right\}.$$

A matrix element $\mathbf{W}_{fi}$ equals

$$\mathbf{W}_{fi} = -\frac{1}{2\pi Z m^*}\int_0^\infty R_{n_f l_f}^* \frac{dU_e}{dr} R_{n_i l_i} r\,dr \int_0^{2\pi} \exp[i(l_i - l_f)\varphi]\frac{\mathbf{r}}{r}d\varphi. \qquad (4.3)$$

The integral with respect to φ in (4.3) is different from zero for $l_f = l_i \pm 1$ only. As it should be, in the case of cylindrical symmetry and in the dipole approximation only transitions with unity change in the azimuthal angular momentum are allowed.

Neglecting, under the condition (2.3), the recoil and using the law of conservation of the projection of the momentum onto the beam axis: $p_{zf} = p_{zi} - \hbar\omega\cos\theta/c$, we can reduce the δ-function expressing the law of energy conservation to the form

$$\delta(\mathscr{E}_i - \mathscr{E}_f - \hbar\omega) = \delta(\omega - \omega_{if})/\hbar(1-\beta\cos\theta),$$

where

$$\omega_{if} = (\varepsilon_{n_i l_i} - \varepsilon_{n_f l_f})/\hbar(1-\beta\cos\theta) \qquad (4.4)$$

is the transition frequency in the laboratory frame. We introduce the notation

$$F(\theta) = \frac{1}{(1-\beta\cos\theta)^3}\left\{1 - \frac{1}{2}\frac{(1-\beta^2)\sin^2\theta}{(1-\beta\cos\theta)^2}\right\}, \qquad (4.5)$$

$$W_{n_i n_f l_i l_f} = -\frac{1}{2m^*}\int_0^\infty R_f^* \frac{dU_e}{dr} R_i r\,dr. \qquad (4.6)$$

After summation over the final-state polarizations and averaging with respect to the initial spin state we obtain for the differential emission probability per unit time

$$dw_{if} = \frac{dW_{if}}{T} = \frac{e^2}{c^3}F(\theta)|W_{n_i n_f l_i l_f}|^2(\delta_{l_f, l_i+1} + \delta_{l_f, l_i-1})$$
$$\times \delta(\omega - \omega_{if})\delta\left(p_{zi} - p_{zf} - \frac{\hbar\omega}{c}\cos\theta\right)\frac{d\omega}{\omega}dO(1-f_f)\frac{d\Gamma_f}{Z}. \qquad (4.7)$$

In the initial state the phase-volume element

$$d\Gamma_i = \frac{2}{2\pi\hbar}Z\,dp_{zi}$$

contains $f_i d\Gamma_i$ particles. The intensity of emission from a phase volume element $d\Gamma_i$ is obtained multiplying dw_{if} by the quantum energy $\hbar\omega$ and by the number of electrons $f_i d_i \Gamma_i$, summing over the final states of the discrete spectrum and integrating with respect to dp_{zf}:

$$dJ_i = \sum_{n_f=0}^\infty \sum_{l_f=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dp_{zf}\frac{dw_{if}}{dp_{zf}}\hbar\omega f_i\,d\Gamma_i.$$

After summation over the spectrum of initial states we find the intensity of emission per unit length of the beam in a unit frequency interval into a unit solid angle

$$\frac{dJ}{dZ\,d\omega\,dO} = \frac{e^2F(\theta)}{\pi^2c^3\hbar}\sum_{n_i=0}^\infty\sum_{l_i=-\infty}^{+\infty}\sum_{n_f=0}^\infty\sum_{l_f=-\infty}^{+\infty}\int_{-\infty}^{+\infty}dp_{zi}\int_{-\infty}^{+\infty}dp_{zf}f_i\cdot$$
$$\cdot(1-f_f)|W_{n_i n_f l_i l_f}|^2(\delta_{l_f,l_i+1} + \delta_{l_f,l_i-1})\delta(\omega-\omega_{if})\delta\left(p_{zi} - p_{zf} - \frac{\hbar\omega}{c}\cos\theta\right). \qquad (4.8)$$

