where $v_E(0)=c\overline{E}_{10}(0)/B_s$ is the electric drift velocity at $\tau=0$.

2) *Stage of a growing field in the kinetic regime corresponding to the time interval* $\tau_1<\tau<\tau_2$. In the above mentioned case of a constant rate of injection

$$\tau_2>\frac{\Delta v}{v_0 q^2}\left(\frac{\partial I}{\partial u_0}\right)^{-1}. \qquad (51)$$

In particular, for a distribution of the form $I=\overline{a}e^{-(v-1)^2/b^2}$ which was considered in Ref. 10, $\tau_2\approx b^2/\overline{a}$; $\Delta v/v_0\approx b$.

We note also that $\overline{E}_{10}(\tau_2)\approx B_0 v_0 b^2/c$. In such a field the electric drift velocity $v_E(\tau_2)\approx v_0 b^2$ so that an ion during a period of oscillations drifts along the small torus radius over a distance of the order of $\Delta a=Rb^2$.

3) *Stage of the hydrodynamical evolution starting for* $\tau>\tau_2$. For a study of this state of the evolution of a monochromatic Alfvén wave one must develop a non-linear theory of the hydrodynamic Alfvén instability which was discussed in the linear approximation in Ref. 9.

According to what has been said in the foregoing, the presence of a source of resonant particles thus affects considerably the non-linear evolution of a monochromatic wave. The main effect displayed when there is a source present consists in a systematic change in the wave amplitude which is qualitatively different from the effect of damped oscillations of the amplitude when there is no source present.

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[1)] This interpretation is due to V. D. Shapiro.

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The phenomenon of parametric trapping of electromagnetic waves in an inhomogeneous plasma

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A theory is developed of absolute parametric aperiodic instability in a spatially inhomogeneous plasma, when the electromagnetic waves generated in the plasma are trapped by the plasma near the peaks of the pumping-wave field.

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1. The present paper is devoted to the theory of the phenomenon of electromagnetic-wave trapping by a plasma. The essence of such a phenomenon consists in the fact that the secondary electromagnetic waves parametrically excited under the action of a pumping electromagnetic wave do not get out of the plasma, but are trapped inside it. It may be expected that such a phenomenon is one of the causes of the reduction in reflection of electromagnetic waves by a plasma during some short interval of time.

As a specific example of the appearance of the trapping phenomenon, below we consider parametric instability in an inhomogeneous plasma, during which the pumping wave gets transformed into an electromagnetic wave and perturbations aperiodically growing in time.[1)] It is shown in the process that, after the intensity of the electric field of the pump exceeds some threshold value in the spatially inhomogeneous plasma, the development of absolute parametric instability becomes possible. The growing—in time—plasma perturbations are localized near the peaks of the electric field of the pumping wave. The regions of such localization are small compared to the characteristic dimension of the pump inhomogeneity in the plasma. Therefore, the secondary electromagnetic waves do not, with exponential accuracy, get out of the small regions of their localization, and are, consequently, not scattered by the plasma. This allows us to assert that, as a result of the parametric instability studied below, there occurs trapping of the electromagnetic radiation by the plasma. Such trapping is essentially connected with the spatial structure of the pump field, and cannot, in the considered model, arise in the field of a running wave. However, parametric trapping turns out to be possible even in the case of very small reflection coefficients.

The theoretical model used below presupposes an electron density of the inhomogeneous plasma linearly depending on the coordinate x:

$$n(x)=n_c(1+xL_N^{-1}),\quad x\geqslant -L_N, \tag{1.1}$$

where $n_c=\omega_0^2 m_e/4\pi e^2$ is the critical density of the plasma, e and m_e are the electron charge and mass, and ω_0 is the pump frequency. Let us represent the pumping-wave's electric field, which we shall assume to be directed along the y axis, in the form

$$E_0(x,t)={}^1/_2[E_0(x)\exp(-i\omega_0 t)+E_0^*(x)\exp(i\omega_0 t)]. \tag{1.2}$$

In the simplest case of negligibly small absorption of the pumping wave in the plasma we have, in accordance with (1.1) and according to Ref. 3, that

$$E_0(x)=AE(0)\,\mathrm{Ai}\,(-\xi),\quad \xi=-(x/L_E), \tag{1.3}$$

where $\mathrm{Ai}(-\xi)$ is the Airy function, $E(0)$ is the intensity of the electric field of the pump in a vacuum, $A=2(\omega_0 \times L_N/c)^{1/6}$ is the distension factor for the field in the inhomogeneous plasma, $L_E=(\lambda\!\!\!^{-}{}_0^2 L_N)^{1/3}$ is the scale of the field inhomogeneity, $\lambda\!\!\!^{-}{}_0=c/\omega_0$ being the wavelength of the pump in a vacuum. In the case of total neglect of the absorption the pump field (1.2) corresponds to a standing electromagnetic wave.

