All these facts indicate once more that the smectic phase A can be in two stable textures: homotropic, which practically corresponds to the liquid-single-crystal state of the sample, and domain texture, corresponding to liquid-polycrystalline state. In this case tension destroys the liquid single crystal state and converts the layer into a liquid-polycrystalline state, while the electric field orients the molecules along the field and makes the sample a liquid single crystal.

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Translated by J. G. Adashko

Contribution to the theory of high-frequency properties of ferromagnets

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(Submitted 24 May 1977)
Zh. Eksp. Teor. Fiz. 73, 1911–1924 (November 1977)

We develop a unique method, based upon a special representation for the spin operators, to calculate the high-frequency, kinetic, and thermodynamic properties of magnetic substances. We use this method to calculate the high-frequency susceptibility, the spectrum and the damping the spin waves, the thermodynamic potential, and the magnetization of a Heisenberg ferromagnet. We show that the components $\chi_{xx}(\mathbf{k},\omega)$ and $\chi_{yy}\mathbf{k},\omega)$ in this representation reduce simply to single-particle Green functions of the quasi-particles.

PACS numbers: 75.10.Jm, 75.30.Cr, 75.30.Ds

1. INTRODUCTION

A large number of papers has been devoted to calculating the high-frequency properties of ferromagnets. The theory of these effects is mainly based upon the following approaches: a phenomenological one, going back to Landau and Lifshitz's well known paper[1]; the kinetic equation method, developed by Akhiezer[2] on the basis of the Holstein-Primakoff (HP) representation[3] for spin operators; the Green function method, using the HP representation of the Dyson-Maleev (DM) representation[4]; finally, the Green function method using the Bogolyubov-Tyablikov,[5] the Vaks-Larkin-Pikin,[6] or the Izyumov-Kasan-Ogly[7] spin operators.

By now we have reached an understanding of the basic regularities connected with the high-frequency properties of magnetic substances such as resonance effects, parametric excitation of spin waves, and also thermodynamic properties at low temperatures. There are, however, a number of problems which are important not only for constructing a theory of magnetically ordered crystals, but also for a correct understanding of experiments. Among these there is, first of all, the problem of a consistent evaluation of the components of the high-frequency (hf) magnetic susceptibility tensor and the problem, closely connected with it, of determining the spectra and interaction of quasi-particles. Traditionally this problem is solved by changing from spin operators to Bose operators using the HP representation. The dynamical interaction of the spins is then replaced by the interaction of the Bose particles. The HP Hamiltonian can serve as a basis for a consistent calculation of the thermodynamic properties of a magnetic substance. As far as the interaction of the spin system with an external field is concerned, the situation here is appreciably more complicated.

An external field acts directly on the spin variables which are in an essentially non-linear way connected with the HP spin wave operators. We shall show that the calculation of the hf susceptibility tensor $\chi(\mathbf{k},\omega)$ cannot be reduced to a calculation of the single-particle Green functions of the HP spin waves.

Although the Green function method for spin operator[6,7] is consistent, its use for describing the properties of a magnetic substance at low temperatures is hardly justified due to its cumbersomeness. In that temperature region it is more convenient to use Bose operators. One must then give up the preference for the representation in which one of the spin operator components is linearly connected with the Bose operators. This is just the situation in the case of the DM representation and the representation suggested in Ref. 8. In Ref. 9 the component $\chi_{+-}(\mathbf{k},\omega)$ of the hf susceptibility tensor was calculated on the basis of the Dyson representation. [4]

We develop in the present paper a unique method for calculating high-frequency, kinetic, and thermodynamic properties of magnetic substances based upon the representation of Ref. 8. We use this method to calculate for a Heisenberg ferromagnet the hf susceptibility tensor, the spectrum and damping of the spin waves, the thermodynamic potential, and the magnetization. We show that the components $\chi_{xx}(\mathbf{k}, \omega)$ and $\chi_{yy}(\mathbf{k}, \omega)$ of the hf susceptibility tensor reduce in this representation merely to the single-particle Green functions of the quasi-particles for which one can apply well known calculation methods. [10]

The expression found in the present paper for the components of the hf susceptibility tensor $\chi(\mathbf{k}, \omega)$ has the correct asymptotic properties both in the region of small wavevectors (ω arbitrary) and in the region of small ω($\mathbf{k}$ arbitrary). In the low-temperature region we construct a consistent perturbation theory in the parameter $1/S$ (S is the atomic spin) and we find up to order S^{-2} the renormalization of the spin wave spectra and their damping. As in the HP method it is necessary for consistent calculations in powers of the expansion in $1/S$ to take into account processes which involve not only four, but also six spin waves. The same results have also been obtained by us by employing the HP and DM methods in the standard way and we show their equivalence. We show that with an accuracy up to the same processes the thermodynamic potential is the same as the results obtained earlier by Dyson[4] and Oguchi. [11]

2. SPIN WAVE HAMILTONIAN IN A FERROMAGNET

We consider an isotropic ferromagnet in an external magnetic field H which is described by the Heisenberg Hamiltonian

$$\mathscr{H}=-\frac{1}{2}\sum_{f\neq f'} I_{ff'}\mathbf{S}_f\mathbf{S}_{f'}-g\mu_0 H\sum_f S_f^z, \tag{1}$$

where $I_{ff'}$ is an exchange integral, μ_0 the Bohr magneton, g the Landé factor, while the spin operators $\mathbf{S}_f$ satisfy the well known commutation relations. For a study of the high-frequency and thermodynamic properties of a ferromagnet it is convenient to change from the spin operators to Bose operators. Usually this change is accomplished by using the HP representation. We shall use the representation[8]

$$\tilde{S}_f^+=S_f^z+iS_f^x=[(S+S_f^y)(S-S_f^y+1)]^{1/2}\exp\left(i\frac{a_f^++a_f}{(2S)^{1/2}}\right),$$
$$\tilde{S}_f^-=(\tilde{S}_f^+)^+,\quad S_f^y=i(S/2)^{1/2}(a_f^+-a_f), \tag{2}$$

where S is the magnitude of the atomic spin. Although this representation is more complicated than the HP representation, it has a number of advantages. First of all, it provides a linear connection between the S_f^y component of the spin operator and the Bose operators. The component $\chi_{yy}(\mathbf{k}, \omega)$ of the hf susceptibility tensor therefore reduces in this representation to single-particle magnon Green functions. We recall that

$$\chi_{ij}(\mathbf{k},\omega)=-\frac{(g\mu_0)^2}{v_0}G_{ij}^{(r)}(\mathbf{k},\omega), \tag{3}$$

where $G_{ij}^{(r)}(\mathbf{k}, \omega)$ is the Fourier transform of the equal-time retarded Green function

$$G_{ij}^{(r)}(\mathbf{R}_f-\mathbf{R}_{f'},t)=-i\theta(t)\langle[S_f^i(t),S_{f'}^j(0)]\rangle, \tag{4}$$

$\mathbf{S}_f(t)$ is the spin operator of the f-th atom in the Heisenberg representation, $i, j=(x, y, z)$; v_0 is the volume of the unit cell.