In the quasiclassical limit $n_i, n_f, l \gg 1$ the matrix element (4.3) goes over into a Fourier component [formula (4.4) in[3]]. The transition frequency (4.4) takes the form

$$\omega_{if} = (\Delta n \omega_r + \Delta l \omega_\varphi)/(1-\beta\cos\theta),$$

where ω_r and ω_φ are the fundamental frequencies of the radial and azimuthal motion $\Delta n = n_i - n_f$, and $\Delta l = l_i - l_f$. Passing from summation over n_i and l_i to integration with respect to the energy E and angular momentum M:

$$\sum_{n_i=0}^{\infty} \to \frac{1}{\hbar}\int_{E_m}^{\infty} \frac{dE}{\omega_r}, \quad \sum_{l_i=-\infty}^{+\infty} \to \frac{1}{\hbar}\int_{-\infty}^{+\infty} dM,$$

it is easy to see that (4.8) formulas of the quantum theory of radiation goes over into the corresponding classical formula (4.9) from[3] with the only difference that the angular dependence (4.5) differs from Eq. (3.5) of[3] by the factor $1-\beta\cos\theta$. This difference is due to the curcumstance that in[3] the radiation intensity pertains to that length dZ of the beam which is occupied by emitters at an instant of time $t = t' + R(t')/c$ that is delayed relative to the emission instant t'. To allow for the influence of the emission on the structure and evolution of the beam, formula (4.5) has to be used.

5. SPATIAL OSCILLATOR

The calculation of the spectral intensity of collisionless radiation can be carried through completely in the case where most of the emitters are situated near the minimum of the potential energy $U_e(r)$ so that in the expansion of U_e in powers of r we can limit ourselves to the quadratic term:

$$U_e(r) = U_e(0) + \frac{1}{2}\frac{d^2U_e(0)}{dr^2}r^2. \qquad (5.1)$$

The Schrödinger equation for the spatial oscillator (5.1) is solved more conveniently in Cartesian than in cylindrical coordinates. After separation of variables the problem reduces to the solution of independent equations for two linear oscillators describing the motion of charge along the x and y axis respectively.[9] The energy spectrum is of the form

$$\varepsilon_{n_x n_y} = U_e(0) + \hbar\omega_0(n_x + n_y + 1),$$

where n_x and n_y are quantum numbers labeling the energy levels and

$$\omega_0 = \left(\frac{1}{m^*}\frac{d^2U_e(0)}{dr^2}\right)^{1/2} \qquad (5.2)$$

is the frequency of the oscillators.

Computation of the matrix elements of the acceleration reduces to finding the matrix elements of the radius vector:

$$|W_{n_i n_f}|^2 = \frac{1}{4m^*}\left|\frac{d^2U_e(0)}{dr^2}\right|^2 \{(x_{n_i n_f})^2 + (y_{n_i n_f})^2\},$$

which are different from zero only for transitions that change one of the quantum numbers by unity. For example,

$$x_{n_i n_f} = \left(\frac{\hbar}{2m^*\omega_0}\right)^{1/2} \delta_{n_{yi} n_{yf}} \{(n_{xi})^{1/2}\delta_{n_{xf}, n_{xi}-1} + (n_{xi}+1)^{1/2}\delta_{n_{xf}, n_{xi}+1}\}.$$

For the oscillator the transition frequency (4.4) does not depend on the quantum numbers: $\omega_{if} = \omega_0(1-\beta\cos\theta)^{-1}$.

The quantum energy $\hbar\omega$ is a positive quantity. Emission corresponds to transition to a lower level $n \to n-1$. In the expression for the intensity of radiation only those terms should be kept which correspond to transitions with lowering of the quantum numbers.

