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"Instantons" of higher order

D. E. Burlankov and V. N. Dutyshev

Gor'kiĭ State University
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Solutions have been obtained for the Yang-Mills equations for the gauge group $SU(2)$ in a Euclidean space having a topological characteristic larger than one.

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1. Classical solutions of the Yang-Mills (YM) equations in Euclidean four-space have been the object of numerous theoretical investigations in recent months. The Euclidean signature of the metric allows one to reduce the order of the YM equations and to represent them in the form of duality relations. The solutions of these equations, "instantons," describe in the quasiclassical approximation tunneling transitions between various vacuum states of the YM field in normal (pseudoeuclidean) Minkowski space.[1] More precisely, the instantons are saddle points in the calculation of various Green's functions by means of functional integration methods.

The solution of the YM equations found by Belavin *et al.*[2] in a Euclidean four-space having spherical symmetry (under $O(4)$) (the instanton) turned out to be invariant under the group $O(5)$[3] on account of the conformal invariance of the YM equations. In the action for the free YM field

$$S=\frac{1}{16\pi e^2}\int\langle F_{ij}F_{kl}\rangle g^{ik}g^{jl}g^{1/2}\,d^4x, \tag{1}$$

the metric tensor enters only in the combination

$$(g^{ik}g^{jl}-g^{il}g^{jk})\,g^{1/2}, \tag{2}$$

which is invariant under the substitution

$$g_{ij}\to\lambda(x)\,g_{ij}. \tag{3}$$

Therefore, any solution is a solution not on a particular Riemannian manifold, but on a class of conformally equivalent such manifolds. Since stereographic projection establishes a conformal mapping between the 4-sphere and Euclidean four-space, the solutions of the YM equations in Euclidean space, when mapped by the stereographic projection onto the sphere, will be solutions of the YM equations on the 4-sphere (S^4) and vice versa. The instanton turned out to be a field "constant" (according to the definition introduced by one of us[4]) on the sphere S^4, and is therefore invariant with respect to its symmetry group $O(5)$.

According to the classification of Belavin *et al.*[2] the instanton has topological characteristic 1; at the same time these authors have predicted from topological considerations that there exist instantons with higher topological characteristics, globally defined by the action integral (1) over the whole four-space: for solutions of higher topological type I_n the action integral must be an integral multiple of the integral for the instanton I_1, with the integer n (the multiplicity) characterizing the topological type of the field (the degree of the mapping of S^4 onto $SU(2)\simeq S^3$ which determines the gauge).[1)] But in mapping of the higher degree the symmetry is necessarily lowered, since for such mappings there appear in four-space submanifolds where the solutions branch, which destroy the spherical symmetry. However, if instead of the Euclidean space one considers the solutions on the sphere S^4, one can choose the branching manifold (which has dimension $d-2$, if d is the dimension of space) as a sphere S^2, so that out of the symmetry group $O(5)$ one retains the sufficiently high symmetry $O(3)\otimes O(2)$, allowing one to find the solution.

2. On the sphere S^4 we introduce a coordinate system which explicitly reflects this symmetry, defining

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the metric element of the sphere in the form

$$
\begin{aligned}
ds^2 &= d\chi^2+\cos^2\chi\, d\tau^2+\sin^2\chi\,(d\theta^2+\sin^2\theta\, d\varphi^2)\\
&= \frac{dx^2}{1-x^2}+(1-x^2)\,d\tau^2+x^2(d\theta^2+\sin^2\theta\, d\varphi^2),\\
& \qquad x=\sin\chi.
\end{aligned}
\tag{4}
$$

The angle χ varies from 0 to $\pi/2$ and τ varies from 0 to 2π. The $O(2)$ symmetry corresponds to a translation in τ and the $O(3)$ symmetry to a rotation in the angles θ and φ.

