Spectrum of electrons interacting with a strong ion-sound turbulence

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(Submitted October 15, 1976)
Zh. Eksp. Teor. Fiz. 72, 1414–1426 (April 1977)

Heating of electrons in a plasma by resonant ion-sound oscillations is considered under conditions when the effective turbulent collision frequency exceeds the frequency of the ion-sound oscillations, $\nu_{\rm eff} > kv_{\rm Te}(m_e/m_i)^{1/2}$. It is shown that under these conditions the plasma heating cannot be described in the weak-turbulence approximation. A collision integral that describes the heating under conditions of strong ion-sound turbulence is obtained. It is found that the stationary solutions of the equation for the regular part of the electron distribution function Φ exist for sufficiently wide spectra of the ion-sound oscillations and have a power-law dependence on the particle velocity v (or on the energy $mv^2/2$).

PACS numbers: 52.50.Gj

1. Plasma heating by ion-sound turbulence has been discussed numerous times.[1–3] In these investigations, however, no attention was paid to effects connected with strong turbulence, which can change significantly the collision integral even at relatively low energy levels of the ion-sound oscillations $W^s/nT_e > m_e/m_i$. A strong-turbulence theory connected with broadening of resonances[4,5] has already been used in the problem of ion-sound turbulence.[6] What was investigated there was the effect of turbulent oscillations on the nonlinear increment of the ion-sound instability.

The purpose of the present paper is to employ a previously developed theory[6] to analyze the collision integral of particles with resonant ion-sound oscillations. In first-order approximation, this collision integral describes almost-elast scattering of the particles by the oscillations, while the turbulent broadening of the resonance, as already shown,[6] has little effect on the indicated scattering processes. We call attention below to the fact that the heating processes in the plasma are due to the rather low inelasticity of these collisions, so that the collision integral that describes the heating is smaller by a factor $(\omega_s/kv_{Te})^2 \sim m_e/m_i$. Weakly as the turbulent broadening influences the scattering processes, it can strongly affect the heating processes. The broadening of the resonances leads to an uncertainty in the frequencies of the ion-sound oscillations of the order of $\nu_{\rm eff}$. A strong ion-sound turbulence corresponds to a condition when this frequency $\nu_{\rm eff}$ exceeds the frequency ω_s of the ion-sound oscillations. It is then natural to expect the heating to be determined not by the parameter $(\omega_s/kv_{Te})^2$ but by $(\nu_{\rm eff}/kv_{Te})^2$. On the other hand, the earlier estimate[6] was $\nu_{\rm eff} \sim \omega_{pe}(W^s/nT_e)^{1/2}$. It is therefore natural to expect the heating to be determined by the parameter $(\nu_{\rm eff}/kv_{Te})^2 \sim W^s/nT_e$, if $kv_{Te} \sim \omega_{pe}$, i.e., the turbulent broadening of the resonances affects strongly the particle heating at $W^s/nT_e > m_e/m_i$. We assume at the same time that $W^s/nT_e < 1$. Thus, the present theory of heating is constructed under the following assumptions:

$$m_e/m_i \ll W^s/nT_e \ll 1, \quad \omega_s \ll \nu_{\rm eff} \ll kv_{Te}. \tag{1}$$

These conditions have been written out for thermal particles. Actually the collision integral derived below is valid for both thermal particles and particles on the tail of the distribution (epithermal). The applicability criterion contains therefore in the general case not the average velocity but the particle velocity, and the applicability condition is that the heating due to the turbulent broadening of the resonances exceed the heating due to the inelasticity of the collisions.

2. Following the previously developed approach,[6] we break up the electron distribution function f_0 into regular and turbulent parts Φ and f, respectively:

$$f_0 = \Phi + f, \quad \langle f \rangle = 0. \tag{2}$$

In the absence of an external electric field, the equation for Φ is ($e = |e|$)

$$\frac{\partial \Phi}{\partial t} + \mathbf{v}\frac{\partial \Phi}{\partial \mathbf{r}} = -\frac{\partial}{\partial \mathbf{v}}\left\langle \frac{e}{m}\nabla\varphi f\right\rangle = I(\mathbf{v}). \tag{3}$$

