Power-law distributions occurring in a plasma turbulent reactor

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(Submitted April 20, 1977)
Zh. Eksp. Teor. Fiz. 73, 2157–2168 (December 1977)

We study the fluxes along the spectrum for stationary power-law distributions of the radiation and for the relativistic electrons in a plasma turbulent reactor (PTR), which is a model for cosmic ray sources. We show that the flux of relativistic electrons is proportional to the flux along the spectrum, and we determine the normalization constants and the directions of the fluxes. We discuss the problem of the sources producing these fluxes.

PACS numbers: 94.40.Cn

1. INTRODUCTION. KOLMOGOROV CHARACTER OF THE PTR POWER-LAW SPECTRA

It is well known that power-law energy (frequency) distributions are characteristic for many cosmic ray sources. Cosmic rays, the radio emission of discrete sources, and also the optical, x-ray and γ-ray emission in a number of cases show this behavior (see, e.g., Refs. 1 to 4).

The explanation of the power-law spectra of radio emission as being due to spontaneous synchrotron mechanism[3] led to a connection, which agrees well with experiments, between the exponents of the spectral density of the radiation $\omega^{-\alpha}$ and the particle distribution $\varepsilon^{-\gamma}$, $2\alpha=\gamma-1$, which follows from the way the characteristic frequency ω depends on the energy ε of the emitting relativistic electron

$$\omega=\omega_0(\varepsilon/mc^2)^2,\quad \varepsilon\gg mc^2 \tag{1.1}$$

provided the radiation leaves the source freely, that is, for small optical thicknesses. By virtue of the general character of the Doppler transformation of the frequency, the relation (1.1) is realized regardless of the actual scattering mechanism (Fig. 1), which determines only the quantity ω_0, which has the meaning of the cyclotron frequency for synchrotron radiation, of the plasma frequency or frequency of the low-frequency photon for the inverse Compton effect, and so on. In this case the power-law spectrum of the particles is given.

Power-law distributions of both particles and radiation could be obtained for a system of relativistic electrons and radiation which are scattered and contained by a turbulent plasma or by random magnetic fields.[1)]

In a plasma turbulent reactor (PTR) stationary power-law distributions are established as a result of the mutual cancellation of induced and spontaneous processes under conditions of large optical thickness, i.e., for lower frequencies and energies than those in the spontaneous-emission region. Condition (1.1) appears in the frequency dependence of the averaged scattering probability only in the form $u(\omega/\varepsilon^2)$,[6] where we have already assumed that ω is measured in units ω_0 and ε in units mc^2. Since we are considering one and the same process, each of the kinetic equations for the photon distribution $N_{\mathbf{k}}$ and the electron distribution $n_{\mathbf{p}}$ contains the same combination of distribution functions $\omega N\partial n/\partial\varepsilon+n$, multiplied by u and the density of electron states ε^2:

$$S(\omega,\varepsilon)=\varepsilon^2 u\left(\frac{\omega}{\varepsilon^2}\right)\left(\omega N\frac{\partial n}{\partial\varepsilon}+n\right). \tag{1.2}$$

Equation (1.2) has been written down in the isotropic case, $N_{\mathbf{k}}=N_\omega$, $n_{\mathbf{p}}=n_\varepsilon$ to which we restrict ourselves in the present paper. The PTR equations which take into account the differential nature of the electron energy transfer ($\omega\ll\varepsilon$) have the form[6]

$$\dot n_\varepsilon=\frac{1}{\varepsilon^2}\frac{\partial}{\partial\varepsilon}\int_0^\infty d\omega\, S(\omega,\varepsilon), \tag{1.3}$$

$$\dot N_\omega=\frac{1}{\omega^3}\int_0^\infty d\varepsilon\, S(\omega,\varepsilon). \tag{1.4}$$

For the sake of convenience we give in Appendix II a quantum-mechanical derivation of the PTR equations which also holds in the isotropic case.[2)]

The PTR equations admit two stationary power-law solutions.[6] One of them ($\gamma=2$, $\alpha=-\frac{5}{2}$), found by Norman and ter Haar, corresponds to a constant flux of relativistic electron number along the spectrum.[6] The other ($\gamma=3$, $\alpha=-\frac{5}{2}$) found by Tsytovich and Chikhachev[5,6] corresponds, as shown below, to a constant energy flux along the spectrum both for electrons and for photons.[3)]

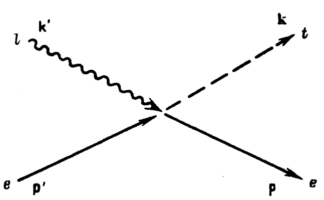

FIG. 1. The direct and inverse Compton-effect processes with the conversion of a plasmon l (or synchrotron mode) into a photon t, which determine the spectra of the relativistic electrons e and the photons in a PTR.

