2. The interaction of stars with extended gas clouds—the products of the disruption of stars in tidal or collisional interactions—leads to an even more rapid filling of the loss cone with stars, i.e., it increases their flow through the tidal sphere. This can be further enhanced by the collective interaction of stars, the importance of which has been pointed out by A. M. Fridman. These questions warrant a separate investigation.

1) For bound stars with orbits lying within the radius $r_{coll} \approx r_* M/m$, at which the kinetic energy of a star $\sim GMm/r$ is comparable with its binding energy $\sim Gm^2/r_*$, disruption as a result of direct collisions of stars may become an important process.[2] However, comparison of r_{coll} and r_{crit} (Eq. (37)) shows that in real clusters $r_{coll} < r_{crit}$, i.e., a star remains in the region $r < r_{coll}$ for only a small fraction of the time and collisions of stars, being very rare, will not change the dependence $n(r)$ in the region $r < r_{coll}$.

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Self-similar motions of a photon gas and the Friedmann model

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Self-similar motions of a gas with equation of state $p = \epsilon/3$ are considered. It is shown that all solutions with a weak discontinuity have a nonsingular horizon. The Friedmann solution belongs to a one-parameter family of solutions that are continuous at the symmetry center; these are described. The solutions with strong shock waves correspond to the problem of initial focusing of the gas toward the center in the case of a supercritical intensity of the discontinuity. The results of qualitative and numerical investigations of the corresponding dynamical system are presented. New cosmological solutions with strong and weak discontinuities are obtained.

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In the early stages in the expansion of the Universe, the dominant contribution to the matter energy was made by electromagnetic radiation, which follows almost unambiguously from the discovery of the microwave blackbody radiation with temperature 2.83 °K.[1] The simplest model of a radiation-filled Universe—the Friedmann–Lemaître model with flat comoving space—admits a simple analytic expression for the metric and the energy density in Lagrangian coordinates[2]:

$$ds^2 = d\tau^2 - \tau a_0 [dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2)],\qquad \varepsilon = 3c^2/32\pi G\tau^2. \tag{1.1}$$

This is a self-similar (similarity) solution. It is therefore natural to consider the status of the Friedmann solution among the other self-similar spherically symmetric solutions and also study the physical and analytic properties of other self-similar solutions, including those with shock waves.

In the present paper, spherically symmetric self-similar motions are studied in the orthogonal coordinate system of an observer:

$$ds^2 = c^2 e^\nu dt^2 - e^\gamma dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1.2}$$

Self-similar spherical motions of a gravitating gas in the framework of general relativity were considered for the first time by Skripkin.[3] Because of a not entirely fortuitous choice of the variables, the correct conditions on the gas-dynamic shock waves in the framework of general relativity lacked the simplicity of these conditions in special relativity,[2] but Skripkin succeeded in deriving a condition on the discontinuity after which the gas goes over into a state of rest: $\varepsilon = 3(7\varkappa r^2)^{-1}$. Skripkin reduced Einstein's equations to an equation of second order with radicals.[3] From the fixed velocity of the shock wave, he calculated the parameters of the gas after the discontinuity and used these data to construct numerically the integral curve outside the static region. In[4], Stanyukovich, Sharkhekeev, and Gurovich obtained a system of ordinary differential equations for self-similar motions. In a comoving coordinate system, Gurovich[5] investigated the self-similar problem for the maximally hard equation of state $p=\varepsilon$. In the framework of the theory of small perturbations of the Friedmann models developed by Lifshitz,[6] self-similar perturbations have been considered by Ruzmaikina.[7] In a comoving coordinate system, the system of ordinary differential equations for self-similar motions of a gravitating gas in general relativity was investigated by Cahill and Taub,[8] who showed in particular by means of the junction conditions at the discontinuities that known solutions could be joined.

For a photon gas in the framework of special relativity, one of the present authors[9] investigated all self-similar motions with axial and central symmetry; self-similar dynamical systems in the case of a central symmetry were considered by Stanyukovich and Skripkin in[10,11]. In connection with the problem of multiparticle production, Khalatnikov[12] reduced the problem to the analysis of a linear equation in the case of one-dimensional nonstationary motions of an ultrarelativistic gas.

In the present paper, which is based on[13], we derive a closed system of two first-order ordinary differential equations that is convenient for qualitative investigation (Sec. 1). We show that the conditions on shock waves for a gas with equation of state $p=\varepsilon/3$ for special variables in general relativity have the same form as in special relativity. We establish the existence of a nonsingular horizon, which we call a light horizon, for certain solutions, outside which the coordinate system (1.2) becomes meaningless. For the cosmological solutions, this is due to the fact that, relative to the symmetry center, the particles outside the light horizon have a superluminal velocity. The integral curves on which the gas velocity is zero at the symmetry center form a one-parameter family, which includes the Friedmann solution. Therefore, all solutions that do not have a source or a vacuum at the symmetry center must go over to this family either by means of a shock or by means of a weak discontinuity. It is interesting to note that all the solutions with weak discontinuities have a light horizon, on which the gas velocity with respect to the system (1.2) is equal to the velocity of light but the pressure is finite. In these solutions, the complete spacetime manifold cannot be covered by the coordinate mesh (1.2).