The second advantage of the representation (2) is connected with the fact that it enables us explicitly to take into account the symmetry of the Hamiltonian. [12] When the system of coordinates is rotated through an angle φ around the y-axis, which is at right angles to the quantization axis, the operators $a_{\mathbf{k}}^+$ and $a_{\mathbf{k}}$ transform as follows: $a_{\mathbf{k}}^+\to a_{\mathbf{k}}+\varphi(1/2SN)^{1/2}\Delta(\mathbf{k})$, $a_{\mathbf{k}}\to a_{\mathbf{k}}+\varphi(1/2SN)^{1/2}\Delta(\mathbf{k})$ (N is the number of atoms). As the original Hamiltonian (1) for $H=0$ is invariant under rotations of the system of coordinates (i.e., should be independent of φ) we get a number of relations (see Eqs. (12)) for the spin wave spectrum and the amplitudes of the magnon interaction. These relations guarantee the correct behavior of the quasi-particle energy at small $\mathbf{k}$ to any order of perturbation theory. On the other hand, they enable us to control the correctness of the calculations in each stage.

To find the spin wave Hamiltonian we expand $\tilde{S}_f^+$ and $\tilde{S}_f^-$ in the representation (2) in terms of S^{-1} up to terms in S^{-2}:

$$\begin{aligned}\tilde{S}^+=S\Bigg\{&\left(1-\frac{1}{24S^3}\right)+i\frac{a^++a}{(2S)^{1/2}}-\left(1+\frac{1}{4S^2}\right)\frac{a^+a}{S}+\frac{1}{8S^3}(a^{+2}+a^2)\\&+i\frac{a^{+3}+a^3}{6S(2S)^{1/2}}-i\frac{a^{+2}a+a^+a^2}{2S(2S)^{1/2}}-\left(\frac{1}{3}+\frac{1}{4S}\right)\frac{a^{+4}+a^4}{4S^2}\\&+\left(\frac{1}{3}+\frac{1}{2S}\right)\frac{a^{+3}a+a^+a^3}{2S^2}-\frac{3}{8}\frac{a^{+2}a^2}{S^3}-i\frac{a^{+5}+a^5}{20S^2(2S)^{1/2}}+i\frac{a^{+4}a+a^+a^4}{12S^2(2S)^{1/2}}\\&+\frac{13}{6!}\frac{a^{+6}+a^6}{S^3}-\frac{7}{5!}\frac{a^{+5}a+a^+a^5}{S^3}+\frac{5}{48}\frac{a^{+4}a^2+a^{+2}a^4}{S^3}-\frac{5}{36}\frac{a^{+3}a^3}{S^3}\Bigg\}.\end{aligned} \tag{5}$$

Substituting (2) into the Hamiltonian (1) and using the expansion (5) we get[1)]

$$\mathscr{H}=\bar{E}_0+\mathscr{H}_0+V, \tag{6}$$

where

$$\bar{E}_0=-S\left(\frac{1}{2}J_0+\omega_0\right)N+\frac{1}{24S^2}(J_0+\omega_0)N; \tag{7}$$

$$\mathscr{H}_0=\sum_{\mathbf{k}}\varepsilon_{\mathbf{k}}a_{\mathbf{k}}^+a_{\mathbf{k}},\quad \varepsilon_{\mathbf{k}}=J_0-J_{\mathbf{k}}+\omega_0; \tag{8}$$

$$\begin{aligned}&V=V_2+V_4+V_6,\\&V_2=\frac{1}{4S^2}(J_0+\omega_0)\sum_{\mathbf{k}}a_{\mathbf{k}}^+a_{\mathbf{k}}-\frac{1}{8S^2}(J_0+\omega_0)\sum_{\mathbf{k}}(a_{\mathbf{k}}^+a_{-\mathbf{k}}^++a_{\mathbf{k}}a_{-\mathbf{k}});\end{aligned} \tag{9}$$

$$\begin{aligned}V_4=\sum_{1234}\{&\psi(12;34)a_1^+a_2^+a_3a_4\Delta(1+2-3-4)\\&+\psi(123;4)(a_1^+a_2^+a_3^+a_4+\text{H. c.})\Delta(1+2+3-4)\\&+\psi(1234)(a_1^+a_2^+a_3^+a_4^++\text{H. c.})\Delta(1+2+3+4)\};\end{aligned} \tag{10}$$

$$\begin{aligned}V_6=\sum_{\substack{123\\456}}\{&\psi(123;456)a_1^+a_2^+a_3^+a_4a_5a_6\Delta(1+2+3-4-5-6)\\&+\psi(1234;56)(a_1^+a_2^+a_3^+a_4^+a_5a_6+\text{H. c.})\Delta(1+2+3+4-5-6)\\&+\psi(12345;6)(a_1^+a_2^+a_3^+a_4^+a_5^+a_6+\text{H. c.})\Delta(1+2+3+4+5-6)\\&+\psi(123456)(a_1^+a_2^+a_3^+a_4^+a_5^+a_6^++\text{H. c.})\Delta(1+2+3+4+5+6)\}.\end{aligned} \tag{11}$$

Here

$$\psi(1234;56) = -\frac{1}{48N^2S^2}\{5(J_0+\omega_0)+J_5+J_6-J_{1-5}-J_{1-6}-J_{2-5}-J_{2-6}-J_{3-5}-J_{3-6}+J_{4-5}-J_{4-6}-{}^1/_2(J_{1+2+3}+J_{1+2+4}+J_{1+3+4}+J_{2+3+4})+{}^1/_4(J_{1+2-5}+J_{1+2-6}+J_{1+3-5}+J_{1+3-6}+J_{1+4-5}+J_{1+4-6}+J_{2+3-5}+J_{2+3-6}+J_{2+4-5}+J_{2+4-6}+J_{3+4-5}+J_{3+4-6}\},$$

$$\psi(1234) = -\frac{1}{48NS}\left\{J_1+J_2+J_3+J_4-4(J_0+\omega_0)-\frac{3}{S}(J_0+\omega_0)\right\},$$

$$\psi(12;34) = -\frac{1}{8NS}\left\{J_{1-3}+J_{1-4}+J_{2-3}+J_{2-4}-(J_1+J_2+J_3+J_4)-\frac{3}{S}(J_0+\omega_0)\right\},$$

$$\psi(123;4) = -\frac{1}{12NS}\left\{2(J_0+\omega_0)-J_1-J_2-J_3+J_4+\frac{3}{S}(J_0+\omega_0)\right\}.$$

$$\psi(123;456) = -\frac{1}{72N^2S^2}\{J_{1+2-4}+J_{1+2-5}+J_{1+2-6}+J_{1+3-4}+J_{1+3-5}+J_{1+3-6}+J_{2+3-4}+J_{2+3-5}+J_{2+3-6}+J_{1+2+3}-10(J_0+\omega_0)\}.$$

We do not need the amplitudes $\psi(12345;6)$ and $\psi(123456)$ and we have not written them out. Here J_k is the Fourier transform of the exchange integral $J_{ff'}$; $\omega_0 = g\mu_0 H$, $1 \equiv \mathbf{k}_1$, $2 \equiv \mathbf{k}_2$, etc.