2. The localization of the plasma perturbations in the vicinities of the peaks of the pump field (1.3) requires the wavelength of such perturbations to be small in comparison with the wavelength of the pump. Denoting the wave vector, $\xi^{1/2}L_E^{-1}$, of the pumping wave by $k_0(x)$ and the x component of the wave vector of the perturbations by k_x, we have the inequality $k_x\gg k_0$. This inequality, for one thing, allows us to use the geometrical-optics approximation; for another, it allows us, in writing down the eikonal equation for the parametric instability due to the nonlinear transformation of the pumping wave into an electromagnetic wave and an aperiodic perturbation, to use immediately the pertinent result of the theory of the spatially homogeneous plasma (see Ref. 4):

$$\begin{aligned}\varepsilon^l(\omega,\mathbf{k})&={}^1/_4 v_E^2(x)(k_x^2+k_z^2)\delta\varepsilon_e^l(\omega,\mathbf{k})\\ &\times[1+\delta\varepsilon_i^l(\omega,\mathbf{k})]\{[\mathbf{k}^2c^2-(\omega-\omega_0)^2\varepsilon^{tr}(\omega-\omega_0,\mathbf{k})]^{-1}\\ &+[\mathbf{k}^2c^2-(\omega+\omega_0)^2\varepsilon^{tr}(\omega+\omega_0,\mathbf{k})]^{-1}\}.\end{aligned} \tag{2.1}$$

Here

$$\mathbf{k}=(k_x(x),k_y,k_z),\quad v_E(x)=(AeE(0)/m_e\omega_0)\,\mathrm{Ai}\,(-\xi)$$

is the oscillation velocity of an electron in the pumping-wave field, and

$$\varepsilon^l(\omega,\mathbf{k})=1+\sum\delta\varepsilon_\alpha{}^l(\omega,\mathbf{k}),\quad \varepsilon^{tr}(\omega,\mathbf{k})=1+\sum\delta\varepsilon_\alpha{}^{tr}(\omega,\mathbf{k}),$$

where $\delta\varepsilon_\alpha^l$ and $\delta\varepsilon_\alpha^{tr}$ are the partial longitudinal and transverse permittivities of the α-type particles.

Setting as our problem the determination of the instability boundary, let us set $\omega=i\gamma=0$. Then the eikonal equation (2.1) can, under the assumption that the wavelength of the plasma perturbations significantly exceeds the ion Debye radius r_{Di}, be written in the form

$$\begin{gathered}k_x^4+2p(x)k_\perp^2k_x^2+k_\perp^4q(x)=0;\\ p(x)=1-\frac{v_E^2(x)\omega_{Le}^2(x)}{4c^2k_\perp^2v_{Te}^2\beta},\quad q(x)=1+\frac{\varkappa^4}{k_\perp^4}-\frac{v_E^2(x)\omega_{Le}^2(x)k_z^2}{2k_\perp^4c^2v_{Te}^2\beta}.\end{gathered} \tag{2.2}$$

Here

$$v_{Te}^2=T_e/m_e,\quad k_\perp^2=k_y^2+k_z^2,\quad \beta=1+r_{Di}^2r_{De}^{-2},$$
$$\varkappa^2=2\gamma_t\omega_0/c^2,\ \gamma_t={}^1/_2\nu_{ei}\omega_{Le}^2(x)\omega_0^{-2}$$

is the damping constant of the transverse wave in the plasma, ν_{ei} is the electron-ion collision rate for a completely ionized plasma, ω_{Le} is the Langmuir electron frequency, and r_{De} is the electron Debye radius. In writing down Eq. (3.3), we assume that the inequality $\varkappa^2\gg|\mathbf{k}|k_0(x)$ is fulfilled at the places where the decrement γ_t is substantial.

The four solutions, $\pm k_1$, $\pm k_2$, of the eikonal equation have the form

$$k_{1,2}(x)=k_\perp[-p\pm(p^2-q)^{1/2}]^{1/2}. \tag{2.3}$$

These solutions can be purely real in some finite x-value ranges, where the plasma perturbations turn out to be spatially localized. In fact, there exist a few essentially different possibilities of appearance of such regions of localization. In the simplest case a region of localization is bounded by reversal points; one of the solutions, $k_x(x)$, is real inside such a region, vanishes on passing through the reversal points, and then becomes purely imaginary. From (2.1) it is clear that the reversal points are determined by the condition

$$q(x)=0. \tag{2.4}$$

Another possibility is connected with the limitation of the region of localization by the transformation points, when inside this region all the roots of the eikonal equation (2.2) are purely real, at the transformation points these roots turn out to be equal, while outside the transformation region the roots become complex. The last property ensures the exponential decay of the plasma perturbations outside the region of localization. The transformation points are determined by the equation

$$p^2(x)=q(x). \tag{2.5}$$

Finally, the third possibility is connected with the limitation of the region of localization both by the reversal points for one of the roots of the eikonal equation (2.2) and by the transformation points, at which the roots turn out to be equal (cf. Ref. 5).