The collision integral (3) can be written in the form

$$I(\mathbf{v}) = -i\frac{e}{m}\int\left(\mathbf{k}\frac{\partial}{\partial \mathbf{v}}\right)G_k\,dk, \quad k = \{\mathbf{k}, \omega\}, \tag{4}$$

where

$$\langle \varphi_k f_{k_1}(\mathbf{v})\rangle = G_k(\mathbf{v})\,\delta(k+k_1), \tag{5}$$

φ_k and f_k are respectively the Fourier components of the potential and of the turbulent part of the distribution function. In the strong-turbulence theory,[6] a new perturbation theory was developed, in which the expansion parameter was not the turbulence potential φ, but $\hat{g}\varphi$, where $\hat{g}$ is an operator defined by an integral equation derived in[6] and having a maximum value estimated at $(g\varphi)_{\max} \sim g/\nu_{\rm eff}$. The equation for $\hat{g}_k$ is of the form

 0038-5646/78/4504-0743$02.40

$$\int dv'\Big[(\omega-kv)\delta(v-v') + \left(\frac{e}{m}\right)^2\frac{\partial}{\partial v_i}\int k_{1i}k_{1j}|\varphi_{k_1}|^2\,dk_1\,\hat g_{k-k_1}(v,v')\frac{\partial}{\partial v_j'}\Big]\hat g_k(v',v'')=\delta(v-v''). \tag{6}$$

The operator $\hat g$ can be expressed here in matrix form:

$$\hat g\Phi=\int g(v,v')\Phi(v')dv'.$$

The expansion of the distribution function f in terms of the parameter $\hat g\varphi$ is

$$f_k=f_k^{(0)}+f_k^{(1)}+f_k^{(2)}, \tag{7}$$

where

$$f_k^{(0)}(v)=\hat g_k(v)\left\{\frac{e}{m}\varphi_k\left(\mathbf k\frac{\partial\Phi}{\partial \mathbf v}\right)\right\}, \tag{8}$$

$$f_k^{(1)}(v)=\hat g_k(v)\left(\frac{e}{m}\right)^2\frac{\partial}{\partial v_i}\int dk_1\,dk_2\,k_{1i}k_{2j}\delta(k-k_1-k_2) \times\hat g_{k_2}(v)(\varphi_{k_1}\varphi_{k_2}-\langle\varphi_{k_1}\varphi_{k_2}\rangle)\frac{\partial\Phi}{\partial v_j}, \tag{9}$$

$$f_k^{(2)}(v)=\hat g_k(v)\left(\frac{e}{m}\right)^3\frac{\partial}{\partial v_i}\int dk_1\,dk_2\,dk_3\,k_{1i}k_{2j}\delta(k-k_1-k_2-k_3) \times\hat g_{k-k_1}(v)\frac{\partial}{\partial v_j}(\varphi_{k_1}\varphi_{k_2}\varphi_{k_3}-\langle\varphi_{k_1}\varphi_{k_2}\rangle\varphi_{k_3}-\varphi_{k_1}\langle\varphi_{k_2}\varphi_{k_3}\rangle-\langle\varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\rangle)\hat g_{k_3}(v)\left(\mathbf k_3\frac{\partial\Phi}{\partial\mathbf v}\right) \tag{10}$$

In accordance with these three terms, we obtain three terms of the collision integral:

$$I^{(0)}(v)=-i\left(\frac{e}{m}\right)^2\int\left(\mathbf k\frac{\partial}{\partial\mathbf v}\right)|\varphi_k|^2\hat g_k(v)\left(\mathbf k\frac{\partial\Phi}{\partial\mathbf v}\right)dk, \tag{11}$$

$$I^{(1)}(v)=2i\left(\frac{e}{m}\right)^3\int\left(\mathbf k\frac{\partial}{\partial\mathbf v}\right)\hat g_k(v)\left(\mathbf k_1\frac{\partial}{\partial\mathbf v}\right)\hat g_{k-k_1}(v)\left\{\frac{|\varphi_{k_1}|^2|\varphi_{k-k_1}|^2}{k^2\tilde\varepsilon_{-k}}+\frac{|\varphi_k|^2|\varphi_{k-k_1}|^2}{k_1^2\tilde\varepsilon_{k_1}}+\frac{|\varphi_k|^2|\varphi_{k_1}|^2}{|k-k_1|^2\tilde\varepsilon_{k-k_1}}\right\}\left((\mathbf k-\mathbf k_1)\frac{\partial\Phi}{\partial\mathbf v}\right)dk\,dk_1, \tag{12}$$