In [4] one can find a description of how to write the potentials in the presence of various symmetries. For this it is necessary to accompany each infinitesimal coordinate change described by the Killing vector field $p_\alpha^i(x)$ (α labels the Killing field) by an infinitesimal gauge transformation

$$
|U_\alpha|=|1|+|q_\alpha(x)|, \tag{5}
$$

with the requirement that these simultaneous transformations should form a group, which leads to the following structure equations for the fields q_α:

$$
p_\alpha q_\beta-p_\beta q_\alpha+[q_\alpha,\ q_\beta]=C_{\alpha\beta}{}^{\gamma} q_\gamma, \tag{6}
$$

where

$$
p_\alpha=p_\alpha{}^i\frac{\partial}{\partial x^i}, \tag{7}
$$

and the structure constants are the same as for the Killing fields:

$$
p_\alpha{}^j p^i_{\beta,j}-p_\beta{}^j p^i_{\alpha,j}=C_{\alpha\beta}{}^{\gamma} p_\gamma{}^i. \tag{8}
$$

In the case of the $O(3)\otimes O(2)$ symmetry, choosing the four Killing operators

$$
\begin{aligned}
p_1&=\sin\varphi\frac{\partial}{\partial\theta}+\operatorname{ctg}\theta\cos\varphi\frac{\partial}{\partial\varphi},\\
p_2&=\cos\varphi\frac{\partial}{\partial\theta}-\operatorname{ctg}\theta\sin\varphi\frac{\partial}{\partial\varphi},\\
p_3&=-\frac{\partial}{\partial\varphi},\quad p_4=\frac{\partial}{\partial\tau},
\end{aligned}
\tag{9}
$$

where p_1, p_2, p_3 are the (antihermitean) angular momentum operators with the usual commutation relations and p_4 commutes with them, we choose the fields q_α to satisfy (6) in the form

$$
\begin{aligned}
& q_i=\tau_i={}^1/_2 i\sigma_i, \qquad i=1,2,3,\\
& q_4=n\tau_r,\quad \tau_r=\sin\theta(\tau_1\cos\varphi+\tau_2\sin\varphi)+\tau_3\cos\theta.
\end{aligned}
\tag{10}
$$

Here σ_i are the Pauli matrices.

The invariance of the YM potentials with respect to transformations with the fields p_α^i and q_α is determined by the equations

$$
\delta_\alpha A_i=\partial_i q_\alpha+[A_i,\ q_\alpha]-p^j_{\alpha,i}A_j-p_\alpha{}^j A_{i,j}=0. \tag{11}
$$

Substituting the Killing fields (9) and the gauge compensation fields (10), we obtain the form of the potentials A:

$$
\begin{aligned}
& A_x=0,\quad A_\tau=-\Phi(x)\tau_r,\\
& A_\theta=\tau_\varphi'-W(x)\tau_\varphi,\quad A_\varphi=-(\tau_\theta'-W(x)\tau_\theta)\sin\theta,
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
&\tau_\theta=\tau_\theta'\cos n\tau-\tau_\varphi'\sin n\tau,\\
&\tau_\varphi=\tau_\theta'\sin n\tau+\tau_\varphi'\cos n\tau,\\
&\tau_\theta'=\cos\theta(\tau_1\cos\varphi+\tau_2\sin\varphi)-\tau_3\sin\theta,\\
&\tau_\varphi'=\tau_1\sin\varphi-\tau_2\cos\varphi.
\end{aligned}
\tag{13}
$$

The YM field strengths can be expressed in terms of two functions, Φ and W of the variable $x=\sin\chi$,

$$
\begin{aligned}
& F_{\tau x}=\Phi'\tau_r,\quad F_{\theta\varphi}=-(1-W^2)\tau_r\sin\theta,\\
& F_{\tau\varphi}=(\Phi+n)W\tau_\varphi\sin\theta,\quad F_{x\theta}=W'\tau_\varphi,\\
& F_{\tau\theta}=(\Phi+n)W\tau_\theta,\quad F_{x\varphi}=-W'\tau_\theta\sin\theta.
\end{aligned}
\tag{14}
$$