In conclusion of this section, let us introduce, in accordance with the assumption, which proves below to be justified, that the dimensions of the region of localization are small compared to the scale L_E, the following approximation (cf. Ref. 6):

$$v_E^2(x)=v_E^2(m)[1-\xi_m(\xi-\xi_m)^2], \tag{2.6}$$

where, according to (1.3),

$$v_E^2(m)=A^2\,\mathrm{Ai}^2(-\xi_m)e^2E^2(0)(m_e\omega_0)^{-2},$$

while ξ_m corresponds to the extremum points of the Airy function.

3. Let us begin the determination of the instability boundary with the case of localization of the plasma perturbations by reversal points. A reversal point exists for perturbations with an x wave-vector component, $k_1(x)$, determined by the formula (2.3). According to the approximation (2.6), Eq. (2.4) leads to the following expression for the coordinates of such points:

$$\xi_t^{(\pm)}-\xi_m=\pm\Delta\xi_t=\pm(\xi_m V)^{-1/2}(V-k_\perp^4-\mathfrak{k}^4)^{1/2}, \tag{3.1}$$

where

$$V=v_E^2(m)\,\omega_{Le}^2(m)\,k_z^2/2\beta v_{Te}^2c^2. \tag{3.2}$$

For the localization of the plasma perturbations to be possible, the coordinates of the reversal points should not assume purely imaginary values. This leads to the inequality

$$V>k_\perp^4+\mathfrak{k}^4, \tag{3.3}$$

which corresponds to the requirement that the pumping-wave intensity be sufficiently high and guarantees the existence of purely real reversal points.

Let us note further that, according to the formula (2.3) for $k_1(x)$, the function $q(x)$ should not be positive between the reversal points. Accordingly, the asymptotic dependence of the perturbations on the coordinates, which corresponds to spatial localization, is possible only when the function $p(x)$ turns out to be nonnegative at the reversal points. Then using the expressions

$$2k_z^2k_\perp^2p(\xi)=k_z^4-k_y^4-\mathfrak{k}^4+k_\perp^4q(\xi), \tag{3.4}$$

$$k_\perp^4q(\xi)=-[V-k_\perp^4-\mathfrak{k}^4][1-(\xi-\xi_m)^2(\Delta\xi_t)^{-2}], \tag{3.5}$$

which can be written down with the aid of the formulas

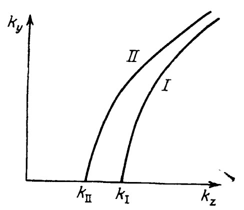

FIG. 1. Instability regions in wave-number space.

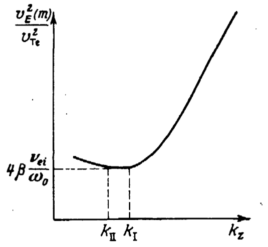

FIG. 2. Threshold of plasma instability with respect to parametric capture.

(3.1) and (3.2), we can assert that it is possible to speak of localization by the reversal points only for perturbations with wave-vector components satisfying the condition

$$k_z^4\geqslant k_y^4+\mathfrak{k}^4. \tag{3.6}$$

According to Fig. 1, the region of such **k** values lies to the right of the curve I. When (3.3) and (3.6) are fulfilled, we have for the plasma perturbations the following dispersion equation:

$$k_\perp L_E\int_{-\Delta\xi_t}^{+\Delta\xi_t}d(\xi-\xi_m)[-p(\xi)+(p^2(\xi)-q(\xi))^{1/2}]^{1/2}=\pi\left(n+\frac{1}{2}\right) \tag{3.7}$$

corresponding to the geometrical-optics approximation, and determining the threshold value of the pump-field intensity (n is a whole number).