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In the present paper we introduce explicitly particle and energy fluxes along the spectrum and we find stationary power-law solutions of the PTR equations by requiring these fluxes to be constant. The electron distribution is normalized by the fluxes along the spectrum (Secs. 2 and 3). This fact entails important physical consequences which refer to the model as a whole. Indeed, it turns out that all solutions found in Refs. 5 and 6 refer to the "inertial" range which is positioned between the (wave and particle) sources and their sinks (the region where there is dissipation or where the particles and radiation leave freely), which are far from one another in energy space.

Under the same general assumptions which led to the two solutions it turns out that the particle flux is directed in the direction of larger energies (Sec. 2) while the total energy flux of electrons and photons is directed towards smaller energies (Sec. 3). The density of relativistic electrons turns out to be proportional to the magnitude of the flux along the spectrum and it is thus necessary for the existence of power-law solutions that there exist sources (or sinks) which produce fluxes along the spectrum and which are positioned with respect to the inertial range in accordance with the flux directions which are found. We note that the turbulent plasma by itself cannot produce such sources (see, however, Sec. 4) and neither can the magnetic field; both occur in the PTR equations in the form of some statistically uniform and stationary external field that allows the basic interaction process in the PTR, viz., the direct and the inverse Compton scattering of an electron with the conversion of a plasmon into a photon (Fig. 1), or the synchrotron process equivalent to it.

The extraneous character of the source manifests itself also in the fact that the power-law solutions of the PTR are, as is shown in Appendix I, in fact solutions of an inhomogeneous system which contains point sources (sinks).

2. DIRECTION OF THE PARTICLE FLUX

It is convenient to write the PTR Eqs. (1.3) and (1.4) in the form of continuity equations

$$\dot{n}_p+\frac{1}{\varepsilon^2}\frac{\partial}{\partial\varepsilon}J_0(\varepsilon)=0,\quad \dot{N}_k+\frac{1}{\omega^2}\frac{\partial}{\partial\omega}J_0(\omega)=0, \tag{2.1}$$

where the particle flux in energy space

$$J_0(\varepsilon)=-\int_0^\infty d\omega\, S(\omega,\varepsilon) \tag{2.2}$$

and the photon flux in frequency space

$$J_0(\omega)=-\int_0^\omega \frac{d\omega}{\omega}\int_0^\infty d\varepsilon\, S(\omega,\varepsilon) \tag{2.2'}$$

are expressed in terms of the quantity $S(\omega,\ \varepsilon)$ given by (1.2), where the remaining notation is also given. The existence of power-law solutions is connected with the symmetry of the system whereby the matrix element given here depends on ω/ε^2 only. For a power-law distribution for the waves,

$$N=A\omega^{-1/2}, \tag{2.3}$$

which makes the equation homogeneous and makes the cancellation of induced and spontaneous processes possible,[4] and for a power-law particle distribution,

$$n=a\varepsilon^s;\quad f(\varepsilon)\equiv n_p\frac{d^3p}{d\varepsilon}\sim\varepsilon^{-\gamma},\quad \gamma=-(s+2); \tag{2.4}$$

the fluxes take the form

$$J_0(\varepsilon)=-a\varepsilon^{s+4}[sAB(^1/_2)+B(0)],\quad B(\nu)\equiv\int_0^\infty dx\,x^\nu u(x)>0, \tag{2.5}$$

$$J_0(\omega)=-a\frac{\omega^{(3+s)/2}}{3+s}\left[sAB\left(-\frac{4+s}{2}\right)+B\left(-\frac{5+s}{2}\right)\right]. \tag{2.5'}$$

From this it is clear that $s=-4$ ($\gamma=2$) and

$$A=A_0\equiv B(-^1/_2)/4B(0) \tag{2.6}$$

correspond to a stationary solution of the set (2.1) with a constant, non-vanishing electron flux

$$J_0(\varepsilon)=J_0^\varepsilon\equiv a[B(^1/_2)B(-^1/_2)-B^2(0)]/B(0) \tag{2.6'}$$

and a vanishing photon flux $J_0(\omega)=0$. The constant terms in the definitions of the fluxes are chosen such that the fluxes vanish for equilibrium distributions (for details see Appendix I). At the same time the normalization constant in the photon distribution (2.3) guarantees the vanishing of the photon flux along the spectrum (i.e., the existence of a stationary solution of Eq. (2.4)) and is given unambiguously by (2.6); the electron distribution is according to (2.6′) parametrized by the particle flux J_0^ε.