The interaction V_6 gives to first order in perturbation theory the same contribution as the interaction V_4 in second order, if we classify the smallness of the corrections in terms of the parameter S^{-1}. The occurrence of higher-order terms in S^{-1} in V_2 and V_4 is caused by bringing the Hamiltonian (6) to the normal form. One verifies easily that the interaction amplitudes V_4 and V_6 for $H = 0$ satisfy the symmetry relations[12]:

$$\begin{gathered}4\psi(1230)+\psi(123;0)=0,\quad 3\psi(023;4)+2\psi(23;40)=0,\\ 6\psi(123450)+\psi(12345;0)=0,\quad 5\psi(02345;6)+2\psi(2345;60)=0,\\ 4\psi(0234;56)+3\psi(234;560)=0,\end{gathered} \qquad (12)$$

guaranteeing the invariance of the Hamiltonian under rotations. One can obtain similar relations also for the case of a magnetic substance with an arbitrary number of sublattices with a ground state which can be either collinear or non-collinear. These relations guarantee the correct behavior of the spectra and the damping coefficients of quasi-particles in the small wavevector region. We shall consider this problem in detail elsewhere.

We write part of the quadratic Hamiltonian in the form of the interaction (9) because we want to retain the parameter S^{-1} as the expansion parameter. Although the interaction (9) leads to new kinds of diagrams, all diagrams can in this case be classified in terms of the parameter S^{-1}.

One sees easily that the Hamiltonian (6) does not conserve the number of quasi-particles and the "simple" vacuum is not an eigenvector of the Hamiltonian (6). Using standard perturbation theory we find the vacuum wavefunction Φ_0 up to and including terms of order S^{-2}:

$$\Phi_0 = \left(1-\frac{1}{8\cdot 4!\,S^2}\right)\Big|0\Big\rangle + \frac{1}{48S^2N^{1/2}}\sum_{\mathbf{k}} a_{\mathbf{k}}^+ a_{-\mathbf{k}}^+\Big|0\Big\rangle - \frac{1}{48SN^{3/2}}\left(1+\frac{3}{4S}\right)\sum_{1234} a_1^+a_2^+a_3^+a_4^+|0\rangle\,\Delta(1+2+3+4)$$
$$+\frac{1}{2\cdot 6!\,S^2N^{5/2}}\sum_{\substack{123\\456}} a_1^+a_2^+\ldots a_6^+|0\rangle\,\Delta(1+2+3+4+5+6)$$
$$+\frac{35}{4\cdot 8!\,S^2N_s^{7/2}}\sum_{12,\ldots,8} a_1^+a_2^+\ldots a_8^+|0\rangle\,\Delta(1+2+3+4)\,\Delta(5+6+7+8). \qquad (13)$$

The energy corresponding to this wavefunction equals

$$E_0 = -{}^1/_2 SJ_0N - g\mu_0 HSN, \qquad (14)$$

which is the same as the exact ground state energy of the ferromagnet. One also finds easily the average spin along the quantization axis

$$\langle\Phi_0|S_f^z|\Phi_0\rangle = S \qquad (15)$$

up to and including terms of order S^{-2}. We note finally that

$$\langle\Phi_0|S_{\mathbf{k}}^+S_{-\mathbf{k}}^+|\Phi_0\rangle = 0 \qquad (16)$$

with the same accuracy. Here $S_{\mathbf{k}}^+ = S_{\mathbf{k}}^x + iS_{\mathbf{k}}^y$.

Therefore, Φ_0 gives all those results which are obtained when we consider the Heisenberg ferromagnet exactly.

3. EQUATIONS FOR THE GREEN FUNCTIONS

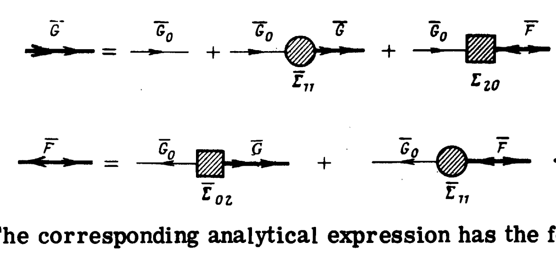
We get in the representation (2) for the $\chi_{yy}(\mathbf{k},\omega)$ component of the hf susceptibility tensor the expression

$$\chi_{yy}(\mathbf{k},\omega) = -\frac{(g\mu_0)^2}{v_0}\frac{S}{2}\langle\langle a_{\mathbf{k}}^+ - a_{-\mathbf{k}}|a_{-\mathbf{k}}^+ - a_{\mathbf{k}}\rangle\rangle_\omega^{(r)}, \qquad (17)$$

and the problem thus reduces to calculating the Green functions.

$$\langle\langle a_{\mathbf{k}}|a_{\mathbf{k}}^+\rangle\rangle \equiv G^{(r)}(\mathbf{k},\omega),\quad \langle\langle a_{\mathbf{k}}^+|a_{-\mathbf{k}}^+\rangle\rangle \equiv F_{20}^{(r)}(\mathbf{k},\omega),$$
$$\langle\langle a_{\mathbf{k}}^+|a_{\mathbf{k}}\rangle\rangle \equiv G^{(r)}(-\mathbf{k},-\omega),\quad \langle\langle a_{\mathbf{k}}|a_{-\mathbf{k}}\rangle\rangle \equiv F_{02}^{(r)}(\mathbf{k},\omega).$$

We shall, as usual, use the analytical continuation of the temperature-dependent Green functions $\bar{G}(\mathbf{k}, i\omega_n)$ and $\bar{F}(\mathbf{k}, i\omega_n)$ to evaluate the retarded Green functions.[10] For the normal, $\bar{G}(\mathbf{k}, i\omega_n)$, and the anomalous, $\bar{F}(\mathbf{k}, i\omega_n)$, Green functions we can get a set of equations which is analogous to the set of equations arising in the theory of a Bose gas.[13] Diagrammatically this set of equations looks like:



(18)