A simple solution, which is not too close to the curve I of Fig. 1, to the dispersion equation (3.7) can be written down when

$$k_z^4-k_y^4-\mathfrak{k}^4\gg4(n+1/2)^{2/3}\xi_m^{1/3}L_E^{-2/3}k_z^2(k_\perp^4+\mathfrak{k}^4)^{1/3}. \tag{3.8}$$

In this case, for the threshold pump field we have

$$\frac{v_E^2(m)}{v_{Te}^2}=\frac{2\beta c^2(k_\perp^4+\mathfrak{k}^4)}{k_z^2\omega_{Le}^2(m)}\left\{1+\frac{(2n+1)(\xi_m)^{1/2}}{|k_z|L_E}\left[\frac{2k_z^2k_\perp^2}{k_\perp^4+\mathfrak{k}^4}-1\right]^{1/2}\right\}. \tag{3.9}$$

Accordingly, for the dimension of the region of localization of the plasma perturbations we obtain

$$\Delta x_t(m)=2L_E\Delta\xi_t=2L_E\left[\frac{2n+1}{\xi_m^{1/2}|k_z|L_E}\right]^{1/2}\left(\frac{k_z^4-k_y^4-\mathfrak{k}^4}{k_\perp^4+\mathfrak{k}^4}\right)^{1/4}. \tag{3.10}$$

The smallness of this dimension in comparison with $L_E\xi_m^{-1/2}$ is guaranteed owing to the inequality $k_\perp\gg k_0$.

In Fig. 2 we show the limiting curve $v_E^2(m)/v_{Te}^2$ plotted against $k_\perp$ for the case $k_y=0$. To the formula (3.9) corresponds the segment of the limiting curve to the right of $k_{\mathrm{I}}=\mathfrak{k}$, except for a small neighborhood of this point. For the description of such a small neighborhood let us use a consequence of the dispersion equation (3.7):

$$\frac{v_E^2(m)}{v_{Te}^2}=\frac{2\beta c^2(k_\perp^4+\mathfrak{k}^4)}{k_z^2\omega_{Le}^2(m)}\left\{1+\left[\frac{3\pi^{3/2}(n+1/2)(2\xi_m)^{1/2}}{\Gamma^2(1/4)L_E(k_\perp^4+\mathfrak{k}^4)^{1/4}}\right]^{4/3}\right\}, \tag{3.11}$$

which obtains when

$$k_z^4-k_y^4-\mathfrak{k}^4\ll2k_z^2[3\pi^{3/2}(n+1/2)(2\xi_m)^{1/2}\Gamma^{-2}(1/4)L_E^{-1}(k_\perp^4+\mathfrak{k}^4)^{1/2}]^{2/3}. \tag{3.12}$$

Here $\Gamma(\frac{1}{4})=3.6256$. To the formulas (3.11) and (3.12) corresponds the following dimension of the region of localization:

$$\Delta x_t(m)=\frac{2L_E}{\xi_m^{1/2}}\left[\frac{3\pi^{1/2}(n+{}^1/_2)\,(2\xi_m)^{1/2}}{\Gamma^2({}^1/_4)L_E(k_\perp^4+\mathfrak{k}^4)^{1/4}}\right]^{2/3}. \qquad (3.13)$$

The smallest threshold pump value for plasma perturbations localized by reversal points near the m-th extremum of the pump field obtains when $k_z=k_\perp=\mathfrak{k}$, and is determined by the formula

$$\frac{v_E^2(m)}{v_{Te}^2}=4\beta\frac{\nu_{ei}}{\omega_0}\left\{1+\left[\frac{3\pi^{1/2}(n+{}^1/_2)\,(2\xi_m)^{1/2}}{2^{1/4}\Gamma^2({}^1/_4)L_E\mathfrak{k}}\right]^{4/3}\right\}. \qquad (3.14)$$

The right-hand side of this formula differs from the threshold formula obtained in the theory of the homogeneous plasma[2] by the presence of the small term in the curly brackets. Let us emphasize that it is precisely the presence of this small term that determines the dimension and, consequently, the very existence of the region of spatial localization of the plasma perturbations. The smallness of the correction term in the formula (3.14) requires the fulfillment of the condition

$$1\ll\mathfrak{k}^2L_E^2\leqslant(\nu_{ei}/\omega_0)\,(L_N/\lambda_0)^{2/3}, \qquad (3.15)$$

which can be fulfilled in the case of a sufficiently large dimension of the spatial inhomogeneity of the plasma.