We show that it follows from (2.6′) and $a>0$ that $J_0^\varepsilon>0$. To do that we consider the expression $B(\alpha)B(\beta)-B^2(\frac{1}{2}(\alpha+\beta))$, a particular case of which is the factor in the normalization condition (2.6′). Using the definition of the moment (2.5) and symmetrizing we get

$$B(\alpha)B(\beta)-B^2\left(\frac{\alpha+\beta}{2}\right)$$
$$=\frac{1}{2}\int dx\,dy\,u(x)u(y)(xy)^{(\alpha+\beta)/2}\left[\left(\frac{x}{y}\right)^{(\alpha-\beta)/4}-\left(\frac{y}{x}\right)^{(\alpha-\beta)/4}\right]^2\geqslant 0, \tag{2.7}$$

whence it follows that $J_0^\varepsilon>0$, i.e., the flux is in the direction of larger electron energies. Indeed,

$$B(\alpha)B(\beta)-B^2\left(\frac{\alpha+\beta}{2}\right)=\int dx\,dy\,u(x)u(y)[x^\alpha y^\beta-(xy)^{(\alpha+\beta)/2}]$$
$$=\frac{1}{2}\int dx\,dy\,u(x)u(y)[x^\alpha y^\beta+x^\beta y^\alpha-2(xy)^{(\alpha+\beta)/2}],$$

which leads to (2.7). For the existence of the stationary distribution

$$N_k=\frac{B(-^1/_2)}{4B(0)}\omega^{-1/2},\quad n_p=\frac{J_0^\varepsilon B(0)}{B(^1/_2)B(-^1/_2)-B^2(0)}\varepsilon^{-4},\quad \gamma=2, \tag{2.8}$$

we need thus an electron source in the region of low (relativistic) energies and a sink at high energies. The latter can easily be realized due to the fact that high-energy electrons leave the trap. We discuss a possible realization of a source in Sec. 4.

3. DIRECTION OF THE ENERGY FLUX

We now write the PTR equations in the form of continuity equations for the energy:

$$\varepsilon\dot{n}_p+\frac{1}{\varepsilon^2}\frac{\partial}{\partial\varepsilon}J_1(\varepsilon)=0,\quad \omega\dot{N}_k+\frac{1}{\omega^2}\frac{\partial}{\partial\omega}J_1(\omega)=0, \tag{3.1}$$

where we have introduced the energy fluxes along the spectrum

$$J_1(\varepsilon)=-\int^{\varepsilon} d\varepsilon\,\varepsilon\frac{\partial}{\partial\varepsilon}\int_0 d\omega\,S(\omega,\varepsilon),$$

$$J_1(\omega)=-\int^{\omega} d\omega\int_0^{\infty} d\varepsilon\,S(\omega,\varepsilon). \tag{3.2}$$

For the power-law distributions (2.3) and (2.4) these fluxes are equal to

$$J_1(\varepsilon)=-a\frac{s+4}{s+5}\varepsilon^{s+5}\left[sAB\left(\frac{1}{2}\right)+B(0)\right], \tag{3.3}$$

$$J_1(\omega)=-\frac{a}{s+5}\omega^{(s+5)/2}\left[sAB\left(-\frac{4+s}{2}\right)+B\left(-\frac{5+s}{2}\right)\right]. \tag{3.3'}$$

It is clear from (3.3) and (3.3′) that a solution with constant, non-vanishing energy fluxes $J_1(\varepsilon)\to J_1^e$ and $J_1(\omega)\to J_1^p$ is possible when

$$s=-5,\ A=A_1=B(0)/5B(1/2). \tag{3.4}$$

In that case the dependence of the fluxes on ε or ω vanishes, on the one hand, and, on the other hand, the numerators in (3.3) and (3.3′) vanish at the same time as the denominator $(s+5)$. Such behavior is quite typical of power-law distributions (first-order zero in the collision integral of the kinetic equation[11]) and has a simple physical cause: the necessary presence of a point source (sink) at small energies (frequencies) for solutions with fluxes along the spectrum (see Appendix I).

Resolving the indefiniteness in (3.3), (3.3′) we see easily that for the solution (3.4) the normalization constant of the electron distribution can be expressed in terms of the total flux along the spectrum $J_1\equiv J_1^e+J_1^p$:

$$\frac{J_1}{a}=\frac{B'(0)B(1/2)-B(0)B'(1/2)}{2B(1/2)},\quad B'(\nu)=\frac{dB(\nu)}{d\nu}. \tag{3.5}$$

Since $a>0$, J_1 has the same sign as the enumerator in (3.5). We show that the latter is always negative. To do this we consider the expression $B'(\xi)B(\xi+\alpha)-B(\xi)\times B'(\xi+\alpha)$. Using the definition (2.5) of $B(\nu)$ we get

$$B'(\xi)B(\xi+\alpha)-B(\xi)B'(\xi+\alpha)=\int dx\,dy\,u(x)u(y)\,[x^\xi y^{\xi+\alpha}\ln x-x^\xi y^{\xi+\alpha}\ln y]$$
$$=\frac{1}{2}\int dx\,dy\,u(x)u(y)\,(xy)^\xi(y^\alpha-x^\alpha)\ln\frac{x}{y}.$$