$$I^{(2)}(v)=-i\left(\frac{e}{m}\right)^4\int\left(\mathbf k\frac{\partial}{\partial\mathbf v}\right)\hat g_k(v)\left(\mathbf k_1\frac{\partial}{\partial\mathbf v}\right)\hat g_{k-k_1}(v)\left(\mathbf k\frac{\partial}{\partial\mathbf v}\right)\times\hat g_{-k_1}(v)|\varphi_k|^2|\varphi_{k_1}|^2\left(\mathbf k_1\frac{\partial\Phi}{\partial\mathbf v}\right)dk\,dk_1. \tag{13}$$

The derivation of $I^{(1)}(\mathbf v)$ is based on a Poisson equation in which the charge density was determined by the zeroth approximation $f_k^{(0)}$, so that the expression for $I^{(1)}(\mathbf v)$ contains the modified permittivity $\tilde\varepsilon_k$ with a value[6]

$$\tilde\varepsilon_k=\varepsilon_k^i+\frac{\omega_{pe}^2}{nk^2}\int\hat g_k(v)\left(\mathbf k\frac{\partial\Phi}{\partial\mathbf v}\right)dp. \tag{14}$$

The integral $I^{(0)}(\mathbf v)$ corresponds to a refined quasilinear equation describing the elastic collisions of the particles with the ion-sound oscillations, whereas $I^{(1)}(\mathbf v)$ and $I^{(2)}(\mathbf v)$ are of higher order than $\hat g\varphi$ and describe the corrections. Being interested in the heating effect, it is expedient to eliminate the elastic scattering from the very outset, by averaging the integrals over the angle variables. The integral $I^{(0)}(\mathbf v)$ is then small in first-order approximation, so that $I^{(1)}(\mathbf v)$ and $I^{(2)}(\mathbf v)$ must be taken into account. Using the well known properties of the correlation function $|\varphi_k|^2=|\varphi_{-k}|^2$, and also the fact that Reg and Img are respectively odd and even in k, we can show that the integral $I^{(1)}(\mathbf v)$ is odd in $\mathbf v$ and vanishes after averaging over the angles.

The solution of the integral equation (6) for g can be written in the form[6]

$$g_k(v,v')=g^{(0)}_{\eta,k}(v)\delta(v-v')+g^{(1)}_{\eta,k}, \tag{15}$$

where $g^{(1)}_{\eta,k}$ is the correction to the solution $g^{(0)}_{\eta,k}$, and $g^{(0)}_{\eta,k}$ takes the form

$$g^{(0)}_{\eta,k}(v)=-i\int_0^\infty d\tau\exp\{i\eta\tau-{}^1/_3D_0\tau^3\}. \tag{16}$$

Here

$$\eta=\omega-\mathbf{kv},\quad D_0(\mathbf k,\mathbf v)=\left(\frac{e}{m}\right)^2\int dk_1(\mathbf{kk}_1)^2|\varphi_{k_1}|^2\int_0^\infty d\tau\exp\{-i(\omega_1-\mathbf k_1\mathbf v)\tau-{}^1/_3D_0(\mathbf k-\mathbf k_1,\mathbf v)\tau^3\}. \tag{17}$$

It will be shown below that the collision integral $I^{(0)}(\mathbf v)$ averaged over the angles in velocity space and making use of the function $g^{(0)}_{\eta,k}(\mathbf v)$ is of the same order of magnitude as the integral $I^{(2)}(\mathbf v)$ in which $g^{(0)}_{\eta,k}(\mathbf v)$ is used. The averaging over the angles makes it necessary to take into account in the function $g^{(0)}_{\eta,k}(\mathbf v)$ used for the calculation of $I^{(0)}(\mathbf v)$ the correction $\frac{1}{3}D_0\tau^3$ in the exponential, for otherwise $I^{(0)}(\mathbf v)$ vanishes. In the calculation of $I^{(2)}(\mathbf v)$ it suffices to use the function $g^{(0)}_{\eta,k}(\mathbf v)$ without this correction. The correction $g^{(1)}_{\eta,k}$ need not be taken into account in the calculation of either $I^{(0)}(\mathbf v)$ or $I^{(2)}(\mathbf v)$.