The situation with regard to the solutions that have shock waves is very different. We show that there exists a critical intensity of the shock which is such that if a distontinuity with subcritical intensity is realized in the solution then the solution has a light horizon. If the amplitude of the discontinuity is supercritical, the solutions do not have a horizon and, qualitatively, have all the properties of self-similar Cauchy problems of the focusing of a gas toward the center described by Sedov in his book.[14] In the case when a static region is formed, none of the solutions with a shock wave have an event horizon in the system (1.2), but all of them with a weak discontinuity do have one. We find out what are the initial data for which the problems of expansion and focusing have a physical meaning. We describe the phase portraits of the self-similar curves near the light horizon, the sonic line, and the coordinate origin. The following results of numerical calculations are presented: 1) solutions with spherical shock waves with subcritical and supercritical intensities having within them either a piece of the Friedmann–Lemaître Universe or a static region; 2) solutions with a weak discontinuity along the sonic surface, which contain either the Friedmann solution or a static region; 3) numerical solutions for inhomogeneous cosmological models, which also have a light horizon.

§1. DERIVATION OF A CLOSED DYNAMICA SYSTEM AND THE CONDITIONS ON THE SHOCK WAVES

For our purpose, it is convenient to introduce a radial velocity (normalized to the velocity of light) defined at a point as the ordinary velocity with respect to the local Lorentz coordinate system with basis vectors directed along the coordinate lines (1.2). The velocity V is expressed in terms of the components of the four-velocity u^i and the metric coefficients e^γ and e^ν as follows:

$$V=u^r e^{(\gamma-\nu)/2}/u^0.$$

For self-similar solutions, the unknown functions $\nu(r,t)$, $\gamma(r,t)$, $V(r,t)$, and $p(r,t)$ have the form

$$\nu=\nu(\lambda),\ \gamma=\gamma(\lambda),\ \varkappa p=P(\lambda)/r^2;\ \lambda=r/ct.$$

We introduce a new self-similar variable ζ, which is the velocity of the surface $\lambda=\text{const}$ with respect to the local Lorentz coordinate system:

$$\zeta=\lambda e^{(\gamma-\nu)/2}. \tag{1.3}$$

In this case, Einstein's equations have the form

$$(\varkappa T_0^r=R_0^r),\quad \frac{4PV}{1-V^2}=X^{-2}\frac{dX}{d\zeta}\zeta^2(1-L),\quad X=e^\gamma; \tag{1.4}$$

$$\left(\varkappa T_r^r=R_r^r-\frac{1}{2}R\right),\quad -P\left(\frac{3V^2+1}{1-V^2}\right)=1-\frac{1}{X}-\frac{\zeta}{X}(1-L)\frac{d\nu}{d\zeta}; \tag{1.5}$$

$$\left(\varkappa T_0^0=R_0^0-\frac{1}{2}R\right),\quad P\left(\frac{3+V^2}{1-V^2}\right)=1-\frac{1}{X}+X^{-2}\frac{dX}{d\zeta}\zeta; \tag{1.6}$$

$$L=2(X-1)(V^2\zeta+\zeta-2V)/[\zeta(3+V^2)-4V]. \tag{1.7}$$

From Eqs. (1.3) and (1.5) we can calculate the self-similar part of the pressure explicitly if we know the solutions $X(\zeta)$ and $V(\zeta)$:

$$P(\zeta)=\frac{X-1}{X}\cdot\frac{(1-V^2)\zeta}{(3+V^2)\zeta-4V}. \tag{1.8}$$

The coefficient $\nu(\zeta)$ can be found from known $X(\zeta)$ and $V(\zeta)$ by means of quadrature from Eq. (1.5) if $P(\zeta)$ is replaced by the expression (1.8):

$$\zeta\frac{d\nu}{d\zeta}(1-L)=\frac{4V(X-1)}{\zeta(3+V^2)-4V}.$$

If the function $P(\zeta)$ is eliminated from Eqs. (1.4) and (1.6), we obtain

$$\zeta\frac{dX}{d\zeta}(1-L)=\frac{4V(X-1)X}{\zeta(3+V^2)-4V}. \tag{1.9}$$

Eliminating the pressure from the equations of motion $T^{i}_{r;i}=0$ and $T^{i}_{0;i}=0$, we obtain

$$\zeta\frac{dV}{d\zeta}(1-L)=\frac{(1-V^2)[2V-\zeta V^2/2-3\zeta/2+\Omega]}{-(1-V\zeta)^2+3(V-\zeta)^2},$$
$$\Omega=\frac{4(X-1)}{\zeta(3+V^2)-4V}\left[V^3\zeta-3V\zeta+3\zeta^2/2+V^2-\frac{V^4\zeta^2}{2}\right]. \tag{1.10}$$

The dynamical system of equations (1.9) and (1.10) for $X(\zeta)$ and $V(\zeta)$ is closed. The connection between the variables ζ and λ can be obtained by integrating the equation

$$\zeta\frac{d\ln\lambda}{d\zeta}(1-L)=1, \tag{1.11}$$

which follows from Eq. (1.3).