4. In this section we shall consider the possibility of parametric trapping accompanied by the excitation of plasma perturbations localized by the transformation points. According to Eq. (2.5), the coordinates, $\xi_b^{(\pm)}$, of the transformation points are determined by the formula

$$\xi_b^{(\pm)}-\xi_m=\pm\Delta\xi_b=\pm\frac{1}{\xi_m^{1/2}}\left[1-2k_z^2\frac{k_y^2+(k_y^4+\mathfrak{k}^4)^{1/2}}{V}\right]^{1/2}. \qquad (4.1)$$

Since in this case for the coefficients of Eq. (2.2) we have

$$p(x)=p(\xi_b)+\frac{V\xi_m}{2k_\perp^2k_z^2}[(\xi-\xi_m)^2-\Delta\xi_b^2],\quad p(\xi_b)=\frac{k_z^2-(k_y^4+\mathfrak{k}^4)^{1/2}}{k_\perp^2},$$
$$q(x)=p^2(\xi_b)+k_\perp^{-4}V\xi_m[(\xi-\xi_m)^2-\Delta\xi_b^2],$$

the realness of the roots of the eikonal equation (2.2) in the entire region between the transformation points requires the fulfillment of the condition

$$k_z^4\leqslant k_y^4+\mathfrak{k}^4 \qquad (4.2)$$

and the right inequality in the following formula:

$$2k_z^2[k_y^2+(k_y^4+\mathfrak{k}^4)^{1/2}]<V\leqslant k_\perp^4+\mathfrak{k}^4. \qquad (4.3)$$

The left inequality in (4.3) ensures the location of the transformation points on the real ξ axis. When the conditions (4.2) and (4.3) are fulfilled, we can write the following dispersion equation for the plasma perturbations:

$$L_E\int_{-\Delta\xi_b}^{+\Delta\xi_b}d(\xi-\xi_b)\,[k_1(\xi)-k_2(\xi)]=(2n+1)\pi. \qquad (4.4)$$

Let us emphasize that, according to the inequality (4.2), this equation describes perturbations with wave vectors lying to the left of the curve I of Fig. 1. Not too close to such a curve, when

$$\frac{(k_y^4+\mathfrak{k}^4)^{1/2}-k_z^2}{(k_y^4+\mathfrak{k}^4)^{1/2}+k_y^2}\gg2\left\{\frac{(2n+1)\,\xi_m^{1/2}(k_y^4+\mathfrak{k}^4)^{1/4}}{L_E[k_y^2+(k_y^4+\mathfrak{k}^4)^{1/2}]}\right\}^{2/3}, \qquad (4.5)$$

we obtain from Eq. (4.4) for the instability boundary the expression

$$\frac{v_E^2(m)}{v_{Te}^2}=\frac{4c^2\beta}{\omega_{Le}^2(m)}[k_y^2+(k_y^4+\mathfrak{k}^4)^{1/2}]$$
$$\times\left\{1+\frac{(2n+1)\,(2\xi_m)^{1/2}}{L_E(k_y^4+\mathfrak{k}^4)^{1/4}}\left[\frac{(k_y^4+\mathfrak{k}^4)^{1/2}-k_z^2}{(k_y^4+\mathfrak{k}^4)^{1/2}+k_y^2}\right]^{1/2}\right\}. \qquad (4.6)$$

Correspondingly, for the region of localization we have

$$\Delta x_b(m)=2L_E\Delta\xi_b=\frac{2L_E}{\xi_m^{1/2}}\left[\frac{(2n+1)\,(2\xi_m)^{1/2}}{L_E(k_y^4+\mathfrak{k}^4)^{1/4}}\right]^{1/2}\left[\frac{(k_y^4+\mathfrak{k}^4)^{1/2}-k_z^2}{(k_y^4+\mathfrak{k}^4)^{1/2}+k_y^2}\right]^{1/4}. \qquad (4.7)$$

According to the formula (4.6), the threshold pump field decreases as the curve I of Fig. 1 is approached. Therefore, let us give here the results for the case opposite to the case (4.5), when

$$\frac{(k_y^4+\mathfrak{k}^4)^{1/2}-k_z^2}{(k_y^4+\mathfrak{k}^4)^{1/2}+k_y^2}\left[\frac{4L_Ek_\perp^2}{3\pi(2n+1)\xi_m^{1/2}|k_z|}\right]^{2/3}-1\ll1. \qquad (4.8)$$

In this case for the threshold field and the region of localization we have

$$\frac{v_E^2(m)}{v_{Te}^2}=\frac{4c^2\beta}{\omega_{Le}^2(m)}[k_y^2+(k_y^4+\mathfrak{k}^4)^{1/2}]\left\{1+\left[\frac{3\pi(2n+1)\,\xi_m^{1/2}}{2^{11/4}L_E(k_\perp|k_z|)^{1/2}}\right]^{4/3}\right\}, \qquad (4.9)$$

$$\Delta x_b(m)=L_E\left[\frac{3\pi(2n+1)}{2^{5/4}\xi_m^{1/2}L_E(k_\perp|k_z|)^{1/2}}\right]^{2/3}. \qquad (4.10)$$

It can be seen from the formulas (4.7) and (4.10) that the smallness of the region of localization is guaranteed owing to the condition $k_\perp\gg k_0$.