By taking into account the laws of conservation of energy and momentum, the combination $f_i(1-f_f)$ that takes account of the fact the electrons are present in the initial state and that the final state is not occupied can be written in the form

$$f_i(1-f_f) = (f_f - f_i)(\exp(\hbar\omega_0/T_\perp) - 1)^{-1} \qquad (5.3)$$

that is, as the difference between the occupation numbers of the initial and final state, multiplied by the Planck factor. Turning to summation over $k = n_{xi} + n_{yi}$, we have

$$\sum_{n_{xi}=0}^{\infty}\sum_{n_{yi}=0}^{\infty}\sum_{n_{xf}=0}^{\infty}\sum_{n_{yf}=0}^{\infty} f_i(1-f_f)\{n_{xi}\delta_{n_{xf},n_{xi}-1}\delta_{n_{yf},n_{yi}} + n_{yi}\delta_{n_{xf},n_{xi}}\delta_{n_{yf},n_{yi}-1}\}$$
$$= 2[\exp(\hbar\omega_0/T_\perp) - 1]^{-1}\sum_{k=0}^{\infty}(k+1)f(k).$$

The integration with respect to dp_{zf} is eliminated by the δ-function, and the remaining integral with respect to dp_{zi} and the sum over k are exactly the same as in the normalization formula:

$$N_e = \frac{2}{2\pi\hbar}\int_{-\infty}^{+\infty} dp_z \sum_{k=0}^{\infty}(k+1)f(k).$$

The final expression for the spectral intensity of radiation by a spatial oscillator is of the form

$$\frac{dJ}{dZ\,d\omega\,dO} = \frac{e^2N_e}{4\pi c^3m^*}\frac{\hbar\omega_0^3}{\exp(\hbar\omega_0/T_\perp)-1}F(\theta)\delta\left(\omega - \frac{\omega_0}{1-\beta\cos\theta}\right). \qquad (5.4)$$

Formula (5.4) holds for an arbitrary degree of degeneracy and an arbitrary ratio $\hbar\omega_0/T_\perp$. All that is required is that most of the particles be near the minimum of the potential $U_e(r)$, where the expansion (5.1) is valid.

The radiation intensity is concentrated in the case of the spatial oscillator (5.1) in a narrow spectral line near the transition frequency. Its width is determined by the collisions of the emitters, by the Doppler effect, and by the anharmonicity of the collective-interaction potential.

It should be noted that the condition $\delta \gg r_0$, which allows us to represent the intensity of radiation of the beam as a sum of radiation intensities of individual electrons, leads in the case of a narrow line to a restriction more stringent than (3.2), namely

$$N_e \ll \frac{m_e c^2}{e^2} \frac{\omega_0}{\Delta\omega}$$

($\Delta\omega$ is the width of the spectral line) because of the resonant dependence of the electric conductivity σ on the frequency.

6. SPECTRAL INTENSITY OF RADIATION BY STRONGLY DEGENERATE ELECTRONS

The frequency dependence of the intensity of collisionless radiation is determined both by the pecularities of the energy spectrum of the electrons and by the nature of the population of the energy quantum states. The emission spectrum of the electrons is different in the two limiting cases of Boltzmann statistics and strong degeneracy. For potentials $U_e(r)$ that go over smoothly from quadratic as $r \to 0$ to logarithmic as $r \to \infty$ the frequency of the finite motion and the distance between the energy levels both decrease as the energy increases. With increasing frequency ω the main contribution to the spectral intensity of collisionless radiation comes from electrons situated lower and lower on the energy scale. In the case of Boltzmann statistics the spectral intensity of the radiation increases with quantum energy. In the region of low frequencies emitted by electrons at large distances from the axis, the radiation intensity increases with increasing photon energy $\hbar\omega$ according to a power law:[3]

$$\frac{dJ}{dZ\,d\omega\,dO} \sim (\hbar\omega)^{2K_e-1}, \tag{6.1}$$

where K_e is given in (2.10). As the frequency becomes higher, the basic contribution to the radiation comes from electrons situated lower on the energy scale. Near the minimum of U_e the energy spectrum becomes equidistant. In the case of Boltzmann statistics the radiation intensity reaches its maximum at the transition frequency $\omega_{if} = \omega_0(1 - \beta\cos\theta)^{-1}$ and falls off sharply thereafter.

The spectral intensity of collisionless radiation behaves differently in the opposite limiting case of strong degeneracy of the electrons in the vicinity of the beam axis. Owing to the Pauli principle, the emission intensity is proportional not only to the number of emitters, but also to the number of unoccupied final states. If the electron degeneracy is strong ($|\tilde{\alpha}| \gg 1, \tilde{\alpha} < 0$), practically all lower quantum states are occupied. With increasing energy ϵ a transition to the Boltzmann statistics, i.e., from occupied to unoccupied states, takes place in the Fermi-energy region $\epsilon \sim |\tilde{\alpha}|T$ of width $\Delta\epsilon \sim T_\perp$.