Conditions at the discontinuities. The definition of differentiable manifolds includes a fixed class of coordinate meshes within which the transition from one system to another must satisfy given smoothness conditions. In order to cover the possibility of occurrence of gas-dynamic shock waves, the coordinate transformations within a distinguished class of coordinate systems must be doubly differentiable with piecewise smooth third derivatives. Then in the absence of a medium, the discontinuities of the second derivatives of the metric across nonisotropic surfaces can be eliminated by means of the choice of the discontinuities of the third derivatives of the coordinate transformations. The differential operator $(R_{ik}-\frac{1}{2}g_{ik}R)n^k$ contains only the first derivatives of the metric along the normal to the surface of the discontinuity, and is therefore continuous. It then follows from Einstein's equations that the energy and momentum flux through the shock wave is continuous, $[T_{ik}]n^k=0$.

When a coordinate system is distinguished by means of a certain four additional restrictions on the form of the metric it may happen that the resulting coordinate mesh does not belong to the privileged family of coordinate systems. Therefore, in such coordinate systems the metric and its first derivatives may have a discontinuity. In this case, as Sedov shows,[15] both the first and the second quadratic form of the surface of the discontinuity must be continuous across a nonisotropic discontinuity surface. For the coordinate system (1.2) there follows from this continuity of the metric and energy–momentum flux through the discontinuity surface.

Suppose the equation of the shock wave has the form $f(r,t)=0$. We denote by $c\tilde{\zeta}$ the velocity of the shock wave with respect to the orthonormal frame constructed from the local coordinate basis (1.2):

$$c\tilde{\zeta}=e^{(\gamma-\nu)/2}\frac{\partial f}{\partial t}\Big/\frac{\partial f}{\partial r}.$$

The four-normal to the shock wave has the components

$$n_r=-e^{\gamma/2}/(1-\tilde{\zeta}^2)^{1/2},\quad n_0=e^{\nu/2}\tilde{\zeta}/(1-\tilde{\zeta}^2)^{1/2}.$$

From the conditions $n_k[T^k_r]=n_k[T^k_0]=0$ there follow, respectively

$$\left[\frac{P+\varepsilon}{1-V^2}+P-\frac{P+\varepsilon}{1-V^2}\tilde{\zeta}\right]=0, \tag{1.12}$$

$$\left[\frac{P+\varepsilon}{1-V^2}(\tilde{\zeta}-V)-P\tilde{\zeta}\right]=0. \tag{1.13}$$

It is remarkable that these conditions have the same form as in special relativity.[2]

Eliminating the pressure for an ultrarelativistic gas from Eqs. (1.12) and (1.13), we obtain

$$V_2=(3\tilde{\zeta}^2-1-2V_1\tilde{\zeta})/[V_1(\tilde{\zeta}^2-3)+2\tilde{\zeta}]. \tag{1.14}$$

Here, V_1 and V_2 are the velocity of the gas particles before and after the discontinuity in units of the velocity of light. In the coordinate system moving with the discontinuity, the condition (1.14) means that the product of the velocities before and after the discontinuity is equal to the square of the velocity of sound in the photon gas.[2]

§2. THE FRIEDMANN SOLUTION AND QUALITATIVE INVESTIGATION OF THE SYSTEM (1-9)–(1.10)

The *Friedmann solution* can be readily transformed to the coordinate system (1.2) from the system (1.1). One transformation is obvious: $a_0\tau R^2=r^2$; the other transformation is found from the condition of orthogonality of the metric (1.2), for which it is convenient to seek it in the form

$$\tau=r/f(\lambda),\quad \lambda=r/ct.$$

After simple calculations we obtain for the function $f(\lambda)$ and the velocity $V(\lambda)$

$$V=f/2=\lambda/[1+(1-\lambda^2)^{1/2}]. \tag{2.1}$$

The metric coefficients e^{γ} and e^{ν} in the Friedmann solution are equal to each other:

$$e^{\gamma}=e^{\nu}=[1+(1-\lambda^2)^{1/2}]/2(1-\lambda^2)^{1/2}. \tag{2.2}$$

The pressure as a function of the coordinates is given by

$$p=c^6t^2(1-(1-\lambda^2)^{1/2})^2/8\pi Gr^4. \tag{2.3}$$

The coordinates (1.2) can cover only part of the Friedmann solution with infinite comoving space.