We note that the condition (1) $\nu_{eff}/kv_{Te}\ll 1$ (this is in fact the expansion parameter in expressions (7)–(13)) imposes a restriction on the electron velocity. Namely, since[6] $\nu_{eff}\sim D_0^{1/3}$, it follows that the parameter $D_0/(kv)^3$ must be small. This takes place when

$$v>v_{Te}(W^s/nT_e)^{1/4}=v_*. \tag{18}$$

All the results that follow are valid only under this condition.

3. To average the collision integrals over the angle variables θ_0 and φ_0 in velocity space, we shall use the identity

$$\left(\mathbf k\frac{\partial}{\partial\mathbf v}\right)\psi=\operatorname{div}_v\mathbf k\psi=\frac{1}{v^2}\frac{\partial}{\partial v}(v^2k_v\psi)+\frac{1}{v\sin\theta_0}\frac{\partial}{\partial\theta_0}(\sin\theta_0k_{\theta_0}\psi)+\frac{1}{v\sin\theta_0}\frac{\partial}{\partial\varphi_0}(k_{\varphi_0}\psi). \tag{19}$$

After the averaging we are left with only the first term. In the expression for $g^{(0)}_{\eta,k}(\mathbf v)$, the dimensionless quantities $(\omega_s/kv)^2$ and $D_0/(kv)^3$ are small parameters, since $(\omega_s/kv_{Te})^2\sim m_e/m_i$ and $D_0/(kv)^3\sim(W^s/nT_e)(v_{Te}/v)^4$. The calculation of the collision integral $I^{(0)}(\mathbf v)$ averaged over θ_0 and φ_0, with $g^{(0)}_{\eta,k}(\mathbf v)$ in the form

$$g^{(0)}_{\eta,k}(v)\approx-i\int_0^\infty e^{-ikv\tau}d\tau \tag{20}$$

yields zero. Allowance for $i\omega_s\tau$ and neglect of $\frac{1}{3}D_0\tau^3$ in the exponential leads to the small quantity $(\omega_s/kv_{Te})^2\sim m_e/m_i$. We are interested in the opposite case, when account must be taken of the parameter $D_0/(kv)^3\sim(W^s/nT_e)(v_{Te}/v)^4$. The calculation of D_0 with neglect of the terms $i\omega_s\tau$ and $\frac{1}{3}D_0\tau^3$ in the exponential in (17) yields

$$D_0=\frac{\alpha(1-x^2)}{v},\quad\alpha=\pi^2\left(\frac{e}{m}\right)^2k^2\int k_1I_{k_1}dk_1. \tag{21}$$

Here

$$x=\frac{\mathbf{kv}}{kv},\quad I_{\mathbf{k}}=k^2\int d\omega|\varphi_k|^2,\quad k=|\mathbf{k}|.$$

Substituting in expression (11) for $I^{(0)}(\mathbf{v})$ the first right-hand term of (19) and the expression (16) for $g^{(0)}_{\eta,\mathbf{k}}$ without allowance for $i\omega_s\tau$ in the exponential and in (21), we obtain

$$I^{(0)}(\mathbf{v})=-2\pi\left(\frac{e}{m}\right)^2\frac{1}{v^2}\frac{\partial}{\partial v}v^2\int k^2I_{\mathbf{k}}dk\,F(p)\frac{\partial\Phi}{\partial v}\frac{p}{kv},\tag{22}$$

Here

$$F(p)=\int_{-1}^{1}x^2\,dx\int_0^\infty dy\,e^{-ipxy-(1-x^2)y^3},\quad p=kv\left(\frac{3v}{\alpha}\right)^{1/3}\gg1.\tag{23}$$

In the derivation of (22) and throughout the sequel it is assumed that the regular part of the distribution function Φ is isotropic, i.e., $\Phi=\Phi(v)$.