The smallest threshold field for the excitation of plasma perturbations localized by transformation points occurs for the value $k_y=0$. In Fig. 1, the region of such instability is located to the left of the curve II, which corresponds to the equation to zero of the left-hand side of the inequality (4.8). The outermost right point of such a region on the abscissa axis is given by the formula

$$k=k\equiv\mathfrak{k}\left\{1-\frac{1}{2}\left[\frac{3\pi(2n+1)\,\xi_m^{1/2}}{4\mathfrak{k}L_E}\right]^{2/3}\right\}. \qquad (4.11)$$

The same point in Fig. 2 limits the region located to the left of it, and corresponding to the localization of the perturbations by the transformation points.

5. In the small region between the curves I and II in Fig. 1 the locking up of the plasma perturbations in the inhomogeneous plasma occurs in the following manner. From the reversal point $\xi_t^{(+)}$ a wave with wave vector k_2 propagates towards the region of higher ξ values. At the transformation point $\xi_b^{(+)}$ ($\xi_b^{(+)}>\xi_t^{(+)}$) the wave is transformed into a wave with wave vector k_1, which then propagates right up to the transformation point $\xi_b^{(-)}$.

Here again there occurs a transformation into a wave with wave vector k_2, which propagates to the reversal point $\xi_t^{(-)}$ and reflects from it. After this the process of transformation and reflection of the waves is repeated. For the thus generated standing wave, bearing in mind the even dependence of the functions p and q on $\xi - \xi_m$, we can write a dispersion equation in the following form (cf. Ref. 5):

$$k_\perp L_E \left\{ \int_0^{\Delta\xi_b} d(\xi-\xi_m)\left[-p+(p^2-q)^{1/2}\right]^{1/2} - \int_{\Delta\xi_t}^{\Delta\xi_b} d(\xi-\xi_m) \times\left[-p-(p^2-q)^{1/2}\right]^{1/2}\right\} = \pi\left(n-\frac{1}{4}\right). \qquad (5.1)$$

The location, necessary for such a dispersion equation, of the reversal and transformation points on the real ξ axis requires the fulfillment of the inequality (3.3), which is opposite to the right inequality in (4.3). At the same time the condition for realness of $k_2(x)$ leads to the condition (4.2). Therefore, firstly, the dispersion equation (5.1) is valid in the region to the left of the curve I in Fig. 1. Secondly, in the region to the left of the curve II in Fig. 1 it cannot lead to threshold pump-field values smaller than the values arising from the dispersion equation (4.4). Therefore, with the aid of the quantization rules (5.1) we can determine the threshold—for instability—pump field only between the curves I and II of Fig. 1, thereby joining the asymptotic formulas (3.11) and (4.9). In Fig. 2, Eq. (5.1) depicts the segment of the curve between the points $k_{\rm I}$ and $k_{\rm II}$.

Notice that the limit of the formula (3.11) and of the curve I in Fig. 1 and the formula (4.9) differ from each other in the numerical coefficients in front of the small correction terms in the curly brackets. In this case the values of such coefficients differ only by roughly 50%. This allows us to limit ourselves here to the establishment of the fact that the dispersion equation (5.1) allows us to find the threshold pump field in the region between the curves I and II in Fig. 1.

6. When the intensity of the pump field exceeds the values corresponding to the instability threshold, the plasma perturbations grow in time with increment γ. Near the threshold, when $\gamma \ll |\mathbf{k}| v_{Ti}$, v_{Ti} being the thermal velocity of the ions, the eikonal equation has the form (2.2) with γ_t replaced by $\gamma_t+\gamma$. Therefore, the formulas for the increment can be directly obtained from the above-found formulas for the parametric-instability threshold. Thus, for example, with the aid of the formula (3.9), we obtain

$$\gamma = -\gamma_t + \left\{-\frac{k_\perp^4 c^4}{4\omega_0^2} + \frac{v_E^2(m) c^2 k_z^2}{8\beta v_{Te}^2}\left[1-\frac{(2n+1)\xi_m^{1/2}}{|k_z| L_E}\left(4\beta\frac{v_{Te}^2}{v_E^2(m)} k_\perp^2 \lambda\!\!\!^-{}_0^2 - 1\right)^{1/2}\right]\right\}^{1/2}. \qquad (6.1)$$