The corresponding analytical expression has the form

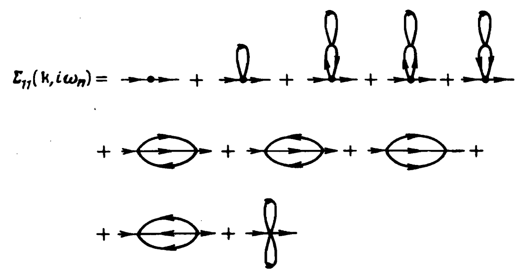
$$[\bar{G}_0^{-1}(\mathbf{k},i\omega_n)-\bar{\Sigma}_{11}(\mathbf{k},i\omega_n)]\bar{G}(\mathbf{k},i\omega_n)-\bar{\Sigma}_{20}(\mathbf{k},i\omega_n)\bar{F}_{20}(\mathbf{k},i\omega_n)=1,$$
$$\bar{\Sigma}_{02}(\mathbf{k},i\omega_n)\bar{G}(\mathbf{k},i\omega_n)+[-\bar{G}_0^{-1}(-\mathbf{k},-i\omega_n)+\bar{\Sigma}_{11}(-\mathbf{k},-i\omega_n)]\bar{F}_{20}(\mathbf{k},i\omega_n)=0. \qquad (18a)$$

Bearing in mind that $\bar{G}_0^{-1}(\mathbf{k}, i\omega_n) = i\omega_n - \varepsilon_k$ we get

$$\bar{G}(\mathbf{k},i\omega_n) = \frac{i\omega_n+\varepsilon_{\mathbf{k}}+\bar{\Sigma}_{11}(-\mathbf{k},-i\omega_n)}{[i\omega_n-\varepsilon_{\mathbf{k}}-\bar{\Sigma}_{11}(\mathbf{k},i\omega_n)][i\omega_n+\varepsilon_{\mathbf{k}}+\bar{\Sigma}_{11}(-\mathbf{k},-i\omega_n)]+\bar{\Sigma}_{20}(\mathbf{k},i\omega_n)\bar{\Sigma}_{02}(\mathbf{k},i\omega_n)} \qquad (19)$$

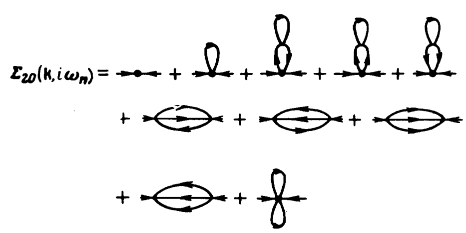
$$
= \frac{\begin{matrix} F_{20}(\mathbf{k}, i\omega_n) \\ -\Sigma_{02}(\mathbf{k}, i\omega_n) \end{matrix}}{[i\omega_n - \varepsilon_k - \Sigma_{11}(\mathbf{k}, i\omega_n)][i\omega_n + \varepsilon_k + \Sigma_{11}(-\mathbf{k}, -i\omega_n)] + \Sigma_{20}(\mathbf{k}, i\omega_n)\Sigma_{02}(\mathbf{k}, i\omega_n)}
$$

It is well known that the mass operators $\overline{\Sigma}_{11}(\mathbf{k}, i\omega_n)$, $\overline{\Sigma}_{20}(\mathbf{k}, i\omega_n)$ and $\overline{\Sigma}_{02}(\mathbf{k}, i\omega_n)$ can be written in the form of a power series in the interaction, where each term in the expansion corresponds to a well defined Feynman diagram. Up to and including terms of order S^{-2} the series for $\overline{\Sigma}_{11}(\mathbf{k}, i\omega_n)$ is described by the diagrams



(20)

By replacing in this series the external outgoing line by an incoming one we get the series for $\overline{\Sigma}_{20}(\mathbf{k}, i\omega_n)$:



(21)

As all interaction amplitudes are real one can prove that $\overline{\Sigma}_{02}(\mathbf{k}, i\omega_n) = \overline{\Sigma}_{20}(\mathbf{k}, i\omega_n)$. In the diagram series (20) and (21) we retained the external lines, as the analytical expression for the diagrams depends on their directions although the lines themselves do not occur in the analytical expression. The appearance of diagrams such as

(22)

is caused by the interaction V_2.

4. HIGH-FREQUENCY MAGNETIC SUSCEPTIBILITY TENSOR. MAGNON SPECTRUM AND DAMPING

By using standard methods[10] generalized to the interactions (9) to (11) we get the following expression for the mass operator of the retarded Green function $\Sigma_{11}(\mathbf{k}, \omega) = \overline{\Sigma}_{11}(\mathbf{k}, \omega + i\delta)$:

$$
\begin{aligned}
\Sigma_{11}(\mathbf{k}, \omega) &= \frac{1}{4S^2}(J_0 + \omega_0) + 4\sum_{\mathbf{q}} \psi(\mathbf{kq}; \mathbf{kq})\, n_{\mathbf{q}} \\
&- \frac{16}{T}\sum_{\mathbf{p},\mathbf{q}} \psi(\mathbf{kp}; \mathbf{kp})\,\psi(\mathbf{pq}; \mathbf{pq})\, n_{\mathbf{q}} n_{\mathbf{p}} (n_{\mathbf{p}} + 1) \\
&- 18\sum_{\mathbf{p},\mathbf{q}} \psi(-\mathbf{ppk}; \mathbf{k})\,\psi(-\mathbf{ppq}; \mathbf{q})\, n_{\mathbf{q}} \frac{2n_{\mathbf{p}} + 1}{2\varepsilon_{\mathbf{p}}} \\
&+ 8\sum_{123} \psi^2(12; 3\mathbf{k}) \frac{(n_1 + 1)(n_2 + 1)n_3 - n_1 n_2 (n_3 + 1)}{\omega + \varepsilon_3 - \varepsilon_1 - \varepsilon_2 + i\delta}\Delta(1 + 2 - 3 - \mathbf{k}) \\
&+ 18\sum_{123} \psi^2(12\mathbf{k}; 3) \frac{n_1 n_2 (n_3 + 1) - (n_1 + 1)(n_2 + 1)n_3}{\omega + \varepsilon_1 + \varepsilon_2 - \varepsilon_3 + i\delta}\Delta(1 + 2 + \mathbf{k} - 3) \\
&+ 6\sum_{123} \psi^2(123; \mathbf{k}) \frac{(n_1 + 1)(n_2 + 1)(n_3 + 1) - n_1 n_2 n_3}{\omega - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 + i\delta}\Delta(1 + 2 + 3 - \mathbf{k}) \\
&+ 96\sum_{123} \psi^2(123\mathbf{k}) \frac{n_1 n_2 n_3 - (n_1 + 1)(n_2 + 1)(n_3 + 1)}{\omega + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + i\delta}\Delta(1 + 2 + 3 + \mathbf{k}) \\
&+ 18\sum_{\mathbf{p},\mathbf{q}} \psi(\mathbf{kpq}; \mathbf{kpq})\, n_{\mathbf{p}} n_{\mathbf{q}}. \qquad (23)
\end{aligned}
$$