The asymptotic form of the function $F(p)$ is calculated in Appendix I. We obtain ultimately for the collision integral $I^{(0)}(\mathbf{v})$ the expression

$$I^{(0)}(\mathbf{v})=-16\pi^3\left(\frac{e}{m}\right)^4\frac{1}{v^2}\frac{\partial}{\partial v}\frac{1}{v^3}\frac{\partial\Phi}{\partial v}\int_{k_{min}}^{k_{max}}I_{\mathbf{k}}\,dk\int_{k_{min}}^{k_{max}}k_1I_{\mathbf{k}_1}dk_1.\tag{24}$$

Recognizing that the second approximation $I^{(2)}(\mathbf{v})$ is small in terms of the parameter $\hat{g}\varphi$, we use the function $g^{(0)}_{\eta,\mathbf{k}}(\mathbf{v})$ in the form (20) to calculate $I^{(2)}(\mathbf{v})$. The calculation of $I^{(2)}(\mathbf{v})$ is given in Appendix II. The result is

$$I^{(2)}(\mathbf{v})=\frac{8}{3}\pi^3\left(\frac{e}{m}\right)^4\left(\frac{1}{v^3}\frac{\partial^4\Phi}{\partial v^4}-\frac{2}{v^4}\frac{\partial^3\Phi}{\partial v^3}+\frac{3}{v^5}\frac{\partial^2\Phi}{\partial v^2}-\frac{3}{v^6}\frac{\partial\Phi}{\partial v}\right)$$
$$\times\left\{\int_{k_{min}}^{k_{max}}\frac{1}{k}I_{\mathbf{k}}\,dk\int_{k_{min}}^{k}k_1^2I_{\mathbf{k}_1}dk_1+\int_{k_{min}}^{k_{max}}k^2I_{\mathbf{k}}dk\int_k^{k_{max}}\frac{1}{k_1}I_{\mathbf{k}_1}dk_1\right\}.\tag{25}$$

To integrate (24) and (25) with respect to k and k_1 it is necessary to know the turbulence spectrum of the ion-sound plasmons. In many cases this spectrum is approximated by the expression

$$I_{\mathbf{k}}=\frac{W_0}{k}\ln\frac{k_0}{k}\cdot\begin{cases}1 & (k>k_*)\\ 0 & (k<k_*)\end{cases}.\tag{26}$$

The spectrum (26) corresponds to the case of generation of ion-sound plasmons. Integration of (24) and (25) with respect to k and k_1 ($k_{min}=k_*$, $k_{max}=k_0$) yields

$$I^{(0)}(\mathbf{v})=-8\pi^3\left(\frac{e}{m}\right)^4W_0^2\left(\frac{1}{v^5}\frac{\partial^2\Phi}{\partial v^2}-\frac{3}{v^6}\frac{\partial\Phi}{\partial v}\right)$$
$$\times\left[k_0\ln^2\frac{k_0}{k_*}-k_*\left(\ln^3\frac{k_0}{k_*}+\ln^2\frac{k_0}{k_*}\right)\right],\tag{27}$$

$$I^{(2)}(\mathbf{v})=\frac{4}{3}\pi^3\left(\frac{e}{m}\right)^4W_0^2\left(\frac{1}{v^3}\frac{\partial^4\Phi}{\partial v^4}-\frac{2}{v^4}\frac{\partial^3\Phi}{\partial v^3}+\frac{3}{v^5}\frac{\partial^2\Phi}{\partial v^2}-\frac{3}{v^6}\frac{\partial\Phi}{\partial v}\right)\cdot$$
$$\times\left[5k_0-4k_*\left(\ln^2\frac{k_0}{k_*}+\ln\frac{k_0}{k_*}+1\right)-\frac{k_*^2}{k_0}\left(2\ln\frac{k_0}{k_*}+1\right)\right].\tag{28}$$

On the other hand, if there is no generation of ion-sound oscillations in the considered wave-number region, then the spectrum is approximated by the expression[2,7]

$$I_{\mathbf{k}}=\frac{W_0}{k}\ln\frac{k}{k_0}\cdot\begin{cases}1 & (k<k_*)\\ 0 & (k>k_*)\end{cases}.\tag{29}$$

Integration of (24) and (25) with respect to k and k_1 ($k_{min}=k_0$, $k_{max}=k_*$) yields

$$I^{(2)}(\mathbf{v})=\frac{4}{3}\pi^3\left(\frac{e}{m}\right)^4W_0^2\left(\frac{1}{v^3}\frac{\partial^4\Phi}{\partial v^4}-\frac{2}{v^4}\frac{\partial^3\Phi}{\partial v^3}+\frac{3}{v^5}\frac{\partial^2\Phi}{\partial v^2}-\frac{3}{v^6}\frac{\partial\Phi}{\partial v}\right)$$
$$\times\left[-4k_0-\frac{k_0^2}{k_*}\left(\ln\frac{k_*}{k_0}+1\right)+k_*\left(2\ln^2\frac{k_*}{k_0}-5\ln\frac{k_*}{k_0}+5\right)\right].\tag{30}$$

Equation (27) remains in force for $I^{(0)}(\mathbf{v})$ (with allowance for the fact that $k_0<k_*$).