Phase portraits of the singular points. A. We linearize the system (1.9)–(1.10) around the straight line $\zeta=0$, $V=0$. We denote $V/\zeta=q$. Then the system (1.9)–(1.10) can be reduced to the form

$$\frac{d\zeta}{\zeta}=\frac{2dq[3-4q-2(X-1)(1-2q)]}{(3-4q)[3(1-2q)-4(X-1)(1-q)]}=\frac{dX[3-4q-2(X-1)(1-2q)]}{4q(X-1)X}. \tag{2.4}$$

It follows from this that in the first approximation in V and ζ the integral curves around the straight line $\zeta=0$, $V=0$ will lie on the surfaces $X(V/\zeta,\alpha)$, where $X(q,\alpha)$ is an integral curve of the equation

$$dX/dq=8q(X-1)X/(3-4q)[3(1-2q)-4(X-1)(1-q)]. \tag{2.5}$$

The qualitative picutre of the integral curves of Eq. (2.5)

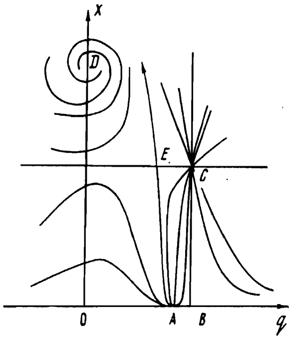

FIG. 1. Integral curves of Eq. (2.5). The singular points have the coordinates $A(7/10, 0)$, $B(3/4, 0)$, $C(3/4, 1)$, $D(0, 7/4)$, $E(1/2, 1)$.

is shown in Fig. 1. If, as we move along an integral curve, we are to attain a zero value for ζ with direction $V/\zeta = q_0$, it is necessary, in accordance with (2.5), that the point X_0, q_0 be a singular point of Eq. (2.5). An additional investigation shows that one of these points has the coordinates $X_0 = 1$, $q_0 = \frac{1}{2}$, and the other $X_0 = 7/4$, $q_0 = 0$.

The one-parameter family of integral curves that enter the point $X_0 = 1$, $q_0 = 0$, has the asymptotic behavior

$$V=\frac{1}{2}\zeta+\left(\frac{7}{40}-\frac{P_1}{5}\right)\zeta^3+\frac{1}{14}\left(\frac{61}{40}-\frac{2P_1^2}{5}-\frac{5P_1}{2}\right)\zeta^5+\dots,$$
$$\varkappa p=P_1t^{-2}[1+(1-2P_1)\zeta^2+\dots]. \qquad (2.6)$$
$$X=1+P_1\zeta^2+P_1\left(\frac{4}{5}-\frac{P_1}{5}\right)\zeta^4+P_1\left(\frac{207}{280}-\frac{17}{35}P_1+\frac{4}{35}P_1^2\right)\zeta^6+\dots$$

For the parameter value $P_1 = \frac{1}{4}$, the expressions (2.6) are the first terms in the expansion of the Friedmann solution (2.1)–(2.3). The exact solution $X_0 = 7/4$, $V = 0$, $p = (7\varkappa r^2)^{-1}$ corresponds to a static configuration of the gas.

The complete set of solutions of Eqs. (1.9)–(1.10) depends on two parameters, while the solutions (1.6) depend on only one; therefore, all the remaining solutions must go over to one of the curves (2.6) or the static solution by means of a weak discontinuity along the sonic line or by means of a strong discontinuity (if, of course, there is neither a source nor vacuum at the center).

B. We now study the behavior of the integral curves of the system (1.9)–(1.10) near the light cone $V = 1$, $\zeta = 1$. If the system (1.9)–(1.10) is linearized with respect to $\zeta - 1$ and $V - 1$ and we write $(V-1)/(\zeta-1) \equiv q$, then we can reduce the system to the form

$$\frac{d\zeta}{\zeta-1}=\frac{(4-q-2X)(q^2+1-4q)\,dq}{q(2-q)(3q-q^2+1)}=\frac{(4-q-2X)\,dX}{2(X-1)X}. \qquad (2.7)$$

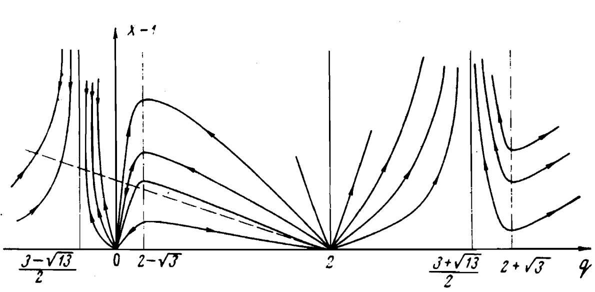

FIG. 2. Integral curves of Eq. (2.8).