Similarly we have for $\Sigma_{20}(\mathbf{k}, \omega)$:

$$
\begin{aligned}
\Sigma_{20}(\mathbf{k}, \omega) &= -\frac{1}{4S^2}(J_0 + \omega_0) + 6\sum_{\mathbf{q}} \psi(-\mathbf{kkq}; \mathbf{q})\, n_{\mathbf{q}} - \frac{24}{T}\sum_{\mathbf{pq}} \psi(-\mathbf{kkp}; \mathbf{p}) \\
&\times \psi(\mathbf{qp}; \mathbf{qp})\, n_{\mathbf{q}} n_{\mathbf{p}} (n_{\mathbf{p}} + 1) - 12\sum_{\mathbf{pq}} \psi(-\mathbf{kk}; -\mathbf{pp})\,\psi(-\mathbf{ppq}; \mathbf{q})\, n_{\mathbf{q}} \frac{2n_{\mathbf{p}} + 1}{2\varepsilon_{\mathbf{p}}} \\
&- 72\sum_{\mathbf{pq}} \psi(-\mathbf{kk} - \mathbf{pp})\,\psi(-\mathbf{ppq}; \mathbf{q})\, n_{\mathbf{q}} \frac{2n_{\mathbf{p}} + 1}{2\varepsilon_{\mathbf{p}}} \\
&+ 12\sum_{123} \psi(12; 3\mathbf{k})\,\psi(-\mathbf{k}12; 3) \frac{(n_1 + 1)(n_2 + 1)n_3 - n_1 n_2 (n_3 + 1)}{\omega + \varepsilon_3 - \varepsilon_1 - \varepsilon_2 + i\delta}\Delta(1 + 2 \\
&- 3 - \mathbf{k}) + 12\sum_{123} \psi(\mathbf{k}12; 3)\,\psi(12; 3 - \mathbf{k}) \frac{n_1 n_2 (n_3 + 1) - (n_1 + 1)(n_2 + 1)n_3}{\omega + \varepsilon_1 + \varepsilon_2 - \varepsilon_3 + i\delta}\Delta(1 \\
&+ 2 + \mathbf{k} - 3) + 24\sum_{123} \psi(123; \mathbf{k})\,\psi(-\mathbf{k}123) \frac{(n_1 + 1)(n_2 + 1)(n_3 + 1) - n_1 n_2 n_3}{\omega - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 + i\delta} \\
&\times \Delta(1 + 2 + 3 - \mathbf{k}) + 24\sum_{123} \psi(123\mathbf{k})\,\psi(123; -\mathbf{k}) \frac{n_1 n_2 n_3 - (n_1 + 1)(n_2 + 1)(n_3 + 1)}{\omega + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + i\delta} \\
&\times \Delta(1 + 2 + 3 + \mathbf{k}) + 24\sum_{\mathbf{pq}} \psi(-\mathbf{kkpq}; \mathbf{pq})\, n_{\mathbf{p}} n_{\mathbf{q}}. \qquad (24)
\end{aligned}
$$

In these formulas $n_{\mathbf{k}} = (\exp\{\varepsilon_{\mathbf{k}}/T\} - 1)^{-1}$, and T is the temperature in energy units. These expressions enable us to find the components of the tensor $\chi(\mathbf{k}, \omega)$, the poles of which, $\omega = \omega(\mathbf{k})$, determine the spin wave spectrum taking into account their interaction with one another, and also their damping coefficient.

Using Eqs. (17) and (19) we get

$$
\chi_{xx}(\mathbf{k}, \omega) = \chi_{yy}(\mathbf{k}, \omega) = \frac{(g\mu_0)^2 S}{v_0}
\times \frac{\varepsilon_{\mathbf{k}} + {}^1/_2[\Sigma_{11}(\mathbf{k}, \omega) + \Sigma_{11}(-\mathbf{k}, -\omega)] + \Sigma_{20}(\mathbf{k}, \omega)}{[\omega - \varepsilon_{\mathbf{k}} - \Sigma_{11}(\mathbf{k}, \omega)][\omega + \varepsilon_{\mathbf{k}} + \Sigma_{11}(-\mathbf{k}, -\omega)] + \Sigma_{20}^2(\mathbf{k}, \omega)}, \qquad (25)
$$

$$
\omega(\mathbf{k}) = \varepsilon_{\mathbf{k}} + \Sigma_{11}(\mathbf{k}, \varepsilon_{\mathbf{k}}) - \frac{1}{2\varepsilon_{\mathbf{k}}}\Sigma_{20}^2(\mathbf{k}, \varepsilon_{\mathbf{k}}), \qquad (26)
$$

or, using (23) and (24)

$$
\begin{aligned}
\omega(\mathbf{k}) &= \varepsilon_{\mathbf{k}} - \frac{1}{SN}\sum_{\mathbf{q}} (J_0 + J_{\mathbf{k}-\mathbf{q}} - J_{\mathbf{q}} - J_{\mathbf{k}})\, n_{\mathbf{q}} + \frac{1}{2N^2S^2} \\
&\times \sum_{\mathbf{pq}} \frac{(J_{\mathbf{p}} + J_{\mathbf{q}} - J_{\mathbf{p}-\mathbf{k}} - J_{\mathbf{q}-\mathbf{k}})(J_{\mathbf{k}} + J_{\mathbf{p}+\mathbf{q}-\mathbf{k}} - J_{\mathbf{k}-\mathbf{p}} - J_{\mathbf{k}-\mathbf{q}})}{J_{\mathbf{p}} + J_{\mathbf{q}} - J_{\mathbf{k}} - J_{\mathbf{p}+\mathbf{q}-\mathbf{k}} + i\delta} \\
&\times [(n_{\mathbf{p}} + 1)(n_{\mathbf{q}} + 1)n_{\mathbf{k}-\mathbf{p}-\mathbf{q}} - n_{\mathbf{p}} n_{\mathbf{q}} (n_{\mathbf{k}-\mathbf{p}-\mathbf{q}} + 1)] \\
&- \frac{1}{TS^2N^2}\sum_{\mathbf{pq}} (J_0 + J_{\mathbf{k}-\mathbf{p}} - J_{\mathbf{k}} - J_{\mathbf{p}})(J_0 + J_{\mathbf{p}-\mathbf{q}} - J_{\mathbf{p}} - J_{\mathbf{q}})\, n_{\mathbf{q}} n_{\mathbf{p}} (n_{\mathbf{p}} + 1). \qquad (26a)
\end{aligned}
$$