4. We now write down a general expression for the collision integral $I(\mathbf{v})=I^{(0)}(\mathbf{v})+I^{(2)}(\mathbf{v})$. For the spectrum (26) at $k_*\ll k_0$ we have

$$I(\mathbf{v})=\frac{4}{3}\pi^3\left(\frac{e}{m}\right)^4W_0^2k_0\left\{5\left(\frac{1}{v^3}\frac{\partial^4\Phi}{\partial v^4}-\frac{2}{v^4}\frac{\partial^3\Phi}{\partial v^3}+\frac{3}{v^5}\frac{\partial^2\Phi}{\partial v^2}-\frac{3}{v^6}\frac{\partial\Phi}{\partial v}\right)\right.$$
$$\left.-6\left(\frac{1}{v^5}\frac{\partial^2\Phi}{\partial v^2}-\frac{3}{v^6}\frac{\partial\Phi}{\partial v}\right)\ln^2\frac{k_0}{k_*}\right\}.\tag{31}$$

For the spectrum (29) at $k_0\ll k_*$ we have

$$I(\mathbf{v})=\frac{4}{3}\pi^3\left(\frac{e}{m}\right)^4W_0^2k_*\left\{\left(\frac{1}{v^3}\frac{\partial^4\Phi}{\partial v^4}-\frac{2}{v^4}\frac{\partial^3\Phi}{\partial v^3}+\frac{3}{v^5}\frac{\partial^2\Phi}{\partial v^2}-\frac{3}{v^6}\frac{\partial\Phi}{\partial v}\right)\right.$$
$$\left.\times\left(2\ln^2\frac{k_*}{k_0}-5\ln\frac{k_*}{k_0}+5\right)-6\left(\frac{1}{v^5}\frac{\partial^2\Phi}{\partial v^2}-\frac{3}{v^6}\frac{\partial\Phi}{\partial v}\right)\left(\ln^3\frac{k_*}{k_0}-\ln^2\frac{k_*}{k_0}\right)\right\}.\tag{32}$$

It is easily seen that Eq. (3) with $I(\mathbf{v})$ in the form (31) or (32) has a stationary power-law solution $\Phi=\Phi_0v^{-\gamma}$. We present the values of γ as functions of the width of the turbulence spectrum $\ln(k_{max}/k_{min})$ for the spectrum (26):

$\ln\frac{k_*}{k_0}$:	3	4	5	6	7	8	9	10	11
γ:	1.3	2.4	3.5	4.6	5.7	6.8	7.9	9	10.1

and for the spectrum (29):

$\ln\frac{k_*}{k_0}$:	3	4	5	6	7	8	9	10	11
γ:	1.7	2.1	2.5	2.9	3.1	3.3	3.6	3.9	4.1

We note that the solutions that are stationary in the full sense of the word are those with $\gamma>3$, for only in this case does the total number of electrons $\int\Phi(\mathbf{v})d\mathbf{v}$ not diverge at infinity. This corresponds to sufficiently broad spectra of the ion-sound turbulence ($\ln(k_0/k_*)>4.5$ for the spectrum (26) and $\ln(k_*/k_0)>6$ for the spectrum (29)).

We rewrite the collision integral

$$I(\mathbf{v})=\frac{A}{v^3}\frac{\partial^4\Phi}{\partial v^4}+\frac{B}{v^4}\frac{\partial^3\Phi}{\partial v^3}+\frac{C}{v^5}\frac{\partial^2\Phi}{\partial v^2}+\frac{D}{v^6}\frac{\partial\Phi}{\partial v}\tag{33}$$

in a form more suitable for applications:

$$I(\mathbf{v})=\frac{1}{F(v)}\frac{d}{dv}\left(X\frac{d^2}{dv^2}+Y\frac{d}{dv}+Z\right)\frac{d\Phi}{dv}.\tag{34}$$

