induced charge exchange effect discussed above. It is evident from formula (5) that Q_{ei} must increase with increasing n_e (though not in direct proportion) since $R_1 \approx \gamma^{-1}\ln(\gamma^2/\varkappa_0)$ while $\varkappa_0 \sim R_1 n_e^{2/3}$. In this case several iterations will be needed to determine the true value of R_1.

The rise of the cross section Q_{ei} essentially compensates the fall of the resonance charge exchange cross section due to Coulomb detuning. Figure 1 shows the charge exchange cross sections for a cesium plasma as functions of the electron density: Q_r is the cross section for two-body charge exchange with allowance for deviation from resonance,[2] Q_{ei} is the electron induced charge exchange cross section, and $Q=Q_r+Q_{ei}$. The cross sections were calculated for the parameter values $v_n=5\cdot10^{-3}$ and $v_e=0.3$. Q_{ei} will be 15 times larger if the atoms are at room temperature. The behavior of the total cross section Q shows that the charge exchange cross section for a one-component plasma varies little over a wide range of electron densities and is roughly equal to the resonance charge exchange cross section.

The effect of the plasma electrons on charge exchange discussed above must be taken into account in measuring charge exchange cross sections for plasmas in the adiabatic velocity region.

In conclusion the authors thank O. B. Firsov, V. S. Lisitsa, and A. V. Chaplik for discussing the work.

1) Several years ago, Lisitsa [1] considered the effect of multiparticle interactions but did not discuss charge exchange.

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A new example of a quantum mechanical problem with a hidden symmetry

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The quantum mechanical problem of the behavior of a neutron in the magnetic field of a linear current is considered. An exact solution of this problem is found. It is found that the system possesses the hidden symmetry $O(3)$. The generators of the symmetry group are constructed. The Schrödinger equation is reduced to an explicitly invariant form.

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1. INTRODUCTION

Several quantum mechanical problems are known in which the degeneracy of the energy levels is stronger than anticipated by starting from the usual spatial symmetry of the system.1) The following pertain to such problems: the oscillator, the Kepler problem, the rotator, and several other problems which do not have an immediate physical interpretation. The additional or "accidental" degeneracy which appears in these problems is associated with the existence of a so-called dynamical symmetry. For example, in addition to the usual rotational symmetry the hydrogen atom possesses the symmetry $O(4)$ (for the discrete spectrum) due to the existence of the conserved Runge-Lenz vector.[2]

Investigating problems in which the internal degrees of freedom of a particle interact with an inhomogeneous field, the authors discovered an example of such a system which possesses a dynamical symmetry: a neutron in the magnetic field of a linear current forms bound states whose spectrum is determined by the dynamical symmetry group $O(3)$. The energy levels are determined by the quantum number n:

$$E_n=-\frac{1}{n^2}\frac{(c\mu I)^2M}{2\hbar^2}, \qquad (1)$$

where M and μ denote the neutron's mass and magnetic moment, I is the current, and the constant c depends on the system of units ($c=0.2$ in the practical system of units).

2. SOLUTION OF THE SCHRÖDINGER EQUATION

A neutral nonrelativistic particle with a magnetic moment is described by the Hamiltonian

$$\mathscr{H}=\mathbf{p}^2/2M-\mu\mathbf{H}, \qquad (2)$$

where $\mathbf{H}$ denotes the external magnetic field. Let us consider the field created by a linear current I directed along the z axis:

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$$\mathbf{H}=0.2I(y/r^2,\ -x/r^2,\ 0) \tag{3}$$

(if r is in centimeters and the current is in amperes, $\mathbf{H}$ will be expressed in gausses). For such a configuration the motion along the z axis is free, and we immediately eliminate the part of the Hamiltonian associated with this motion. For a particle of spin $\frac{1}{2}$ the time-independent Schrödinger equation can be written in the form

$$\left[\frac{1}{2M}(p_x^2+p_y^2)-0.2\mu I(\sigma_x y-\sigma_y x)^{-1}\right]\psi=E\psi. \tag{4}$$

It is convenient to change to dimensionless variables, having made the following scale transformation:

$$\varkappa\mathbf{r}=\boldsymbol{\rho},\quad \mathbf{p}/\varkappa\hbar=\boldsymbol{\pi},\quad \varkappa=(-2ME)^{1/2}/\hbar \tag{5}$$

(we are oriented toward the discrete spectrum). In terms of the new variables Eq. (4) takes the form

$$[\pi^2+1+2n(\sigma_x y-\sigma_y x)^{-1}]\psi=0, \tag{6}$$

where $n=-0.2\mu MI/\varkappa\hbar^2$ and the components of the dimensionless vector $\boldsymbol{\rho}$[2)] are denoted by x and y.