Hence it follows that

$$\operatorname{sgn}[B'(\xi)B(\xi+\alpha)-B(\xi)B'(\xi+\alpha)]=-\operatorname{sgn}\alpha. \tag{3.6}$$

In our case $\xi=0$, $\alpha=\frac{1}{2}$, i.e., $J_1<0$: the total energy flux along the spectrum is in the direction of lower energies,

$$N_k=\frac{B(0)}{5B(1/2)}\omega^{-5/2},\quad n_p=\frac{2B(1/2)\,|J_1^e+J_1^p|}{B(0)B'(1/2)-B'(0)B(1/2)}\varepsilon^{-5},\quad \gamma=3. \tag{3.7}$$

At the same time the partial fluxes J_1^e and J_1^p may have different signs, provided only that $J_1^e+J_1^p<0$.

4. THE PROBLEM OF THE SOURCES

The general energy sink must thus for the realization of the solution with $s=-5$ ($\gamma=3$) be at low energies (frequencies) and the source at high energies. Between them there is an "inertial range" with a power-law distribution $\gamma=3$ and in the region of energies which are larger than the energy of the source a power-law distribution with a particle flux ($\gamma=2$) may leave the system, see Fig. 2. It is immediately clear from (2.5) and (3.3) that the solution (2.8) with a constant flux of electrons corresponds to non-vanishing energy fluxes both for the electrons and for the photons. Similarly, for the solution (3.7) the electron and photon fluxes vanish while the energy fluxes along the spectrum are constant. The power-law distributions are thus single-flux distributions and for their realization it is completely necessary that there are (energy or particle) sources and sinks (which correspond to one another) which have a distance between them in energy space.

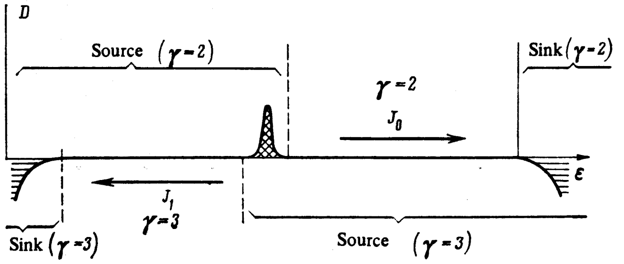

FIG. 2. Schematic picture of the inertial ranges and the regions where the source and the sink play a role in a PTR.

It seems to us that the problem of the sources which produce fluxes becomes the main physical problem in PTR theory which has not yet been discussed in the literature. Usually, it is naturally assumed that the energy source is plasma (Langmuir) turbulence. We note, however, that the plasma in the PTR theory does not appear as a dynamic system, but plays the role of some external field guaranteeing the possibility of bremsstrahlung. In the main order in ω_0/ω and mc^2/ε (see Appendix II) which corresponds to Eqs. (1.2) to (1.4) the total energy of the electron-photon system $\mathscr{E}$ can change only due to extraneous sources producing an energy flux. Indeed,

$$\dot{\mathscr{E}}=\int_0^\infty d\varepsilon\,\varepsilon^3\dot{n}_p+\int_0^\infty d\omega\,\omega^3\dot{N}_k,$$

or, using the kinetic Eqs. (1.3), (1.4)

$$\dot{\mathscr{E}}=\int_0^\infty d\varepsilon\,\varepsilon\frac{\partial}{\partial\varepsilon}\int_0^\infty d\omega\,S+\int_0^\infty d\omega\int_0^\infty d\varepsilon\,S. \tag{4.1}$$

For distributions which decrease rapidly enough at the origin and at infinity, for which we can in (4.1) change the order of integration over ε and ω, while the integrated term which arises when we integrate by parts vanishes, it follows from (4.1) that the total energy is conserved:

$$\dot{\mathscr{E}}=\varepsilon\int_0^\infty d\omega\,S\Big|_0^\infty-\int_0^\infty d\varepsilon\int_0^\infty d\omega\,S+\int_0^\infty d\omega\int_0^\infty d\varepsilon\,S=0.$$

However, the more general distributions which are of interest to us with non-vanishing fluxes along the spectrum do not possess these properties at all. For them there follows from (4.1) and the equations of continuity

(3.1) and (3.2) a general expression for the energy conservation law in a system which is open along the spectrum

$$\dot{\mathscr{E}}=-[J_1(\varepsilon)+J_1(\omega)]|_{\varepsilon,\omega=0}^{\infty} \quad (4.2)$$

—the change of the total energy is determined by the difference of the fluxes leaving and entering the system. Hence it follows at once that stationary power-law solutions of the PTR equations with constant particle fluxes (2.8) and energy fluxes (3.7) lead to $\dot{\mathscr{E}}=0$, by virtue of $J_1(0)=J_1(\infty)$. Similarly, the conservation of the total number of particles

$$\mathfrak{N}=\int d\mathbf{p}\, n_{\mathbf{p}}$$

requires that the particle flux leaving and entering "the spectrum" are equal,

$$\dot{\mathfrak{N}}=-J_0(\varepsilon)|_0^{\infty}. \quad (4.3)$$

Thus energy (and particles) flow into the PTR system of waves and particles from the turbulent plasma, and the fluxes are determined by extraneous sources.