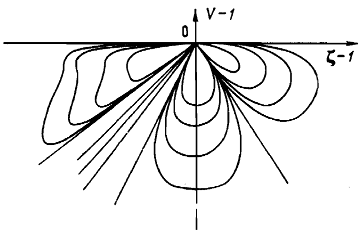

FIG. 3. Integral curves of the system (2.7) projected onto the V, ζ plane.

The integral curves lie (in the first approximation) on the family of surfaces $X = X((V-1)/(\zeta-1), \alpha)$, where $X(q, \alpha)$ is an integral curve of the equation

$$dX/dq=2(X-1)X(q^2+1-4q)/q(2-q)(3q-q^2+1). \qquad (2.8)$$

In Fig. 2, we show the qualitative picture of the integral curves of Eq. (2.8), the arrows indicating the direction of increasing ζ. Figure 3 is the qualitative picture of the integral curves for $X > 1$ and fixed value of the constant of integration in Eq. (2.7) in the projection onto the V, ζ plane. The direction $q = 2$ is singular.

The asymptotic behavior of the ends of the loops (Fig. 3), which approach the straight line $V = \zeta = 1$ as $X \to \infty$ with finite slope $q_0 \neq 0$, is

$$X=\frac{\tilde{\alpha}}{1-\zeta}+O(1), \quad V-1=q_0(\zeta-1)+O[(\zeta-1)^2], \quad P\sim\frac{q_0}{q_0-2}. \qquad (2.9)$$

If the slope is $q = 0$, the asymptotic behavior of the curves is

$$X-1=\alpha(\zeta-1)+O[(1-\zeta)^2], \quad V-1=-\beta(\zeta-1)^2+O[(\zeta-1)^3],$$
$$P={}^1/_2\alpha\beta(\zeta-1)^2, \quad \alpha>0, \quad \beta>0.$$

It is necessary to consider separately the asymptotic behavior of the curves as $q \to \infty$. Analysis shows that there exists a one-parameter family which does not belong to the two-parameter family of curves (2.7) and which has the asymptotic behaviors

$$V=1-\alpha\sqrt{1-\zeta}+({}^3/_4+\alpha^4/16)(1-\zeta)+\dots,$$
$$x(\alpha^2+2)=\alpha(1-\zeta)^{-1/2}+(\alpha^4-36)/16+\dots; \quad P\approx1. \qquad (2.10)$$

When $\alpha = \sqrt{2}$, the expressions (2.10) are the asymptotic behavior of the Friedmann solution as $\zeta \to 1$. Numerical calculations show that the one-parameter family having the asymptotic behaviors (2.6) in the limit $\zeta \to 0$ has the asymptotic behaviors (2.10) as $\zeta \to 1$. These solutions can be appropriately called "submanifolds" of the inhomogeneous cosmological models bounded by the light horizon. It is interesting that for the Friedmann solution the coefficient $\alpha/(\alpha^2+2)$ in the asymptotic behavior for X has the maximal value.

C. We now investigate the structure of weak discontinuities near the sonic lines, which are defined as the zeros of the numerator and the denominator on the right-hand side of (1.10):

$$\zeta_0=(V_0\sqrt{3}+1)(V_0+\sqrt{3})^{-1}, \qquad (2.11)$$
$$X_0-1=(V_0-\sqrt{3})^2[4(1-V_0^2)]^{-1}.$$

As ζ varies from $\sqrt{3}/3$ to $\sqrt{3}/2$, we obtain the sonic lines in the different inhomogeneous cosmological models described above. When ζ_0 is equal to $\sqrt{3}/3$, we obtain from (2.11) the sonic line in the static solution: $X_0=7/4$, $V_0=0$, $\zeta_0=\sqrt{3}/3$. Near this line, the curves from a degenerate node, at which they arrive on the sonic line tangent to the straight line $V=0$, $X=7/4$.

When $\sqrt{3}/3<\zeta_0<\sqrt{3}/2$, the curves near the sonic line (2.11) form a node lying in the plane

$$X-X_0=\delta(V-V_0),\quad \delta=\frac{V_0(3-V_0^2)(\sqrt{3}+V_0)(7-2V_0\sqrt{3}-3V_0^2)}{2(1-V_0^2)^2(\sqrt{3}-3V_0^2\sqrt{3}+4V_0)(1+V_0\sqrt{3})}.$$

Finally, if $\zeta_0=\sqrt{3}/2$, we obtain from (2.11) the sonic line in the Friedmann solution (2.1)–(2.3): $X_0=3/2$, $V_0=\sqrt{3}/3$, $\zeta_0=\sqrt{3}/2$; near it, the curves form a degenerate node. All curves arrive on the sonic line tangent to the straight line

$$X-{}^3/_2=2\sqrt{3}(\zeta-\sqrt{3}/2),\quad V-\sqrt{3}/3=\frac{4}{3}(\zeta-\sqrt{3}/2).$$