Let us briefly describe the method of solution, the details of which are given in the Appendix. Multiplying the equation by the quantity $\sigma_x y-\sigma_y x$ and changing to the momentum representation, we obtain a system of first-order equations for the components of the spinor ψ. Then, by using the invariance of the Hamiltonian under rotations around the z axis one can separate variables in polar coordinates. Upon the elimination of one of the spinor components we obtain a second-order differential equation which a certain change of variable reduces to the equation for hypergeometric functions. The wave functions of the discrete spectrum are characterized by two quantum numbers n and m. The radial quantum number n takes positive integer values; m denotes the eigenvalues of the operator $J_z=L_z+\sigma_z/2$. For a level with quantum number n the value $|m|\leqslant n-\frac{1}{2}$. In the momentum representation the wave functions have the following form ($m>0$ for the example):

$$\psi=iC_{n,m}\left(\sigma_x\frac{\partial}{\partial\pi_y}-\sigma_y\frac{\partial}{\partial\pi_x}\right)\times\begin{pmatrix}(\pi_x+i\pi_y)^{m-1/2}F\left(n,-n,m+\dfrac{1}{2},\dfrac{\pi^2}{1+\pi^2}\right)\\ \dfrac{n}{m+1/2}\dfrac{(\pi_x+i\pi_y)^{m+1/2}}{1+\pi^2}F\left(n+1,1-n,m+\dfrac{3}{2},\dfrac{\pi^2}{1+\pi^2}\right)\end{pmatrix}, \tag{7}$$

where the $C_{n,m}$ are normalization coefficients. The wave functions of the continuous spectrum appear to be similar; however, the authors did not direct their attention to the problem of scattering.

3. THE HIDDEN SYMMETRY

The spectrum of the system under consideration is cited in the Introduction. It was found to be degenerate with respect to the projection of the total angular momentum. In analogy with the quantum oscillator and the Kepler problem, one can imagine that the additional degeneracy in this case is related to the existence of new integrals of the motion.

Let us rewrite Eq. (6) in the form $(K+1)\psi=0$ where K is, apart from a constant factor, the Hamiltonian of our system, subjected to the scale transformation (5):

$$K=\pi^2+2n(\sigma_x y-\sigma_y x)^{-1}. \tag{8}$$

Let us consider the following set of three operators:

$$\begin{aligned}J_z&={}^1/_2\sigma_z+(x\pi_y-y\pi_x),\\ A_x&=-n(\sigma_x y-\sigma_y x)^{-1}y+{}^1/_2(\pi_x J_z+J_z\pi_x),\\ A_y&=n(\sigma_x y-\sigma_y x)^{-1}x+{}^1/_2(\pi_y J_z+J_z\pi_y).\end{aligned} \tag{9}$$

One of these operators represents the projection of the angular momentum on the z axis. The other two satisfy the following commutation relations:

$$\begin{gathered}[A_x,J_z]=-iA_y,\quad [A_y,J_z]=iA_x,\quad [A_x,A_y]=-iKJ_z,\\ [A_x,K]=0,\quad [A_y,K]=0.\end{gathered} \tag{10}$$

Redefining the operators according to the formula

$$J_x=A_x(-K)^{-1/2},\quad J_y=A_y(-K)^{-1/2}, \tag{11}$$

we obtain the dynamical symmetry group $O(3)$ for the discrete spectrum. The following relationship, established by direct calculation, exists between the generators of this group and the Hamiltonian K:

$$K=-n^2/(\mathbf{J}^2+{}^1/_4). \tag{12}$$

An analogous relationship between the Hamiltonian and the Casimir operator of the dynamical symmetry group also exists in the Kepler problem.[1]

Equation (6) can be rewritten in the form $(\mathbf{J}^2+\frac{1}{4}-n^2)\psi=0$. The properties of the irreducible representations of the rotation group are sufficiently widely known. We are interested in the representations with half-integer projections of the angular momentum. They are characterized by a half-integer positive number (the spin). One can easily see that $n=j+\frac{1}{2}$. The only question is whether all such representations occur in the spectrum of our system and in what multiplicity do they occur. From the results of Sec. 2 it is known that all of the half-integer representations are realized exactly once. A method of proving this fact independently of Sec. 2 consists in an examination of the functions which tend to zero under the action of the "lowering" operator J_x-iJ_y.