On the other hand, for the excitations localized by the transformation points we have, according to (4.6), the expression

$$\gamma = -\gamma_t + \left\{-\frac{k_y^4 c^4}{4\omega_0^2} + \frac{\omega_0^2}{4}\left[\frac{v_E^2(m)}{4 v_{Te}^2 \beta}\left\{1-(2n+1)2(2\xi_m)^{1/2} \times \frac{v_{Te}}{v_E(m)}\left(\frac{\lambda\!\!\!^-{}_0}{L_N}\right)^{1/3}\left(\frac{v_E^2(m)-4k_\perp^2\lambda\!\!\!^-{}_0^2\beta v_{Te}^2}{v_E^2(m)-4k_y^2\lambda\!\!\!^-{}_0^2\beta v_{Te}^2}\right)^{1/2}\right\} - k_y^2\lambda\!\!\!^-{}_0^2\right]^2\right\}^{1/2}. \qquad (6.2)$$

The formulas (6.1) and (6.2) are applicable in regions of k-vector component values lying on the two different sides of the curve

$$k_z^4 = k_y^4 + 4(\gamma+\gamma_t)^2\omega_0^2/c^4$$

and not too close to it. This curve is the analog of the curve I in Fig. 1.

Notice that, if in the formulas (6.1) and (6.2) we set $\beta=1$, then they turn out to be also valid in the case of a nonisothermal plasma, when $|\mathbf{k}| v_{Ti} < \gamma < |\mathbf{k}| v_s$, where v_s is the velocity of sound. For relatively strong fields, when the pump field satisfies the conditions

$$v_{Ti} \ll \omega_{Li}(m)\frac{v_E(m)}{c}\frac{|k_z|}{k_\perp^2} \ll v_{Te}, \quad \frac{k_\perp c^2}{\omega_0},$$

the increment of the excitations trapped by the reversal points is determined by the following formula:

$$\gamma^2 = \frac{v_E^2(m)\omega_{Li}^2(m)}{2c^2}\frac{k_z^2}{k_\perp^2} - \omega_{Li}^2(m) k_\perp^2 r_{De}^2 - \omega_{Li}^2(m)\frac{v_{Te} v_E(m)}{c^2}\left(\frac{\lambda\!\!\!^-{}_0}{L_N}\right)^{1/3}\frac{|k_z|}{k_\perp}(2\xi_m)^{1/2}\left(n+\frac{1}{2}\right)\left(1-\frac{k_y^2 v_E^2(m)}{2c^2 k_\perp^4 r_{De}^2}\right)^{1/2},$$

where ω_{Li} is the ion Langmuir frequency. Correspondingly, for the region of spatial localization we have

$$\Delta x_t(m) = \frac{2L_E}{\xi_m^{1/2}}\left\{(2n+1)(2\xi_m)^{1/2}\frac{k_\perp}{|k_z|}\frac{v_{Te}}{v_E(m)}\left(\frac{\lambda\!\!\!^-{}_0}{L_N}\right)^{1/3}\right\}^{1/2}\left\{1-\frac{k_y^2 v_E^2(m)}{2c^2 k_\perp^4 r_{De}^2}\right\}^{1/4}.$$

In this case the condition

$$1-\frac{k_y^2 v_E^2(m)}{2c^2 k_\perp^4 r_{De}^2} \gg \left(\frac{\lambda\!\!\!^-{}_0}{L_N}\right)^{2/9}\left[\frac{(2\xi_m)^{1/2}(n+{}^1/{}_2)|k_z| v_E(m)}{k_\perp^3 v_{Te}\lambda\!\!\!^-{}_0^2}\right]^{2/3},$$

which, like the condition (3.8), makes it possible to determine the increment in almost the entire region in which the excitations are trapped by the reversal points, should also be fulfilled.

For the excitations trapped by the transformation points we have

$$\gamma^2 = \frac{v_E^2(m)\omega_{Li}^2(m)}{2c^2}\{1-2(2n+1)\psi\} - 2^{1/2}|k_y| v_s\frac{v_E(m)\omega_{Li}(m)}{c}\{1-(2n+1)\psi\},$$

where

$$\psi = (2\xi_m)^{1/2}\left(\frac{\lambda\!\!\!^-{}_0}{L_N}\right)^{1/3}\frac{v_{Te}}{v_E(m)}\left(1-\frac{2^{1/2}k_z^2\lambda\!\!\!^-{}_0 v_{Te}}{v_E(m)|k_y| - 2^{1/2}k_y^2\lambda\!\!\!^-{}_0 v_{Te}}\right)^{1/2}.$$