The role of the turbulent plasma may, however, be important and must therefore be included explicitly in the system. As a source one might, for instance, invoke the collapse[12] of Langmuir waves (cf. Ref. 7) leading to statistically uniform acceleration of the particles in the plasma, or other acceleration mechanisms caused by plasma turbulence, while a source may be a mechanism of collisions with the basic plasma particles at low energies or the departure of particles from the reactor at high energies. The power of the source will then be determined by the power dissipated by the plasma turbulence. In turn, the plasma must be excited by an extraneous mechanism, for instance, a pulsar[3] or convection, and so on. However, this way of transferring energy from plasma turbulence to the PTR requires a separate detailed discussion and at the present moment constitutes a patent difficulty in the PTR theory, in particular for the apparently quite suitable (see Ref. 7) spectrum with $\gamma=3$ for which a source at high energies is necessary. At the same time it does not contradict present-day ideas about the structure of cosmic sources.[3,6]

The discussion of the analogy with Kolmogorov turbulence may, apparantly, be of wider interest than the particular PTR problem. This character manifests itself also in other astrophysical systems with power-law distributions[13] and, possibly, can lead to an answer to the question whether there are rather general reasons for the formation of such universal, essentially non-equilibrium distributions as are the power-law distributions in the universe.

The authors express their gratitude to S. Ya. Braude and S. A. Kaplan for stimulating discussions.

APPENDIX I

POINT SOURCE IN THE KINETIC EQUATIONS AND POWER-LAW SOLUTIONS

The stationary solutions (2.8), (3.7) lead to a δ-function-like singularity (point source (sink) of particles or of energy) on the right-hand side of the kinetic equation,[5)] i.e., they are solutions of an inhomogeneous equation, if we consider it not only inside the inertial range, but also along the whole axis. We use an electrostatic analogy[14] and write the kinetic equation (for the sake of simplicity we consider one equation) with a source D

$$\dot{N}_{\mathbf{p}}=I_{\text{coll}}\{N_{\mathbf{p}}\}+D \quad (\text{I}.1)$$

in the form of a continuity equation in **p**-space

$$\dot{N}_{\mathbf{p}}+\operatorname{div}\mathbf{E}(\mathbf{p})=4\pi\rho(\mathbf{p}), \quad (\text{I}.2)$$

where

$$\operatorname{div}\mathbf{E}(\mathbf{p})=-I_{\text{coll}}\{N_{\mathbf{p}}\},\quad 4\pi\rho(\mathbf{p})=D \quad (\text{I}.3)$$

(the explicit form of I_{coll} is here unimportant; it is only important that is conserves the particle number). The quantity $\mathbf{E}(\mathbf{p})$ has the meaning of a particle flux density in **p**-space, and $\rho(\mathbf{p})$ that of a particle source density. In the stationary case (I.2) takes the form of the Poisson equation

$$\operatorname{div}\mathbf{E}(\mathbf{p})=4\pi\rho(\mathbf{p}). \quad (\text{I}.4)$$

Since we can impose on the flux density an additional condition curl $\mathbf{E}=0$, the electrostatic analogy is clear. It is necessary for the existence of a stationary solution of Eq. (I.1) that there be consistency between the source and the sink, which because of particle conservation,

$$\int d\mathbf{p}\, I_{\text{coll}}=0$$

reduces to the equation

$$\int d\mathbf{p}\, D(\mathbf{p})=0.$$

This enables us to consider a singularity (localized source or sink) at the origin only. In the far zone (inertial range)

$$\rho(\mathbf{p})=e\delta(\mathbf{p}),\quad \mathbf{E}=\frac{e}{p^2}\frac{\mathbf{p}}{p}. \quad (\text{I}.5)$$

The flux J per unit solid angle through the surface of a sphere of radius p (or, what is the same, the flux in modulo-p or energy space) is by Gauss's theorem simply the charge, i.e., the power of the source

$$J=\frac{1}{4\pi}\int_{p=\text{const}} d\mathbf{s}\,\mathbf{E}=\frac{1}{4\pi}\int_{p'<p} d\mathbf{p}'\operatorname{div}\mathbf{E}(\mathbf{p}')=e. \quad (\text{I}.6)$$