4. EXPLICITLY INVARIANT FORM OF THE SCHRÖDINGER EQUATION

In the previous section we expressed the Hamiltonian for our problem in terms of the operators J_k which commute with the Hamiltonian and generate the algebra $O(3)$. Based on this fact one can assert that the wave functions in the appropriate variables are spherical functions of the complete angular momentum. In this section we shall indicate these variables and find an explicitly invariant form of the Schrödinger equation.

Equation (4) can be rewritten by considering momentum space as a stereographic projection of a threedimensional sphere. For the Kepler problem this method was first utilized by Fock.[2] However, in contrast to the Kepler problem the necessity of an additional transformation of the spinors arises here.

In momentum space Eq. (4) has the form

$$(\mathbf{p}^2+p_0^2)\psi(\mathbf{p})-\frac{i\gamma}{2\pi}\int d^2p'\,\frac{-\sigma_x(p-p')_y+\sigma_y(p-p')_x}{(\mathbf{p}-\mathbf{p}')^2}\psi(\mathbf{p}')=0, \tag{13}$$

where $p_0^2=-2ME$ and $\gamma=0.4I\mu M/\hbar$. Introducing coordinates on the sphere

$$n_{x,y}=\frac{2p_0p_{x,y}}{p_0^2+\mathbf{p}^2},\quad n_z=\frac{p_0^2-\mathbf{p}^2}{p_0^2+\mathbf{p}^2},\quad \mathbf{n}^2=1, \tag{14}$$

we obtain

$$(p_0^2+\mathbf{p}^2)\psi(\mathbf{p})=-\frac{in}{2\pi}\int\frac{dn_x'dn_y'}{n_z'}\Big[-\sigma_1\Big(n_2-n_2'\frac{1+n_3}{1+n_3'}\Big)+\sigma_2\Big(n_1-n_1'\frac{1+n_3}{1+n_3'}\Big)\Big]\frac{p_0^2+\mathbf{p}'^2}{(\mathbf{n}-\mathbf{n}')^2}\psi(\mathbf{p}'), \tag{15}$$

where n is defined in the comment associated with formula (6). By the spinor transformation

$$\psi(\mathbf{n})=U^{-1}(p_0^2+\mathbf{p}^2)\psi(\mathbf{p}), \tag{16}$$

where

$$U=\begin{pmatrix}1+n_3 & n_1-in_2\\ -n_1-in_2 & 1+n_3\end{pmatrix}, \tag{17}$$

Eq. (15) reduces to the form

$$\psi(\mathbf{n})=-\frac{n}{4\pi}\int\frac{d^2n'}{n_z'}\Big[1-\frac{2i\sigma[\mathbf{n}\times\mathbf{n}']}{(\mathbf{n}-\mathbf{n}')^2}\Big]\psi(\mathbf{n}'). \tag{18}$$

The expression d^2n'/n_z' can be rewritten in the following form: $\delta(\mathbf{n}'^2-1)d^3\mathbf{n}'$ after which the rotational invariance of Eq. (18) becomes obvious. Let us represent Eq. (18) in the operator form

$$\psi(\mathbf{n})=-nP\psi(\mathbf{n}). \tag{19}$$

By direct verification let us make sure that the functions $\psi_{J,m}^{1,2}$ characterizing definite values of the operators $\mathbf{J}^2$, J_z, and $\mathbf{L}^2$ are eigenfunctions of the operator P. The superscripts 1 (2) correspond to $L=J-\frac{1}{2}$, $L=J+\frac{1}{2}$. We have:

$$\psi_{J,m}^1=\begin{pmatrix}\left(\frac{J+m}{2J}\right)^{1/2}Y_{J-1/2,m-1/2}\\ \left(\frac{J-m}{2J}\right)^{1/2}Y_{J-1/2,m+1/2}\end{pmatrix},$$

$$\psi_{J,m}^2=\begin{pmatrix}-\left(\frac{J-m+1}{2J+2}\right)^{1/2}Y_{J+1/2,m-1/2}\\ \left(\frac{J+m+1}{2J+2}\right)^{1/2}Y_{J+1/2,m+1/2}\end{pmatrix}. \tag{20}$$