Single-valuedness of the fields on the sphere implies that n must be an integer. The absence of singularities at $x=0$ and $x=1$ implies that

$$
W(0)=1,\qquad \Phi(1)=0. \tag{15}
$$

The branching manifold is the sphere S^2 determined by the condition $\chi=\pi/2$. The self-duality conditions on the sphere[2] can be expressed in terms of the Lamé coefficients for the metric (4):

$$
\begin{aligned}
&\frac{1}{h_\tau h_x}F_{\tau x}=\frac{1}{h_\theta h_\varphi}F_{\theta\varphi},\quad \frac{1}{h_\tau h_\theta}F_{\tau\theta}=-\frac{1}{h_x h_\varphi}F_{x\varphi},\\
&\frac{1}{h_\tau h_\varphi}F=\frac{1}{h_x h_\theta}F_{x\theta},
\end{aligned}
\tag{16}
$$

which after the substitution (14) lead to a system of differential equations for the functions Φ and W

$$
\frac{d\Phi}{dx}=\frac{1-W^2}{x^2},\quad \frac{dW}{dx}=-\frac{(\Phi+n)W}{1-x^2} \tag{17}
$$

with the boundary conditions (15).

Introducing new variables

$$
F=\Phi+n+\frac{1}{x},\quad U=\frac{W(1-x^2)^{1/2}}{x},\quad y=\frac{1}{2}\ln\frac{1+x}{1-x}, \tag{18}
$$

the system (17) reduces to the form

$$
\frac{dF}{dy}=-U^2,\quad \frac{dU}{dy}=-FU, \tag{19}
$$

which yields solutions satisfying the boundary conditions (15):

$$
F_n=(n+1)\operatorname{cth}(n+1)y,\quad U_n=\frac{n+1}{\operatorname{sh}(n+1)y}. \tag{20}
$$

In terms of the variable x, these solutions become for the functions Φ and W

$$
\begin{aligned}
\Phi_n&=(n+1)\frac{(1+x)^{n+1}+(1-x)^{n+1}}{(1+x)^{n+1}-(1-x)^{n+1}}-\frac{1}{x}-n,\\
W_n&=2(n+1)\frac{x(1-x^2)^{n/2}}{(1+x)^{n+1}-(1-x)^{n+1}}.
\end{aligned}
\tag{21}
$$

3. We express the total action of the YM field, substituting the field strengths (14) and making use of the self-duality relations (17)

$$S=\frac{1}{16\pi e^2}\int\frac{\langle F_{ij}^2\rangle}{h_i^2h_j^2}d^4\omega\cdot$$

$$=\frac{\pi}{2e^2}\int_0^1\left(\Phi'^2+2\frac{W'^2(1-x^2)}{x^2}+\frac{(1-W^2)^2}{x^4}+\frac{2(\Phi+n)^2}{x^2(1-x^2)}\right)x^2\,dx \tag{22}$$

$$=\frac{\pi}{e^2}\int_0^1(\Phi'(1-W^2)-2W'(\Phi+n)W)\,dx=\frac{\pi}{e^2}(\Phi+n)(1-W^2)|_0^1=\frac{\pi}{e^2}n.$$

We see that the action is indeed a multiple of π/e^2 as predicted by Belavin *et al.*[2]

The YM field strengths have two independent components:

$$H_\parallel=\frac{|F_{x\tau}|}{h_xh_\tau}=\frac{|F_{\theta\varphi}|}{h_\theta h_\varphi}=\frac{1-W^2}{x^2},$$
$$H_\perp=\frac{|F_{\tau\varphi}|}{h_\tau h_\varphi}=\frac{|F_{\tau\theta}|}{h_\tau h_\theta}=\frac{|F_{x\varphi}|}{h_xh_\varphi}=\frac{|F_{x\theta}|}{h_xh_\theta}=\frac{(\Phi+n)W}{1-x^2}. \tag{23}$$