We state now that we are looking for a solution of the kinetic equation in power-law form with an unknown power. The singularity at the origin is then a power-law function (we use here the fact that the collision integral is homogeneous)

$$\operatorname{div}\mathbf{E}=ef(n)/p^n \quad (\text{I}.7)$$

and the flux takes the form

$$J = ef(n)\int \frac{p^2\,dp}{p^n} = \frac{ef(n)}{3-n}\,p^{3-n}. \qquad (I.8)$$

Constancy of the flux (I. 8) corresponds to $n = 3$. However, there then arises in (I. 8) a pole because the denominator vanishes. In order that J be finite the function $f(n)$ must have the form $f(n) = (3-n)\varphi(n)$, where $\varphi(3) = 1$ in accordance with (I. 6). The singularity described by a δ-function arises thus as the limit of a homogeneous power-law expression

$$\delta(\mathbf{p}) = \lim_{n\to 3-0} \frac{3-n}{4\pi p^n}. \qquad (I.9)$$

The appearance of the indeterminate form 0 : 0 in the flux (I. 8) and the first-order zero in I_{coll} (cf. Ref. 11) in the power-law representation of the solution are, according to (I. 3) and (I. 7), connected just with the point singularity (I. 9) at the origin.[14] The expansion of the indeterminate factor in (I. 8) leads to a relation between the flux and the derivative with respect to the index of the collision integral which was obtained by Karas'[11]

$$\left.\frac{\partial I_{\text{coll}}}{\partial n}\right|_{n=3} = \frac{J}{p^3},$$

which is in a number of cases convenient both to determine the exponents and to normalize the spectra. We note that the limiting transition is accomplished for the class of functions which is integrable at the origin ($n \leqslant 3$).

The whole discussion differs for the energy flux by the presence of a factor $\omega_{\mathbf{k}}$ which is included both in I_{coll} and in the source density. The solution with a constant particle flux is thus a stationary solution of the inhomogeneous PTR equations with a point source of particles

$$\dot{n}_{\mathbf{p}} = \frac{1}{\varepsilon^2}\frac{\partial}{\partial\varepsilon}\int_0^\infty d\omega\, S + 4\pi J_0^e\delta(\mathbf{p}), \qquad \dot{N}_{\mathbf{k}} = \frac{1}{\omega^3}\int_0^\infty d\varepsilon\, S, \qquad (I.10)$$

while the solution with a constant energy flux is a stationary solution of the equations

$$\varepsilon\dot{n}_{\mathbf{p}} = \frac{1}{\varepsilon}\frac{\partial}{\partial\varepsilon}\int_0^\infty d\omega\, S + 4\pi J_1^e\delta(\mathbf{p}),$$

$$\omega\dot{N}_{\mathbf{k}} = \frac{1}{\omega^2}\int_0^\infty d\varepsilon\, S + 4\pi J_1^p\delta(\mathbf{k}). \qquad (I.11)$$

The fluxes J_1^e and J_1^p do not directly determine the parameters of the distribution, but are connected with the quantity

$$\left.\frac{d}{ds}(sA)\right|_{s=s_1}.$$

$$\frac{J_1^e}{a} = B\left(\frac{1}{2}\right)\left.\frac{d}{ds}(sA)\right|_{s=s_1}. \qquad (I.12)$$

$$\frac{J_1^p}{a} = -B\left(\frac{1}{2}\right)\left.\frac{d}{ds}(sA)\right|_{s=s_1} + \frac{1}{2}\left[B'(0) - \frac{B(0)B'(^1/_2)}{B(^1/_2)}\right],$$

whence follows (3. 5) and the condition $J_1^e + J_1^p < 0$.

We note that the choice of the integration limit in Eqs. (3. 2) for the energy fluxes is determined by the nature of the limiting transition in (I. 9) and corresponds to such a choice of constant in the definition of the flux that it equals the power of the source. Correspondingly, the lower limits in (3. 2) must be zero ($s + 5 \to +0$) and the limit in (2. 2′) be equal to infinity which then leads to Eqs. (3. 3) and (3. 3′) for the energy fluxes ($s \geqslant -5$) and (2. 5) and (2. 5′) for the particle fluxes ($s < -3$).

APPENDIX II

DERIVATION OF THE PTR EQUATIONS BY MEANS OF THE QUANTAL KINETIC EQUATION

We consider for the sake of argument the processes of the direct and inverse Compton scattering of electrons with the conversion of a plasmon into a photon (Fig. 1). The quantum analogy enables us to write down immediately the kinetic equation for the photon distribution in the form

$$\dot{N}_{\mathbf{k}}{}^t = \int d\mathbf{k}'\, d\mathbf{p}\, d\mathbf{p}'\, W^{lt}_{\mathbf{k}'\mathbf{p}';\mathbf{k}\mathbf{p}} f^{lt}_{\mathbf{k}'\mathbf{p}';\mathbf{k}\mathbf{p}}, \qquad (II.1)$$

where the probability for the process is

$$W^{lt} = U^{lt}\delta(\omega_{\mathbf{k}}{}^t + \varepsilon_{\mathbf{p}} - \omega_{\mathbf{k}'}{}^l - \varepsilon_{\mathbf{p}'})\,\delta(\mathbf{k} + \mathbf{p} - \mathbf{k}' - \mathbf{p}'), \qquad (II.2)$$

and f^{lt} is a combination of the distribution functions which for $n_p \ll 1$ and $N^l_{\mathbf{k}'} \gg 1$ ($N^l_{\mathbf{k}'}$ is the plasmon spectral density) has the form

$$f^{lt}_{\mathbf{k}'\mathbf{p}'|\mathbf{k}\mathbf{p}} = N_{\mathbf{k}'}{}^l[n_{\mathbf{p}'} + N_{\mathbf{k}}{}^t(n_{\mathbf{p}'} - n_{\mathbf{p}})]. \qquad (II.3)$$