The eigenvalues of the operator P are given by the formula

$$P\psi_{J,m}^{1,2}=\pm(J+1/2)^{-1}\psi_{J,m}^{1,2}; \tag{21}$$

therefore the kernel of the operator P can be represented in the form

$$\frac{1}{4\pi}-\frac{i\sigma[\mathbf{n}\times\mathbf{n}']}{2\pi(\mathbf{n}-\mathbf{n}')^2}=\sum_{J,m}\frac{1}{J+1/2}[\psi_{J,m}^1(\mathbf{n})\psi_{J,m}^{1*}(\mathbf{n}')-\psi_{J,m}^2(\mathbf{n})\psi_{J,m}^{2*}(\mathbf{n}')]. \tag{22}$$

The functions $\psi_{J,m}^{1(2)}$ are also eigenfunctions for the operator $(1+\boldsymbol{\sigma}\cdot\mathbf{L})$, where

$$(1+\sigma\mathbf{L})\psi_{J,m}^{1(2)}=\pm(J+1/2)\psi_{J,m}^{1(2)}. \tag{23}$$

Comparing Eqs. (21) and (23), we see that

$$P=(1+\sigma\mathbf{L})^{-1}. \tag{24}$$

Let us return to Eq. (19). Since the quantum number n is positive, it follows from the expansion (22) that the wavefunctions are $\psi_{J,m}^2$ and $n=j+\frac{1}{2}$. Thus,

$$\psi_{n,m}(\mathbf{p})=\frac{C_{n,m}}{p_0^2+\mathbf{p}^2}U\psi_{n-1/2,m}^2(\mathbf{n}), \tag{25}$$

where the $C_{n,m}$ are normalization factors.

5. CONCLUSION

We have examined various methods of solving the problem of the behavior of a neutral particle with spin $\frac{1}{2}$ in the field of a wire. The authors did not consider a number of questions associated with this problem. In particular, the continuous spectrum of the system remains uninvestigated. The dynamical symmetry group for this part of the spectrum is $O(2,1)$. In this case one can also write down an explicitly invariant equation of the type (18) where, just as in the Kepler problem, the compact region of integration changes into a noncompact region (the surface of a two-sheet hyperboloid). The wave functions of the continuous spectrum form the basis of a representation of the second fundamental series of the group $O(2,1)$ and are Legendre functions.[3] For problems involving a hidden symmetry one often introduces the so-called noninvariance group, one of whose irreducible representations describes all states of the system.[4] For the modified "Hamiltonian"

$$\tilde{\mathscr{H}}=1/2(\pi^2+1)(\sigma_xy-\sigma_yx), \tag{26}$$

whose spectrum is linear but whose wave functions agree with the wave functions of the starting Hamiltonian with the factor $\sigma_xy-\sigma_yx$, the authors have found such a group. It turned out to be isomorphic to the complex form $O(5)$; however, in the opinion of the authors this group is not very useful for solving physical problems because due to differences in the eigenvalues the evolution determined by the Hamiltonians $\mathscr{H}$ and $\tilde{\mathscr{H}}$ is not the same. In other words, the connection between the generators of $O(5)$ and the usual operators of the type coordinates, momenta, etc., is only simple for fixed energies. For a detailed discussion of these problems in connection with the Kepler problem, see the review by Popov.[1]

One more property of the system is related to the degeneracy of the levels with respect to the quantum number m. If an additional constant magnetic field $\mathbf{H}_0$ is imposed on the system, the levels of the system are split independently of its direction, and moreover the splitting is linear with respect to $|\mathbf{H}_0|$. The analogous phenomenon in the hydrogen atom is called the linear Stark effect.

In conclusion the authors thank S. S. Gershteĭn for valuable discussions.

APPENDIX

Let us consider Eq. (6). Let us make the substitution $\psi = (\sigma_x y - \sigma_y x)\varphi$. Then in the momentum representation the equation can be rewritten in the form

$$i(\pi^2+1)\left(\sigma_x\frac{\partial}{\partial\pi_y}-\sigma_y\frac{\partial}{\partial\pi_x}\right)\varphi+2n\varphi=0. \qquad \text{(A.1)}$$

During the motion of a particle with spin in a field of axial symmetry, the projection of the total angular momentum **J** on the z axis is conserved. We shall consider time-independent states in which J_z has a definite value m; in this connection the spinor has the form

$$\varphi=\begin{pmatrix}(\pi_x+i\pi_y)^{m-1/2}\varphi^+(\pi^2)\\(\pi_x+i\pi_y)^{m+1/2}\varphi^-(\pi^2)\end{pmatrix}. \qquad \text{(A.2)}$$