The maximum of collisionless radiation coincides in the case of Fermi statistics with the transition frequency for electrons with energy of the order of the Fermi energy $\epsilon \sim |\tilde{\alpha}|T_\perp$. In the region of higher frequencies the emission intensity drops as the photon energy increases. In this frequency region the spectral intensity is determined by details of the behavior of the potential U_e in the range $U_e(0) < U_e < U_e(0) + |\tilde{\alpha}|T_\perp$ and cannot be calculated in general form.

The dependence of the radiation intensity on the photon energy in the high-frequency region can be found in the limiting case of strong degeneracy, when the main contribution comes from electrons chiefly in the region of logarithmic potential far from the axis. In the limiting case $\hbar\omega \ll T$ this dependence was found previously.[3] The relation (5.3) allows to calculate the spectral intensity of the radiation in the high-frequency region ($\omega \gg \omega_{max}$) for arbitrary $\hbar\omega/T_\perp \sim 1$:

$$\frac{dJ}{dZ\,d\omega\,dO} = \frac{2^{7/2}}{\pi} \frac{e^2 K_e^{3/2} T_e^3 v_T}{(\hbar c\gamma)^3} F(\theta) \left(\ln\frac{\omega}{\omega_{max}}\right)^{1/2} \frac{\hbar(1-\beta\cos\theta)/T_\perp}{\exp[\hbar\omega(1-\beta\cos\theta)/T_\perp]-1}, \tag{6.2}$$

where K_e is defined in (2.10),

$$v_T = (2T_e/m_e)^{1/2}, \quad \gamma = (1-\beta^2)^{-1/2},$$

$\omega_{max} = (v_T/r_e)(1 - \beta^2)^{1/2}(1 - \beta\cos\theta)^{-1}$; r_e is the characteristic dimension of the region occupied by the electrons. We note that with N_e and N_i given, the temperature of the electrons T_e appears in (6.2) only in the exponential of the Planck factor and under the logarithm sign.

7. MICROPINCH IN A VACUUM DIODE

We consider the micropinch in a vacuum diode[4] in the light of the theory of equilibrium and collisionless radiation of relativistic beams. For lack of a nonstationary theory of beam evolution, the description of the process itself is only qualitative.

In the course of discharge the electrons impinging on the anode destroy it. In the vicinity of the anode positive ions are formed as a result of ionization of the atoms by the beam electrons. The electrons produced in the ionization process are subjected to strong repulsion by the beam electrons and fly away in the radial direction. But the ions are, on the contrary, subjected to attraction and are accumulated near the beam axis. When the number of ions per unit beam length N_i gets into the region (2.9), both the ions and the electrons at large distances from the axis are acted on by the remaining charges with attraction forces. Under the action of these forces the beam collapses and a pinch effect occurs. In the course of compression the temperatures of the electrons and ions increase. Simultaneously with the heating, the charges lose energy to the radiation, whose intensity increases with the compression in inverse proportion to the square of the beam radius.[3]

If the electrons or ions are heated in the course of compression so strongly that any one of the inequalities (2.11) is violated, then the corresponding energy of compression will be smaller than the energy of the thermal scatter of the particles in the transverse direction, and the compression will be stopped. Otherwise, i.e. if the inequalities (2.11) are not violated, in the process of compression and radiative cooling the beam is "condensed" in the vicinity of the axis. The compression proceeds until further increase of the electron density becomes impossible because of degeneracy. The subsequent evolution of the beam is determined in the main by the radiative cooling of the electrons. The electron temperature drops, and in accordance with (2.6), the transverse dimension of the beam grows.

The collisionless radiation is most intense in the state of strongest compression and heating of the beam. The spectral maximum of the intensity is then in the region of the highest frequencies, the discrete nature of the energy spectrum is most strikingly manifest, and the hardest quanta are emitted. Subsequently, as the radiative cooling proceeds, higher and higher energy levels become populated, as a result of which the maximum of the spectral intensity of radiation is shifted into the region of low frequencies.