Using the fact that k', $k \ll p'$, p and introducing $\Delta\mathbf{k} \equiv \mathbf{k} - \mathbf{k}'$ we integrate with respect to $\mathbf{p}'$ and expand the quantities U and n in terms of $\Delta\mathbf{k}$:

$$\dot{N}_{\mathbf{k}}{}^t = \int d\mathbf{k}' N_{\mathbf{k}'}{}^l \int d\mathbf{p}\, v^{lt}_{\mathbf{k}'\mathbf{p}|\mathbf{k}\mathbf{p}}\left[n_{\mathbf{p}} + N_{\mathbf{k}}{}^t\Delta\mathbf{k}\frac{\partial n}{\partial\mathbf{p}}\right], \qquad (II.4)$$

where

$$v^{lt}_{\mathbf{k}'\mathbf{p}'|\mathbf{k}\mathbf{p}} = U^{lt}_{\mathbf{k}'\mathbf{p}'|\mathbf{k}\mathbf{p}}\delta(\omega_{\mathbf{k}}{}^t - \omega^l_{\mathbf{k}'} - \mathbf{v}\Delta\mathbf{k}). \qquad (II.5)$$

The kinetic equation for $n_{\mathbf{p}}$ takes into account the cancellation of the process corresponding to Fig. 1 and the one obtained from it by the substitution $\mathbf{p} \rightleftarrows \mathbf{p}'$ and hence, contains two terms:

$$\dot{n}_{\mathbf{p}} = \int d\mathbf{k}\, d\mathbf{k}'\, d\mathbf{p}'\,[W^{lt}_{\mathbf{k}'\mathbf{p}'|\mathbf{k}\mathbf{p}} f^{lt}_{\mathbf{k}'\mathbf{p}'|\mathbf{k}\mathbf{p}} + W^{tl}_{\mathbf{k}\mathbf{p}'|\mathbf{k}'\mathbf{p}} f^{tl}_{\mathbf{k}\mathbf{p}'|\mathbf{k}'\mathbf{p}}], \qquad (II.6)$$

where

$$f^{tl}_{\mathbf{k}\mathbf{p}'|\mathbf{k}'\mathbf{p}} = N_{\mathbf{k}'}{}^l[-n_{\mathbf{p}} + N_{\mathbf{k}}{}^t(n_{\mathbf{p}'} - n_{\mathbf{p}})]. \qquad (II.7)$$

Because of the principle of detailed balancing $W^{tl}_{\mathbf{k}\mathbf{p}'|\mathbf{k}'\mathbf{p}} = W^{lt}_{\mathbf{k}'\mathbf{p}|\mathbf{k}\mathbf{p}'}$ and the main term in the expansion in $\Delta\mathbf{k}$ vanishes. After integration over $\mathbf{p}'$ the equation has the form

$$\dot{n}_{\mathbf{p}} = \int d\mathbf{k}' N^l_{\mathbf{k}'} \int d\mathbf{k}[\Phi(\mathbf{p} + \Delta\mathbf{k}, \mathbf{p}) - \Phi(\mathbf{p}, \mathbf{p} - \Delta\mathbf{k})],$$

where

$$\Phi(\mathbf{p}', \mathbf{p}) = v^{lt}_{\mathbf{k}'\mathbf{p}'|\mathbf{k}\mathbf{p}}[n_{\mathbf{p}'} + N_{\mathbf{k}}{}^t(n_{\mathbf{p}'} - n_{\mathbf{p}})].$$

Expanding in $\Delta\mathbf{k}$ we get

$$\dot{n}_{\mathbf{p}}=\frac{\partial}{\partial\mathbf{p}}\int d\mathbf{k}'\,N_{\mathbf{k}'}^{l}\int d\mathbf{k}\,\Delta\mathbf{k}\,v^{lt}_{\mathbf{k}'\mathbf{p}|\mathbf{k}\mathbf{p}}\left[\Delta\mathbf{k}\frac{\partial n}{\partial\mathbf{p}}N_{\mathbf{k}}^{t}+n_{\mathbf{p}}\right]. \qquad (\text{II}.8)$$