We present explicit expressions for the frequency shift of the spin waves for a simple cubic ferromagnet in two limiting cases. If $\omega_0 \ll \varepsilon_{\mathbf{k}} - \omega_0 \ll T \ll T_c$, then $\Delta\omega_{\mathbf{k}} = \Delta\omega_{\mathbf{k}}^{(1)} + \Delta\omega_{\mathbf{k}}^{(2)}$,

$$
\begin{aligned}
\Delta\omega_{\mathbf{k}}^{(1)} &= -\frac{\bar{Q}}{S}\frac{\zeta(5/2)}{2^5\pi^{3/2}}(\varepsilon_{\mathbf{k}} - \omega_0)\left(\frac{T}{SI}\right)^{5/2}, \\
\Delta\omega_{\mathbf{k}}^{(2)} &= \frac{B}{S^2}(\varepsilon_{\mathbf{k}} - \omega_0)\left(\frac{T}{SI}\right)^3, \qquad (27)
\end{aligned}
$$

where I is the nearest-neighbor exchange integral,

$$
B = \frac{\zeta^2(3/2)}{3\cdot 2^6\pi^3} + \frac{1}{3\cdot 2^2\pi^4}\int_0^\infty dx\, x^3 n(x^2)\int_0^\infty dy\, y n(y^2)\ln\left|\frac{y - x}{y + x}\right| \approx 0.9\cdot 10^{-3},
$$

$$
\bar{Q} = 1 + \frac{1}{S}\sum_{\mathbf{q}} \frac{\cos^2 q_x a}{3 - \cos q_x a - \cos q_y a - \cos q_z a}; \quad n(z) = (e^z - 1)^{-1}. \qquad (27a)
$$

We note that the quantity $\overline{Q}$ is the same as the first two terms in the expansion of the Dyson factor $Q(S)$ in terms of S^{-1}. The first term in (27) is the same as that found in Refs. 14 and 7, while the second term is connected with taking processes with not merely four, but also six spin waves consistently into account. If $\omega_0 \ll T \ll \varepsilon_k - \omega_0 \ll T_c$, we have

$$\Delta\omega_k = \Delta\omega_k^{(1)}. \tag{28}$$

For the damping of the spin waves Eq. (26a) gives results which are the same as the results in a paper by Kashcheev and Krivoglaz. [14]

5. ASYMPTOTIC PROPERTIES OF $\chi(k, \omega)$

We remember the asymptotic properties of the magnetic susceptibility tensor of a Heisenberg ferromagnet. Bearing in mind that the exchange Hamiltonian commutes with the total spin of the system one checks easily (see, e.g., Ref. 15) that

$$\chi_{xx}(0,\omega) = \chi_{yy}(0,\omega) = -\frac{(g\mu_0)^2}{v_0}\frac{\langle S_z\rangle}{2}\left\{\frac{1}{\omega+\omega_0+i0} - \frac{1}{\omega-\omega_0+i0}\right\}, \tag{29}$$

where $\langle S_z\rangle$ is the average value of the spin along the quantization axis. If $\omega = 0$ and the wavevector $\mathbf{k}$ sufficiently small we have from Bogolyubov's theorem about the singularities of the correlation functions at small $\mathbf{k}$[16]

$$\chi(\mathbf{k}, 0) \geqslant \mathrm{const}/k^2. \tag{30}$$

We show that the components $\chi_{xx}(\mathbf{k}, \omega)$ and $\chi_{yy}(\mathbf{k}, \omega)$, for which we found general expressions in the preceding section, satisfy both condition (29) and condition (30). We consider first the case $\mathbf{k} = 0$ with ω arbitrary. Expanding (25) in a series in S^{-1} we get

$$\chi_{xx}(0,\omega) = \chi_{yy}(0,\omega) = -\frac{(g\mu_0)^2}{v_0}\cdot\frac{1}{2}\left\{S - \frac{1}{N}\sum_q n_q - \frac{1}{N^2TS}\sum_{pq}(J_0+J_{p-q}-J_p-J_q)n_q n_p(n_p+1)\right\}\left(\frac{1}{\omega+\omega_0+i0} - \frac{1}{\omega-\omega_0+i0}\right). \tag{31}$$

We used the analytical expressions (23) and (24) for $\Sigma_{11}(\mathbf{k}, \omega)$ and $\Sigma_{20}(\mathbf{k}, \omega)$ for $\mathbf{k} = 0$.

We show in the Appendix that the quantity inside the braces is the average value of the spin along the quantization axis, evaluated up to and including terms of order S^{-1}. Hence, Eq. (31) and (29) are the same.

When $\omega = 0$ we get for $\chi_{xx}(\mathbf{k}, 0) = \chi_{yy}(\mathbf{k}, 0)$ the expression

$$\chi_{xx}(\mathbf{k}, 0) = -\frac{(g\mu_0)^2}{v_0}\frac{S}{\varepsilon_k + \Sigma_{11}(\mathbf{k}, 0) - \Sigma_{20}(\mathbf{k}, 0)}$$

One can easily check by using Eqs. (23) and (24) that as the wavevector $\mathbf{k}$ tends to zero, $\Sigma_{11}(\mathbf{k}, 0) - \Sigma_{20}(\mathbf{k}, 0) \leqslant \mathrm{const}\cdot k^2$. Therefore, when $H = 0$

$$\chi_{xx}(\mathbf{k}, 0) = \chi_{yy}(\mathbf{k}, 0) \geqslant \mathrm{const}/k^2.$$

6. COMPARISON WITH OTHER METHODS

We noted in Section 2 that the ground state energy E_0, the average value of the spin along the quantization axis $\langle S_z\rangle$, and the correlation function $\langle S_k^+ S_{-k}^+\rangle$, found when using the representation of Ref. 8, within the approximation considered are the same as the corresponding exact values for those quantities. It is useful also to compare the results obtained in the representation of Ref. 8 with the results obtained in the HP and DM representations.

It is for the same reason as the one given for the case of the Hamiltonian (6) necessary in the HP representation to calculate up to terms which describe processes involving six particles and to include corrections arising from the non-commutativity of the operators. We have[11]

$$\mathscr{H}^{\mathrm{HP}} = E_0 + \mathscr{H}_2 + \mathscr{H}_4 + \mathscr{H}_6 + \ldots,$$

where E_0 and $\mathscr{H}_2$ are given by Eqs. (14) and (8), while

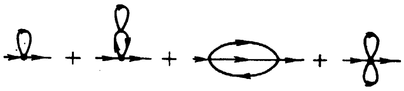
$$\mathscr{H}_4 = \sum_{1234}\psi(12;34)\,a_1^+a_2^+a_3a_4\Delta(1+2-3-4),$$

$$\psi(12;34) = \frac{1}{8NS}\left\{(J_1+J_2+J_3+J_4)\left(1+\frac{1}{8S}\right) - J_{1-3} - J_{1-4} - J_{2-3} - J_{2-4}\right\} \tag{32}$$

$$\mathscr{H}_6 = \sum_{12,\ldots,6}\psi(123;456)\,a_1^+a_2^+a_3^+a_4a_5a_6\Delta(1+2+3-4-5-6),$$

$$\psi(123;456) = \frac{1}{96S^2N^2}\left\{J_1+J_2+J_3+J_4+J_5+J_6 - \frac{2}{3}(J_{1+2-4} + J_{1+2-5} + J_{1+2-6} + J_{1+3-4} + J_{1+3-5} + J_{1+3-6} + J_{2+3-4} + J_{2+3-5} + J_{2+3-6})\right\}. \tag{33}$$