D. The investigation of the system (1.9) and (1.10) gives the following asymptotic expressions for $X(\zeta)$, $V(\zeta)$, and $P(\zeta)$:

$$X=X_0+O(1/\zeta),\ V=V_0+O(1/\zeta),\ P=P_0+O(1/\zeta),$$
$$P_0=(1-V_0^2)(X_0-1)/X_0(3+V_0^2). \tag{2.12}$$

(The concrete expressions for the coefficients of $1/\zeta$ are given in[13]; for $X_0-1=(3+V_0^2)[2(1+V_0^2)]^{-1}$ the solutions can be expanded in powers of $\zeta^{-1/2}$).

At large ζ, the self-similar variable λ is related to the variable ζ (1.11) as follows:

$$\frac{d\zeta}{d\lambda}=\frac{\zeta}{\lambda}\left(1-\frac{2(X_0-1)(V_0^2+1)}{3+V_0^2}\right).$$

To large positive ζ there correspond $\lambda\gg1$ for $(1-L)>0$ or

$$X_0<(5+3V_0^2)[2(1+V_0^2)]^{-1}. \tag{2.13}$$

Then in the limit $\zeta\to\infty$ we obtain a restriction on the possible value of the constant P_0 in (2.12):

$$P_0<(1-V_0^2)(5+3V_0^2)^{-1}. \tag{2.14}$$

Thus, the self-similar Cauchy problem for focusing or expansion for a gravitating photon gas has a solution under the restrictions (2.13) and (2.14). The actual integration of the system (1.9) and (1.10) may restrict these inequalities further. Other singular points are investigated in[13].

§3. DISCUSSION OF RESULTS

To investigate the possibility of analytic continuation of the solutions beyond the light horizon, let us consider the connection between the comoving coordinate system and the observer's coordinate system for the self-similar motions. Suppose the metric in the comoving coordinate system has the form

$$ds^2=a^2d\tau^2-b^2d\xi^2-r^2d\Omega^2.$$

In the comoving coordinate system, the momentum equation gives $p=a^{-4}f(\tau)$ and from the energy equation we obtain $p^{3/4}r^2b=\varphi(\xi)$.[2] We fix the choice of the time τ and the Lagrangian coordinate in such a way as to obtain $f(\tau)=A^4\tau^2$, $\varphi(\xi)=A^3\xi^{1/2}$.

Self-similar solutions in the comoving coordinate system are distinguished by the requirement

$$a=a(m),\ b=b(m),\ r=\xi R(m),\ p=a^{-4}A^4\tau^2, \tag{3.1}$$

where $m=\xi/\tau$. By definition, the velocity four-vector in the comoving coordinate system has only a nonvanishing fourth component: $u^i(0,0,0,a^{-1})$. By the rule for transforming vectors in the coordinate (1.2), we have

$$u^0=\frac{\partial t}{\partial\tau}a^{-1},\quad u^r=\frac{\partial r}{\partial\tau}a^{-1}. \tag{3.2}$$

Dividing the first equation by the second and using the definition of the velocity V from (3.1), we obtain

$$\left.\frac{\partial r}{\partial\tau}\right|_\xi=V\left.\frac{\partial t}{\partial\tau}\right|_\xi\exp\left(\frac{\nu-\gamma}{2}\right). \tag{3.3}$$

From the condition of orthogonality of the comoving coordinate system, we obtain by means of (3.3)

$$V\left.\frac{\partial r}{\partial\xi}\right|_\tau=\exp\left(\frac{\nu-\gamma}{2}\right)\left.\frac{\partial t}{\partial\xi}\right|_\tau. \tag{3.4}$$

Substituting $r=\xi R(m)$ and $t=\xi R(m)/\lambda$ into Eqs. (3.3) and (3.4), we obtain from them

$$\frac{d\ln R}{d\zeta}(1-L)=\frac{V}{(V-\zeta)\zeta},$$
$$\frac{d\ln m}{d\zeta}(1-L)=-\frac{1-V^2}{(V-\zeta)(1-V\zeta)}. \tag{3.5}$$

For the coefficients a and b, using (3.5), we obtain

$$a^2=\frac{m^2R^2X}{\zeta^2}\frac{(1-V\zeta)^2}{1-V^2},\quad b^2=\frac{R^2X}{\zeta^2}\frac{(\zeta-V)^2}{1-V^2}. \tag{3.6}$$

Equations (3.5) and (3.6) determine R, a, and b as a function of m in the parametric form in terms of ζ.