Then

$$(1+\pi^2)\left[\left(m+\frac{1}{2}\right)+\pi^2\frac{\partial}{\partial\pi^2}\right]\varphi^-=n\varphi^+,$$
$$-(1+\pi^2)\frac{\partial}{\partial\pi^2}\varphi^+=n\varphi^-. \qquad \text{(A.3)}$$

By eliminating one of the spinor components, for example φ^-, we obtain:

$$(1+\pi^2)^2\pi^2\varphi^{+\prime\prime}-(1+\pi^2)\left[(1+\pi^2)\left(m+\frac{1}{2}\right)+\pi^2\right]\varphi^{+\prime}+n^2\varphi^+=0 \qquad \text{(A.4)}$$

and making the substititon $w=\pi^2/(1+\pi^2)$, we arrive at the hypergeometric equation

$$(1-w)w\varphi^{+\prime\prime}+(m+{}^1/_2-w)\varphi^{+\prime}-n^2\varphi^+=0. \qquad \text{(A.5)}$$

If n is not an integer, the general solution of Eq. (A.5) for $m>0$ may be written down in the form

$$\varphi^+=C_1F(n,\ -n,\ m+{}^1/_2,\ w)$$
$$+C_2(1-w)^{m+1/2}F(m+{}^1/_2-n,\ m+{}^1/_2+n,\ m+{}^3/_2,\ 1-w), \qquad \text{(A.6)}$$

and the following expression is valid for $m<0$:

$$\varphi^+=D_1F(n,\ -n,\ -m+{}^1/_2,\ 1-w)$$
$$+D_2w^{-m+1/2}F(-m+{}^1/_2-n,\ -m+{}^1/_2+n,\ -m+{}^3/_2,\ w). \qquad \text{(A.7)}$$

Let us consider the quantization procedure for $m>0$. The case $m<0$ is analogous. Let us investigate the asymptotic form of the wave functions for large values of π^2 which corresponds to $w\to 1$. The second term in Eq. (A.6) behaves like $(1-w)^{m+1/2}$ which guarantees integrability of the corresponding wave function. The *first term* tends to the constant value

$$\frac{\Gamma^2(m+{}^1/_2)}{\Gamma(n+m+{}^1/_2)\Gamma(-n+m+{}^1/_2)}+O(1-w), \qquad \text{(A.8)}$$

that is, the corresponding wave function is not normalizable. Considering the asymptotic behavior for small values of π^2 (this corresponds to $w\to 0$) we obtain the opposite effect: the first solution is regular, and for $m>\frac{1}{2}$ the second behaves like

$$\frac{w^{-m+1/2}\Gamma(m-{}^1/_2)\Gamma(m+{}^3/_2)}{\Gamma(m+{}^1/_2+n)\Gamma(m+{}^1/_2-n)}(1+O(w)), \qquad \text{(A.9)}$$

and for $m=\frac{1}{2}$ it behaves like

$$\frac{1}{\Gamma(n+1)\Gamma(1-n)}[\ln w+O(1)]. \qquad \text{(A.10)}$$

For small values of π^2 the total wave function corresponding to the second term is nonintegrable.

It is clear that one can seek integrable (normalizable) solutions only for integer values of n. In this case the general solution is given by[5]

$$\varphi^+=C_1F(-n,\ n,\ m+{}^1/_2,\ w)$$
$$+C_2(-w)^{-n-m-1/2}(1-w)^{m+1/2}F(n+1,\ n+m+{}^1/_2,\ 2n+1,\ 1/w) \qquad \text{(A.11)}$$

for $m+\frac{1}{2}\leqslant n$ and

$$\varphi^+=D_1F(-n,\ n,\ m+{}^1/_2,\ w)$$
$$+D_2w^{1/2-m}F(-m-n+{}^1/_2,\ n-m+{}^1/_2,\ {}^3/_2-m,\ w) \qquad \text{(A.12)}$$

for $m+\frac{1}{2}>n$. Only the wave function corresponding to the first term in Eq. (A.11) is normalizable. The discrete levels of the system thus correspond to the following quantum numbers: $n=1, 2, \ldots,$ where $m+\frac{1}{2}\leqslant n$. Negative values of m are treated in analogous fashion. To sum up, it is found that the energy level characterized by the quantum number n has a $2n$-fold degeneracy: $|m|\leqslant n-\frac{1}{2}$.

1) For detailed knowledge concerning the concept of a hidden symmetry we refer the reader to a review article by Popov.[1]

2) The sign in the formula for n is chosen with allowance for the fact that the magnetic moment of the neutron is negative; therefore $n>0$.

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