Up to and including terms of order S^{-2} the series for the mass operator $\Sigma^{\mathrm{HP}}(\mathbf{k}, \omega)$ is shown below:

$\Sigma^{\mathrm{HP}}(k, i\omega_n) =$  (34)

It corresponds to the analytical expression

$$\Sigma^{\mathrm{HP}}(\mathbf{k},\omega) = 4\sum_q \psi(\mathbf{kq};\mathbf{kq})n_q - \frac{16}{T}\sum_{pq}\psi(\mathbf{kp};\mathbf{kp})\psi(\mathbf{pq};\mathbf{pq})n_q n_p(n_p+1) + 8\sum_{123}\psi^2(12;3\mathbf{k})\frac{(n_1+1)(n_2+1)n_3 - n_1n_2(n_3+1)}{\omega+\varepsilon_3-\varepsilon_1-\varepsilon_2+i\delta}\Delta(1+2-3-\mathbf{k}) + 18\sum_{pq}\psi(\mathbf{kpq};\mathbf{kpq})n_p n_q. \tag{35}$$

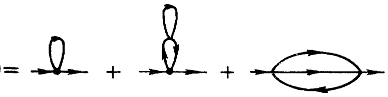
The Hamiltonian of the ferromagnet in the DM representation is

$$\mathscr{H}^{\mathrm{DM}} = E_0 + \mathscr{H}_2 + \mathscr{H}_4,$$

where

$$\mathscr{H}_4 = \frac{1}{4NS}\sum_{1234}(J_1+J_2-J_{1-3}-J_{2-3})\,a_1^+a_2^+a_3a_4\Delta(1+2-3-4), \tag{36}$$

while E_0 and $\mathscr{H}_2$ are, respectively, given by (14) and (8). The counterpart to Eq. (35) for the mass operator $\Sigma^{\mathrm{DM}}(\mathbf{k}, \omega)$ is described by the diagrams

$\Sigma^{\mathrm{HP}}(k, i\omega_n) =$  (37)

These diagrams correspond to the analytical expression[9]

$$\Sigma^{DM}(\mathbf{k},\omega)=4\sum_{\mathbf{q}}\psi(\mathbf{kq};\mathbf{kq})\,n_{\mathbf{q}}-\frac{16}{T}\sum_{\mathbf{pq}}\psi(\mathbf{kp};\mathbf{kp})\,\psi(\mathbf{pq};\mathbf{pq})\,n_{\mathbf{q}}n_{\mathbf{p}}(n_{\mathbf{p}}+1)$$
$$+8\sum_{123}\psi(12;3\mathbf{k})\,\psi(\mathbf{k}3;12)\,\frac{(n_1+1)(n_2+1)\,n_3-n_1n_2(n_3+1)}{\omega+\varepsilon_3-\varepsilon_1-\varepsilon_2+i\delta}$$
$$\times\Delta(1+2-3-\mathbf{k}). \qquad (38)$$

Comparing Eqs. (23), (24) with (35) and (38) we see that the mass operators in the HP and DM formalisms differ from those calculated in the present paper. Nonetheless, one can easily check that the poles of the Green functions are the same in all three representations and are given by Eq. (26a). For the spectrum of the ferromagnet, for the temperature corrections to the spectrum, for the ground state energy, for the average value of the spin along the quantization axis, and—as is shown in the Appendix—for the thermodynamic potential, the results obtained for the spin operator Green functions in the formalism of the HP and DM representations are the same as those obtained in the representation of Ref. 8. As far as the high-frequency properties of the magnetic substance are concerned, the situation here turns out to be more complicated. The components of the hf susceptibility tensor found in Ref. 9, using the DM formalism, have the correct asymptotic properties. Indeed, it follows from Eqs. (36), (38) that $\Sigma^{DM}(0,\omega)=0$, $\Sigma(\mathbf{k},0)\propto k^2$ which also guarantees that conditions (29), (30) are satisfied as according to Ref. 9 $\chi_{+-}(\mathbf{k},\omega)$ reduces to a single-particle Green function. However, although in the case of a ferromagnet the DM representation also leads to correct results, the possibility of the occurrence of uncontrolled errors in other more complicated cases, due to the non-Hermiticity of the representation, makes its practical application complicated.

The main difficulty in calculating the hf susceptibility tensor in the HP representation are the infinite series through which the spin operators are expressed in terms of the Bose operators. For instance, we have for the Green function $\langle\langle S_{\mathbf{k}}^{+}\,|\,S_{\mathbf{k}}^{-}\rangle\rangle$, and thus also for $\chi_{+-}(\mathbf{k},\omega)$ the series

$$\langle\langle S_{\mathbf{k}}^{+}|S_{\mathbf{k}}^{-}\rangle\rangle=2S\Big\{\langle\langle a_{\mathbf{k}}^{+}|a_{\mathbf{k}}\rangle\rangle-\frac{1}{4S}\sum_{123}[\langle\langle a_{\mathbf{k}}^{+}|a_1^{+}a_2a_3\rangle\rangle\,\Delta(1-2-3-\mathbf{k})$$
$$+\langle\langle a_1^{+}a_2^{+}a_3|a_{\mathbf{k}}\rangle\rangle\,\Delta(1+2-3-\mathbf{k})]+\frac{1}{16S^2}\sum_{\substack{123\\1'2'3'}}\langle\langle a_1^{+}a_2^{+}a_3|a_{1'}^{+}a_{2'}a_{3'}\rangle\rangle\,\Delta(1+2$$
$$-3-\mathbf{k})\,\Delta(1'-2'-3'-\mathbf{k})+\ldots\Big\}. \qquad (39)$$

For the calculation of the components of the tensor $\chi_{+-}(\mathbf{k},\omega)$ we cannot restrict the calculation to evaluating the single-particle Green function $\langle\langle a_{\mathbf{k}}^{+}|a_{\mathbf{k}}\rangle\rangle$ assuming the other terms in the series (39) to be corrections. This is due to the fact, as one can check easily, that the mass operator (35) does not vanish for $\mathbf{k}=0$ and arbitrary ω and, hence, the single-particle HP Green function does not have the correct structure (29). To calculate in the HP representation the components of the tensor $\chi(\mathbf{k},\omega)$ we must therefore know the n-particle Green functions and we have to remove series such as (39).

One can avoid the difficulties mentioned here by using the representation proposed in Ref. 8.