Substituting in (3.5) the asymptotic behaviors (2.10) of the inhomogeneous cosmological models near the horizon, we obtain expressions for $R(\zeta)$, $m(\zeta)$, $a^2(\zeta)$, and $b^2(\zeta)$ as series in powers of $(1-\zeta)^{1/2}$:

$$\frac{R}{R_0}=1+\frac{\alpha^2+2}{2\alpha^3}(1-\zeta)^{1/2}+O(1-\zeta),\quad \frac{m}{m_0}=1-\frac{\alpha^2+2}{\alpha^3}(1-\zeta)^{1/2}+O(1-\zeta),$$
$$a^2=m_0^2R_0^2\frac{\alpha^2}{2(\alpha^2+2)}+O((1-\zeta)^{1/2}),\quad b^2=R_0^2\frac{\alpha^2}{2(\alpha^2+2)}+O((1-\zeta)^{1/2}). \tag{3.7}$$

Expressing $(1-\zeta)^{1/2}$ in terms of $m-m_0$, we obtain for R/R_0, a^2, and b^2 analytic series in powers of $m=m_0$ that contain no singularities on the light horizon. Formally, the continuation through the horizon corresponds in this case to a change in the sign in front of $(1-\zeta)^{1/2}$, i.e., to the transition to a different branch, ζ formally remaining less than unity outside the light horizon as well.

The solutions with the asymptotic behaviors (2.9) can also be continued analytically through the light horizon.

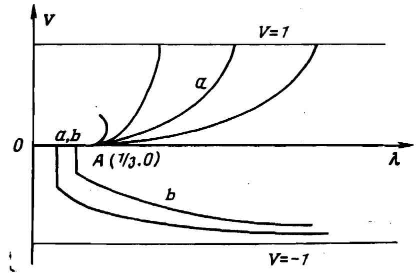

FIG. 4. Graphs of $V=V(\lambda)$ for solutions with weak and strong discontinuities in the presence of static core: a) typical solution with weak discontinuity, b) typical solution with strong discontinuity.

For the functions $R(\zeta)$, $m(\zeta)$, $a^2(\zeta)$, and $b^2(\zeta)$ we obtain in this case expansions in powers of $\zeta-1$:

$$\frac{R}{R_0}=1-\frac{q_0-2}{(q_0-1)\alpha}(\zeta-1)+O[(\zeta-1)^2],$$
$$\frac{m}{m_0}=1+\frac{2q_0(q_0-2)}{(q_0^2-1)\alpha}(\zeta-1)+O[(\zeta-1)^2], \qquad (3.8)$$
$$a^2=m_0^2R_0^2\alpha\frac{(q_0+1)^2}{2q_0}+O(\zeta-1). \quad b^2=R_0^2\alpha\frac{(q_0-1)^2}{2q_0}+O(\zeta-1).$$

Expression $\zeta-1$ in terms of $m-m_0$, we obtain analytic series in powers of $m=m_0$ for the unknown functions. The solution outside the light horizon is given by Eqs. (3.8) for $\zeta>1$.

Thus, the continuous and discontinuous self-similar motions of an ultrarelativistic gas described above indicate that all the continuous self-similar motions of the gas have a cosmological nature since in accordance with (3.7) they can be analytically continued outside the light horizon. This one-parameter class of solutions, which includes the Friedmann–Lemaitre solution, is, generally speaking, a set of inhomogeneous expanding cosmological models (see (2.6) and (2.10)). The parts of the curves of this class up to the sonic line (2.11) occur as central cores in the larger two-parameter class of solutions that have weak discontinuities. (In the ζ, X, V space the sonic spheres are represented by points on the sonic line (2.11).) To every central core one can join any solution of the two-parameter class of solutions across the sonic sphere. The analytic continuation beyond the light horizon of each such solution with weak discontinuity in the comoving coordinate system is given by Eqs. (3.8). These solutions are cosmological inhomogeneous

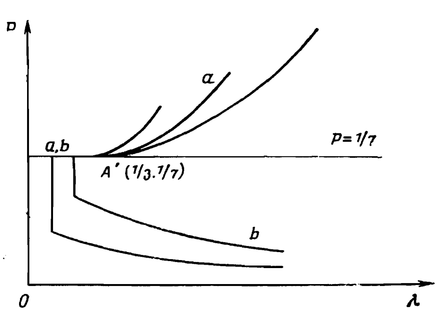

FIG. 5. Graphs of $P=P(\lambda)$ for solutions with weak and strong discontinuities in the presence of a static core.