In this case the region of localization and the corresponding condition of applicability have the form

$$\Delta x_b(m) = \frac{2L_E}{\xi_m^{1/2}}\left\{2(2n+1)(2\xi_m)^{1/2}\frac{v_{Te}}{v_E(m)}\left(\frac{\lambda\!\!\!^-{}_0}{L_N}\right)^{1/3} \times\left[1-\frac{2^{1/2}k_z^2\lambda\!\!\!^-{}_0 v_{Te}}{v_E(m)|k_y| - 2^{1/2}k_y^2\lambda\!\!\!^-{}_0 v_{Te}}\right]^{1/2}\right\}^{1/2},$$

$$\frac{v_E(m)|k_y|\lambda\!\!\!^-{}_0}{2^{1/2}v_{Te}} - k_\perp^2\lambda\!\!\!^-{}_0^2 \gg \left[2(2n+1)(2\xi_m)^{1/2}\frac{v_{Te}}{v_E(m)}\left(\frac{\lambda\!\!\!^-{}_0}{L_N}\right)^{1/3}\right]^{2/3}\left[\frac{v_E(m)|k_y|\lambda\!\!\!^-{}_0}{2^{1/2}v_{Te}} - k_y^2\lambda\!\!\!^-{}_0^2\right]^{1/3}.$$

The expressions for the increment differ by only small corrections from the results of the theory of the homogeneous plasma. At the same time the possibility of spatial localization and the dimension of the region in which the radiation gets trapped are essentially determined by the spatial inhomogeneity. It also follows from the obtained expressions that, for spatial localization to occur, it is necessary that the pump field suppress to a certain extent the thermal motion of the particles, which enlarges the region of localization.

7. The above-presented results indicate the possibility of the phenomenon of parametric trapping in the field of a standing wave. In order to realize what limitations the structure of the field in the actual plasma imposes on such a possibility, let us use a model in which strong absorption of the electromagnetic field occurs in a small region. Then outside such a region, instead of the formula (2.6), we can write

$$v_E^2(x)=\tilde{v}_E^2(m)\left[1-\frac{4R^{1/2}}{(1+R^{1/2})^2}\xi_m(\xi-\xi_m)^2\right].$$

Here $\tilde{v}_E(m)$ is the velocity amplitude of the electron oscillations at the points ξ_m, while R is the energy coefficient of reflection of the pumping wave. It follows from this formula that in, for example, the formulas (3.10), (3.13), (4.7), and (4.10) ξ_m should be replaced by

$$4R^{1/2}(1+R^{1/2})^{-2}\xi_m.$$

In the enumerated formulas the right-hand sides should be small compared to $L_E/\xi_m^{1/2}$. This condition allows us to say how small the reflection coefficient can be and still not exclude the possibility of parametric trapping. In this case such a critical reflection coefficient is always small in comparison with unity only when the wavelength of the secondary electromagnetic waves is sufficiently small compared to the wavelength of the pump in the plasma. Thus, for example, for $k_z^2 \gg \mathfrak{k}^2$, k_y^2, when the formula (3.10) is valid, the reflection coefficient should exceed the value

$$R>16(2n+1)^4\xi_m^2(k_\perp L_E)^{-4}.$$

This quantity can be used to estimate that limiting value that can be assumed during a certain interval of time by the coefficient of reflection of the pumping wave as a result of its transformation into parametrically trapped electromagnetic waves with wave vector **k**. Since, according to (3.9), we have $(k_\perp \lambdabar_0)^4 \sim (v_E(m)/v_{Te})^4 \ll 1$, such a reflection coefficient turns out to be small compared to unity, for example, under conditions when the characteristic plasma-inhomogeneity dimension, L_N, exceeds by three orders of magnitude the pump wavelength $\lambda_0 = 2\pi\lambdabar_0$ in a vacuum.

Finally, let us also touch upon here the question of the fate of the wave trapped by the plasma. Since such a wave in the region of its localization turns out to be a standing wave, upon the attainment of the value, $\tilde{E}$, of the intensity of the localized—in the region—electric field at which the contribution to the electron-oscillation velocity turns out to be of the order of $\tilde{v}_E(m)\sim 2v_{Te}(\beta\nu_{ei}/\omega_0)^{1/2}$, the trapped wave can become the cause of an parametric instability that leads to the trapping of an electromagnetic wave of even shorter wavelength in the region of a peak of the field of the primary trapped wave. Such a process of fractionation of the electromagnetic field trapped by the plasma can go on right down to a scale comparable to the wavelength, λ_0, of the pump.

Since such short-wave radiation is ineffectively absorbed by the plasma particles, it may be inferred that the parametric trapping leads to the reconstruction of the spatial structure of the pumping wave. In this connection, let us note that a soliton-like pumping wave is not unstable against parametric trapping for both the isolated and periodic solitons[7] of the electromagnetic field if their spatial period is less than $\lambdabar_0/\mathscr{E}$, where $\mathscr{E}$ is the soliton amplitude in units of $E_p=(8T_e m_e \omega_0^2/e^2)^{1/2}$.

[1]Such an instability has been studied in the framework of the theory of the spatially homogeneous plasma by Volkov[1] and Kiriĭ.[2]

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