If the accelerating potential difference and the diode current at the moment of the pinch effect is known (16 kV and ~100 kA in[4]), then the following parameters are unambiguously determined: $\beta = 0.25$, $N_e = 0.83 \times 10^{14}\,\mathrm{cm}^{-1}$. The total potential energy of the collective compression is $e^2 N_e \beta^2 \approx 750\,\mathrm{keV}$. One part of this energy, namely $e^2(N_e - N_i)$, pertains to the ion, and the remainder $e^2[N_i - (1-\beta^2)N_e]$ to the electron. This means that in the process of compression the conditions (2.11) are not violated until the electrons or ions are heated to a temperature of the order of hundreds of keV. Under such conditions the beam may be completely condensed in the vicinity of the axis in the course of compression and radiative cooling.

On the basis of x-ray pictures, the transverse dimensions of beams in the state of greatest compression are estimated to be of the order of a few microns[10] with an average density of the order of $10^{21}\,\mathrm{cm}^{-3}$.[4] Unfortunately contemporary experimental techniques do not answer unambiguously the question whether the current is distributed uniformly over the cross section of the beam or is concentrated in one or several veins whose transverse dimensions correspond to the density of the condensed state. Recently a paper by Lochte-Holtgreven[11] was published in which it is reported that in an electric explosion of deuterated liquid filaments strong heating takes place in separate regions of very small size, which results in nuclear fusion if the density exceeds $10^{23}\,\mathrm{cm}^{-3}$.

For an estimate of the energy of the emitted quanta we note that the electron density in the degenerate state is of the order $n_e \sim (m_e T_e)^{3/2}/\hbar^3$. The characteristic beam radius r_0 is determined from the condition $n_e \pi r_0^2 \sim N_e$, i.e., $r_0 \sim N_e^{1/2} \hbar^{3/2} (m_e T_e)^{-3/4}$. In the case of weak current (3.2) we obtain, using (5.2), for the energy of an emitted photon

$$\hbar\omega \sim m_e c^2 \alpha^{1/2} \beta (T_e/m_e c^2)^{3/4}, \quad N_e \ll mc^2/e^2,$$

where α is the fine-structure constant ($U_e \sim e^2 N_e \beta^2$).

In the opposite limiting case of a strong current, the Larmor radius is small in comparison with the size of the beam. The energy of the emitted quantum is equal to the distance between Landau levels. Noting that the magnetic field of the current is of the order $H \sim I/cr_0$, we obtain

$$\hbar\omega = \hbar\Omega \sim \hbar e H/m_e c,$$

or

$$\hbar\omega \sim m_e c^2 \alpha^{1/2} \beta^{1/2} (T_e/m_e c^2)^{3/4} (I/m_e c^3 e^{-1})^{1/2}, \quad N_e \gg m_e c^2/e^2.$$

For temperatures of the order of hundreds of keV and currents of the order of hundreds of kA this corresponds to the region of hard x-radiation.

Thus the emission of hard x-ray quanta in the experiment of Lee[4] is naturally explained as collisionless emission of photons by electrons in transitions between discrete energy levels in the state of strongest compression and heating of the beam.

In the literature there are attempts to explain the hard x-ray emission by linking it with current dips and to the appearance of strong induction fields which can accelerate the electrons to the required energies.[12] A possible cause of current breakdown is assumed to be the anomalous resistance of the plasma, which may be made turbulent by the developing electrostatic instabilities. The electrons accelerated by the induction field could emit hard quanta in collisions with plasma ions and with the surface of metallic electrodes. However, and this is also pointed out by Lee,[13] the induction field connected with dLI/dt (where L is the inductance) is evidently insufficient to explain fully the hard x-radiation of large intensity. In particular, this mechanism does explain why the radiation intensity decreases in power-law fashion with increasing photon energy. Also, it remains unexplained why the electrons accelerated by the induction field emit hard quanta predominantly in the plasma of the anode vapor and not in collisions with the anode itself. This was shown experimentally by Cilliers *et al.*[14] who attempted to determine the spatial localization of the x-ray source. The radiation intensity averated over the time of radiative cooling, observed in,[4] decreases with the decrease of the photon energy in a power-law manner. If we assumed a power-law cooling $T_e = T_0(1 + t/_\tau)$, where $T_0 \gg \hbar\omega$, then on averaging the Planck factor we obtain a power-law decrease of intensity:

$$\left\langle \frac{dJ}{dZ\, d\omega\, dO} \right\rangle \sim \int_0^\infty \frac{dt}{\exp[\hbar\omega(1-\beta\cos\theta)/T_\perp(t)]-1} \sim (\hbar\omega)^{-1/\lambda}.$$