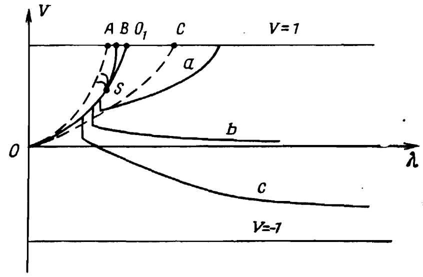

FIG. 6. Graphs of $V=V(\lambda)$ for solutions with weak and strong discontinuities in the presence of a homogeneous Friedmann core. Typical curves with strong discontinuity: 1) Oa, a solution with light horizon, 2) Ob, in which $V\to 0$ as $\lambda\to\infty$, 3) Oc corresponds to focusing toward the center as $\lambda\to\infty$. On the Friedmann curve OO_1 there is a point of weak discontinuity $S(\sqrt{3}/2,\ 1/\sqrt{3})$; OSB is a typical curve with weak discontinuity. Inhomogeneous cosmological solutions: $OA(P_1=0.125)$, $Oc(P_1=0.75)$.

models with weak discontinuity across the sonic characteristic. They have greater generality than the class of regular cosmological solutions (2.6) and (2.10) since they depend on two parameters.

In contrast to the solutions with weak discontinuity, in the solutions with shock waves of sufficiently high intensity there is no light horizon. Within the shock wave, the solution is described by a piece of one of the solutions (2.6) and (2.10) or by the static solution. Thus, homogeneity and expansion of the matter around the observer do not by themselves guarantee homogeneity of the Universe as a whole. At a sufficiently great distance from the center, the shock wave may lead to a complete change in the structure of the solution with the matter pressure decreasing to zero at infinity. In other words, homogeneously expanding regions (pieces of Friedmann universe) can be formed as a result of the focusing of gas toward the center of a star with a pseudo-Euclidean asymptotic behavior preserved at infinity.

In their interesting paper,[16] Carr and Hawking investigated the problem of the self-similar formation of nonstationary black holes in cosmological models that tend at large distances to the flat Friedmann model. They used the comoving coordinate system. They showed that the event horizon for the black hole is inside the light horizon if the latter exists, and that the asymptotic approach to the Friedmann model occurs outside the light horizon. Our investigation corresponds to the opposite case of solutions that are nonsingular within the light horizon.

In the case of subcritical intensity of a strong discontinuity, the solutions behind the shock wave have a light horizon. Such solutions correspond to self-similar inhomogeneous universes with shock waves. Their analytic continuation beyond the horizon is given in Eqs. (3.8).

Figures 4–7 show the results of numerical integration of the systems (1.9) and (1.10) with the self-similar variable $\lambda\equiv r/ct$ plotted along the abscissa. Figures 4 and 5 show the solutions for the velocity and the pressure with strong and weak discontinuities that have a static

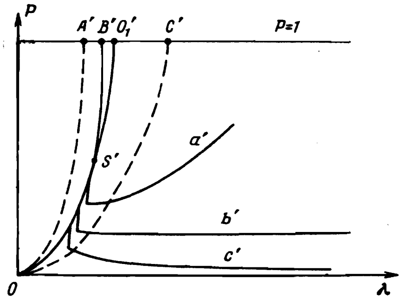

FIG. 7. The graphs of $P=P(\lambda)$ corresponding to the curves of Fig. 6.

core. Figures 6 and 7 show the solutions for the velocity and pressure with strong and weak discontinuities that have a uniformly expanding core (a piece of the Friedmann universe). In the same figures we show the inhomogeneous cosmological solutions (2.6) and (2.10) for the values of 0.125 and 0.75, respectively, of the parameter P_1.

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Particle-like solutions of the scalar Higgs equation

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The properties of particle-like solutions of the scalar Higgs equation are considered. These solutions should correspond in the Vinciarelli-Drell model to gluon-type mesons that contain no quark-antiquark pair.

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I. INTRODUCTION

According to present indications, the effective mass of the quarks inside hadrons is small; in particular, for non-strange quarks it turns out to be of the order of 5–10 MeV.[1] On the other hand, free quarks either do not exist at all or their concentration in matter and their production cross section are small.[2]

This situation makes it necessary to resort to models in which the quarks interact with fields of the boson type, and this interaction is such that a small region $\sim 10^{-13}$ cm is produced, in which the quarks move with velocity $\sim c$ and with a mass on the order of zero, whereas in the remaining space they can either not be present at all, or have their very large effective mass. Models of the first type[3] correspond to absolute confinement, while in the model of the second type the free quarks should be observable. If the data of LaRue, Fairbank, and Hebbard[4] were to be confirmed, then we should give preference to the models of the second type. The simplest of these models is the Vinciarelli-Drell model,[5] in which the quark field q interacts with the scalar Hicks field u described by the equation[1)]

$$\Box u = 4\lambda^2 u(\eta^2 - u^2). \qquad (1.1)$$

For (1.1), vacuum corresponds to $u = \pm\eta$. If the interaction of the quark with this field takes the form $fu\bar{q}q$, then the quark outside the hadron has a mass $f\eta$, and if $f\eta \gg 1$ GeV, then the free quark has a small effective mass.

Since arguments exist favoring the assumption that the interaction of the quarks is due to colored Yang-Mills

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