Equations (II.4) and (II.8) form the set which describes an anisotropic PTR. An additional symmetry arises for isotropic distributions in the physically interesting case $\omega^l \ll \omega^t$. As before it is then necessary to retain ω^l in the matrix element and the conservation law (II.5), leading to (I.1), but we can omit it in the operators

$$\Delta\mathbf{k}\frac{\partial}{\partial\mathbf{p}}\approx\omega^{t}\frac{\partial}{\partial\varepsilon},\qquad \frac{\partial}{\partial\mathbf{p}}\Delta\mathbf{k}\ldots\approx\frac{v}{p^2}\frac{\partial}{\partial\varepsilon}\frac{p^2}{v}\omega^{t}\ldots$$

Thanks to this the same matrix element, averaged over the angles occurs in both kinetic equations:

$$\bar{v}^{lt}=\int d\Omega_{\mathbf{k}'}d\Omega_{\mathbf{k}}v^{lt}_{\mathbf{k}'\mathbf{p},\mathbf{k}\mathbf{p}}=\int d\Omega_{\mathbf{k}'}d\Omega_{\mathbf{p}}v^{lt}_{\mathbf{k}'\mathbf{p},\mathbf{k}\mathbf{p}}, \qquad (\text{II}.9)$$

and we have split off a factor

$$\frac{W^l}{\omega_0}=\int d\mathbf{k}'\,N_{\mathbf{k}'}^{l},$$

where $(2\pi)^3W^l$ is the total energy of the plasma turbulence (or of the random magnetic field in a synchrotron reactor). As a result the set of PTR equations takes the symmetric form:

$$\omega g(\omega)\dot{N}=\int_0^\infty d\varepsilon\,S,\qquad g(\varepsilon)\dot{n}=\frac{\partial}{\partial\varepsilon}\int_0^\infty d\omega\,S, \qquad (\text{II}.10)$$

$$S=w\left[\omega N\frac{\partial n}{\partial\varepsilon}+n\right],\qquad w=g(\varepsilon)g(\omega)\omega\frac{W^l}{4\pi\omega_0}\bar{v}^{lt},$$

$$g(\varepsilon)=p^2/v\approx\varepsilon^2/c^3,\qquad g(\omega)=k^2/c=\omega^2/c^3.$$

Finally, the representation of w in the form

$$w=\frac{mc^2}{\omega_0}\frac{\varepsilon^2}{c^3}u\left[\frac{\omega}{\omega_0}\Big/\left(\frac{\varepsilon}{mc^2}\right)^2\right] \qquad (\text{II}.11)$$

(which changes (II.10) to the form (1.2) to (1.4)[6] after the change to the variables ω/ω_0 and ε/mc^2 and to the functions $N(\omega)\rightarrow\lambda N(\omega/\omega_0)$, $n(\varepsilon)\rightarrow\lambda^{-3}n(\varepsilon/mc^2)$, $\lambda=mc^2/\omega_0$) is determined by the relativistic invariance properties.[15,6]

1)Such a system, which has received the name turbulent reactor or plasma turbulent reactor (PTR) has been considered in detail by Kaplan and Tsytovich[5] and by Norman and ter Haar.[6] The equations describing it in the form suggested in Ref. 6 and which explicitly use the symmetry of the system will be written down below. We note that the turbulence occurring in the name PTR refers to a plasma and not to the radiation or the electrons and does not have a direct relation to the analogy with Kolmogorov turbulence which is discussed below.

2)See Ref. 7 for a detailed reference to the literature on PTR. We note that the basic PTR equations in this review (Ref. 7) are given with an unfortunate mistake (the factor ε^2 in (1.2) is omitted).

3)In Ref. 6 the constancy of the electron energy flux along the spectrum was assumed, but an attempt to prove it was based upon an integral conservation law which was inadequate for the problem (cf. Sec. 4). We note that under the PTR conditions α and γ are not yet connected through the relation $2\alpha=\gamma-1$.

4)The functional simplicity of the stationary Kolmogorov (power-law) spectra are in an essential way connected with the symmetry of the system. In hydrodynamic and plasma systems this is the absence of distinguished scales (self-similarity) leading to the homogeneity of the equations. The possibility to study such systems analytically was first noted by Zakharov[8] and is connected with this homogeneity which enables us to find solutions of the kinetic equations of weak turbulence (see Refs. 9, 10). The symmetry of the PTR equations manifests itself in the ω/ε^2 -dependence of u. The spectrum (2.3) when the electron distribution is (2.4) leads to the fact that $S(\omega,\ \varepsilon)$ also depends on ω only in the combination ω/ε^2 and this leads to the homogeneity of the PTR equations required for the existence of power-law solutions, as first noted by Norman and ter Haar.

5)Of course, the source is a point source from the point of view of the "far zone"—the inertial range in which the solution with arbitrary sources goes over into its (power-law) asymptotic behavior.

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Translated by D. ter Haar