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Effect of nonlinear dissipation on plasma heating in the strong Langmuir turbulence regime

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We consider the one- and three-dimensional problems of plasma heating taking nonlinear effects into account. We study in the one-dimensional case the buildup of Langmuir solitons due to dissipative slowingdown, we obtain the way the spectrum develops in time in the constant pumping regime, and we investigate the self-similar electron heating regime. We consider in the three-dimensional case the effect of nonlinear conversion of the Langmuir oscillations into sound on the plasma heating when acoustic collapse takes place. We estimate the maximum extent of the inertial range corresponding to such a regime. We obtain the self-similar electron distribution for heating due to nonlinear conversion.

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In connection with the problem of the heating of a plasma target by powerful beams of light or of relativistic electrons, the heating of a plasma under strong Langmuir turbulence conditions has recently been studied intensively (see, for instance, Refs. 1 to 5). In those papers the main subject of the study was the resonance mechanism for the formation of hot electron "tails" and the influence of non-linear effects was not taken into account, although the role of non-linear dissipation in the dynamics of Langmuir solitons had been studied earlier.[1,6] Only the recent paper by Galeev *et al.*[7] drew attention to the non-linear conversion process of Langmuir waves into sound which under conditions of constant pumping is generated when collapsing solitons are damped.[5,8] This process is the main one in a typical three-dimensional problem; in the one-dimensional case, and also under conditions of adiabatically slow damping of a collapsing soliton,[3] the sound produced cannot guarantee conversion—in that case the process of the slowing-down of the solitons by trapped particles[1] comes into play and it can, in particular, determine completely the structure of the wave spectrum.[6]

The aim of the present paper is the study of the details of the heating of the particles under conditions when the dynamics of strong Langmuir turbulence is essentially determined by the non-linear dissipation, as well as the determination of the conditions for the existence of such regimes. We consider the one-dimensional case in the framework of the soliton-gas model, and the three-dimensional one in the acoustic collapse approximation, which to some extent distinguishes our considerations from the ones in ref. 7.

1. STRONG ONE-DIMENSIONAL TURBULENCE. SOLITON BUILD-UP REGIME

We consider a one-dimensional model of strong Langmuir turbulence which is a set of Langmuir solitons—localized non-linear Langmuir waves.[9] The frequency of the oscillations of the solitons is close to ω_{pe}. Characteristic parameters are: amplitude E, reciprocal of the width $k_0=eE/\sqrt{6}T$, and velocity v, $0\leqslant v<c_s$. The spectral expansion of the Langmuir soliton field has the form

$$E_k=E\left(k_0\operatorname{ch}\frac{\pi k}{2k_0}\right)^{-1}\approx\sqrt{6}\frac{T}{e}\vartheta\left(1-\frac{\pi k}{2k_0}\right). \quad (1)$$

It is well known that an isolated Langmuir soliton is a stationary and stable structure. Therefore, the transfer of energy from large to small dimensions which is characteristic for the strong turbulence regime proceeds in the case of not too powerful pumping through the fusion of solitons which are close in size.[10] As solitons with appreciably different amplitudes do not interact, the transfer can only take place in relays. In the hydrodynamic approximation a steady-state soliton amplitude distribution is established

$$F(E)\approx\mathrm{const}\cdot E^{-3}, \quad (2)$$

and in the model with discrete levels $E_{n+1}=2E_n$ we have accordingly for the occupation numbers $N(E)$

$$N(E)\approx\mathrm{const}\cdot E^{-2}, \quad (3)$$

which gives the spectral energy density $|E_k|^2\propto k^{-2}$.