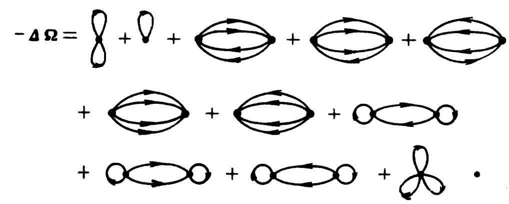
APPENDIX

We study the thermodynamic properties of a ferromagnet using the representation (2). To find all thermodynamic quantities it is sufficient to know the thermodynamic potential Ω of the system. In the low-temperature region one may assume that the thermodynamic potential consists of the thermodynamic potential of a perfect magnon gas

$$\Omega_0=\bar{E}_0+T\sum_{\mathbf{k}}\ln(1-\exp\{-\varepsilon_{\mathbf{k}}/T\}), \qquad (A.1)$$

where $\bar{\Sigma}_0$ is given by Eq. (7) of the main text, and corrections due to the interaction of magnons with one another, $\Delta\Omega$.

The diagrams contributing to $\Delta\Omega$, up to and including terms of order S^{-2} have the form

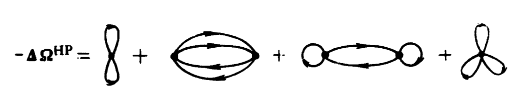


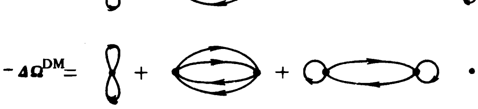
These diagrams are constructed using the Hamiltonian (6). As a result we get for the thermodynamic potential Ω:

$$\Omega=E_0-T\sum_{\mathbf{k}}\ln(1+n_{\mathbf{k}})-\frac{1}{2SN}\sum_{\mathbf{kq}}(J_0+J_{\mathbf{k-q}}-J_{\mathbf{k}}-J_{\mathbf{q}})\,n_{\mathbf{k}}n_{\mathbf{q}}$$
$$-\frac{1}{4S^2N^2}\sum_{\mathbf{kqp}}\frac{(J_{\mathbf{p}}+J_{\mathbf{q}}-J_{\mathbf{p-k}}-J_{\mathbf{q-k}})\,(J_{\mathbf{k}}+J_{\mathbf{p+q-k}}-J_{\mathbf{k-p}}-J_{\mathbf{k-q}})}{J_{\mathbf{p}}+J_{\mathbf{q}}-J_{\mathbf{k}}-J_{\mathbf{p+q-k}}}\,n_{\mathbf{p}}n_{\mathbf{q}}(2n_{\mathbf{k}}+1)$$
$$-\frac{1}{8TS^2N^2}\sum_{\mathbf{kqp}}(J_0+J_{\mathbf{k-p}}-J_{\mathbf{k}}-J_{\mathbf{p}})\,(J_0+J_{\mathbf{q-p}}-J_{\mathbf{q}}-J_{\mathbf{p}})\,n_{\mathbf{k}}n_{\mathbf{q}}n_{\mathbf{p}}(n_{\mathbf{p}}+1), \qquad (A.2)$$

where E_0 is the exact ground state energy (14) of the ferromagnet.

Using the Hamiltonians (32), (33), and (36) we can find expressions for $\Delta\Omega$ in the HP and DM representations. The corresponding diagrams are of the form





Associating analytical expressions to these diagrams one checks easily that Ω^{HP} is equal to Ω^{DM} and Ω given by Eq. (A.2). This means that also all thermodynamic properties of the ferromagnet, considered in the representation of Ref. 8 will be the same as the results obtained in the HP and DM representations. In particular, we get for the equilibrium magnetic moment density

$$\frac{v_0}{g\mu_0}M(T,H)=-\frac{1}{N}\frac{\partial\Omega}{\partial(g\mu_0H)}=S-\frac{1}{N}\sum_{\mathbf{k}}n_{\mathbf{k}}-$$

$$-\frac{1}{TSN^2}\sum_{pq}\{J_0+J_{p-q}-J_q-J_p\}n_pn_q(n_q+1)$$

$$-\frac{1}{4S^2N^3T}\sum_{kqp}\frac{(J_p+J_q-J_{p-k}-J_{q-k})(J_k+J_{p+q-k}-J_{k-p}-J_{k-q})}{J_p+J_q-J_k-J_{p+q-k}}$$

$$\times n_pn_q[2n_k(n_p+n_q+n_k+3)+(n_p+n_q+2)]-\frac{1}{8T^2S^2N^3}\sum_{kqp}(J_0+J_{k-p}-J_k-J_p)$$

$$\times(J_0+J_{q-p}-J_q-J_p)n_kn_qn_p(n_p+1)(2n_k+2n_p+3) \qquad \text{(A. 3)}$$

up to and including terms of order S^{-2}.

In conclusion we give the corrections to the thermodynamic potential and the magnetization in the low-temperature region:

$$\Delta\Omega=-N\frac{\bar{Q}}{S}\frac{3\zeta^2(^5/_2)}{2^{10}\pi^3}T\left(\frac{T}{SI}\right)^4-N\frac{C_1}{S^2}T\left(\frac{T}{SI}\right)^{9/2},$$

$$\frac{v_0}{g\mu_0}\Delta M=-\frac{\bar{Q}}{S}\frac{3\zeta(^5/_2)\zeta(^3/_2)}{2^9\pi^3}\left(\frac{T}{SI}\right)^4-\frac{C_2}{S^2}\left(\frac{T}{SI}\right)^{9/2}, \qquad \text{(A. 4)}$$

where

$$C_1=\frac{1}{2^6\pi^6}\int_0^\infty dx\int_0^x dy\int_0^\infty dz\int_{x-y}^{x+y}dt\, xyzn(x^2)n(y^2)n(z^2)$$

$$\times(t^2-x^2-y^2)^2\ln\frac{2z^2+2zt+t^2-x^2-y^2}{2z^2-2zt+t^2-x^2-y^2}\approx 3.8\cdot 10^{-6},$$

$$C_2=\frac{1}{2^6\pi^6}\int_0^\infty dx\int_0^x dy\int_0^\infty dz\int_{x-y}^{x+y}dt\, xyzn(x^2)n(y^2)n(z^2)[n(x^2)+n(y^2)+n(z^2)+3]$$

$$\times(t^2-x^2-y^2)^2\ln\frac{2z^2+2zt+t^2-x^2-y^2}{2z^2-2zt+t^2-x^2-y^2}\approx 4\cdot 10^{-5}.$$

The first terms are the well known corrections found by Dyson.[4] The second terms are due to taking the interaction processes of four and six magnons into account.

In conclusion the authors express their gratitude to I. E. Dzyaloshinskiĭ for a discussion of this work and to T. M. Eremenko for his help with the evaluation of integrals.

[1)] For the sake of convenience in constructing a perturbation theory we have changed here from the exchange integral $I_{ff'}$ to the parameter $J_{ff'}$ through the relation $J_{ff'}=SI_{ff'}$.

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Translated by D. ter Haar