For $n=1$ the solution (21) reduces to

$$\Phi_1+1=x,\quad W_1=(1-x^2)^{1/2}. \tag{24}$$

Substituting this into (23) we obtain

$$H_\parallel^{(1)}=H_\perp^{(1)}=1. \tag{25}$$

Thus, the solution (24) represents a field for which the field strengths are constant at all points of the 4-sphere. This is the instanton field [2] projected onto S^4 by means of a stereographic projection. For other values of n, as the angle χ varies from 0 to $\pi/2$ the value of $H_\parallel$ varies from $n(n+2)/3$ to 1, and $H_\perp$ varies from $n(n+2)/3$ to 0.

4. In the case of stereographic projection of our solutions into Euclidean space, the result (at $n>1$) will depend on the coordinate χ of the point chosen as the pole of the projection. This coordinate will be denoted by χ_0. If $\chi_0=0$ the projection exhibits spherical symmetry under $O(3)$, which separates the "space" coordinates x_i and "time" t, with the angles χ,τ,σ,φ, being four-dimensional toroidal coordinates, related to the Cartesian coordinates by means of the relations

$$x=\frac{a\sin\chi}{1-\cos\chi\cos\tau}\sin\theta\cos\varphi,\quad y=\frac{a\sin\chi}{1-\cos\chi\cos\tau}\sin\theta\sin\varphi,$$
$$z=\frac{a\sin\chi}{1-\cos\chi\cos\tau}\cos\theta,\quad t=\frac{a\cos\chi\sin\tau}{1-\cos\chi\cos\tau}. \tag{26}$$

Here a is the diameter of the projection sphere.

The metric of the flat four-dimensional space differs from the metric (4) by the conformal multiplier

$$\lambda=\frac{a^2}{(1-\cos\chi\cos\tau)^2}. \tag{27}$$

In this case the solutions obtained here coincide with the solutions of Witten[5] with the potential

$$g=[(a-z)/(a+z)]^{n+1}. \tag{28}$$

If $\chi_0=\pi/2$ the symmetry of the projection is lowered to $O(2)\otimes O(2)$. For other values of χ_0 the field in Euclidean space has an explicit $O(2)$ symmetry, retaining the hidden $O(3)\otimes O(2)$ symmetry.

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1) Translator's note (September 28, 1977): What the authors call "topological characteristic" has been variously called "Pontrjagin index" or "Chern number" in the literature on instantons. For a review of the geometric aspects of this theory, cf. e.g. the translator's "Introduction to the Fiber–Bundle Approach to Gauge Theories," Lecture Notes in Physics, vol. 67, Springer Verlag, Berlin–Heidelberg–New York, 1977. Although in the early literature the "degree of map" interpretation was emphasized, it has become clear that the interpretation as the integral of the second Chern class over S^4 is more natural. Since the article was submitted, connections with self-dual curvatures in $SU(2)$ bundles over S^4 (i.e., self-dual, finite-energy Yang-Mills fields) have been intensely studied in the mathematical literature. A byproduct of this was the proof by M. F. Atiyah, N. J. Hitchin, and I. M. Singer, Proc. Nat. Acad. Sci. (USA) **74**, (1977) that for Chern number n there are exactly $8n-3$ independent instantons, and the description of a series of "Ansätze" by M. F. Atiyah and R. C. Ward, Commun. Math. Phys. **55**, 117 (1977), leading to an explicit construction of solutions, and based on Penrose's twistor techniques, together with a complexification of S^4 to $P^3(C)$, the complex projective space in 3 dimensions (i.e., the space of all complex lines through the origin in C^4, the four-dimensional complex vector space). The solutions found by the authors seem to correspond to Ansatz A_2 of Atiyah and Ward.

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