For the unknown functions $X(v)$, $Y(v)$, and $Z(v)$ we have

$$X=\frac{AF}{v^3},\quad Y=\frac{BF}{v^4}+\frac{3AF}{v^4}-\frac{A}{v^3}\frac{dF}{dv}$$
$$Z=\frac{CF+4BF+12AF}{v^5}-\frac{6A+B}{v^4}\frac{dF}{dv}+\frac{A}{v^3}\frac{d^2F}{dv^2}\tag{35}$$

and the function $F(v)$ satisfies the differential equation

$$Av^3\frac{d^3F}{dv^3}-(9A+B)v^2\frac{d^2F}{dv^2}+(36A+8B+C)v\frac{dF}{dv}-(60A+20B+5C+D)F=0. \tag{36}$$

We seek the solution of (36) in the form $F(v)=v^n$; the equation for n is then

$$An^3-(12A+B)n^2+(47A+9B+C)n-(60A+20B+5C+D)=0, \tag{37}$$

and from (32) we have at $\Phi(v)=\Phi_0 v^{-\gamma}$ an equation for γ

$$A\gamma^3+(6A-B)\gamma^2+(11A-3B+C)\gamma+(6A-2B+C-D)=0. \tag{38}$$

Equations (37) and (38) are presented in canonical form with identical coefficients, whence follows a unique relation between n and γ

$$\gamma=n-6. \tag{39}$$

5. Our results can be used to interpret a number of astrophysical observations, particularly of fast electrons accelerated in chromospheric flares on the sun,[8] which generate hard x rays. The theoretical schemes[8,9] that describe the plasma processes that occur in the current layers usually associated with regions from which solar flares emerge, include an intense ion-sound turbulence. After the passage of the magentic-field annihilation wave, a region is left in which the ion-sound turbulence is damped and the particles are accelerated. It is frequently assumed that the ion-sound turbulence is accompanied by Langmuir turbulence. There are a number of mechanisms capable of causing the Langmuir turbulence.[8] A strong Langmuir turbulence is absorbed, owing to modulation instability, in the tails of the Maxwellian distribution, and produces likewise fast particles. According to Galeev,[10] the energy spectrum f_ε of such particles is given by

$$f_\varepsilon\sim\frac{1}{\varepsilon^{7/4}},\quad \int f_\varepsilon\,d\varepsilon=n,\quad \varepsilon\ll mc^2. \tag{40}$$

This spectrum is not too steep, since the x-ray flare observation data yield $f_\varepsilon\sim 1/\varepsilon^\gamma$ with $\gamma\approx 3$–4. Other schemes in which the loss mechanisms are taken into account[8] can yield steeper spectra.

The mechanism considered in this paper for the generation of fast particles is capable of competing with the mechanism of acceleration by Langmuir oscillations. It appears that under real conditions both mechanisms may be in operation. Acceleration by ion-sound plasmons differs qualitatively from acceleration by Langmuir oscillations. First, the spectra of the particles accelerated by the ion-sound oscillations are always power-law functions:

$$f_\varepsilon\approx 1/\varepsilon^{(\gamma-1)/2},\quad f_\varepsilon\sim v f_v.$$

Second, the energy spectra of the particles can be quite steep. Thus, according to[8], $\ln(k_0/k_*)\approx 10$–12 and $\gamma\approx 9$–10, i.e., $(\gamma-1)/2\approx 4$–4.5. Thus, a strong ion-sound turbulence produces steeper particle spectra, in agreement with the observations.

It is also obvious that the results of the present investigation can be applied to other problems, for example collisionless shock waves in laboratory and cosmic plasmas, and other laboratory experiments in which ion-sound oscillations are excited in one way or another.

APPENDIX I

Let us calculate the asymptotic form of the integral (23). The standard variants of the stationary-phase method cannot be used for this integral, since the phase function $S=-xy$ has a stationary point (0, 0) lying on the level line $y=0$ of the phase.

We consider the auxiliary integral

$$\Phi(p)=\int_0^\infty dy\left(\int_{-\infty}^{\infty} e^{-ipxy}\varphi(x,y)\,dx\right), \tag{I.1}$$

where the function $\varphi(x,y)$ is infinitely differentiable and is finite in (x,y).

Lemma. The following asymptotic expansion is valid as $p\to+\infty$:

$$\Phi(p)\sim\sum_{n=0}^{\infty}a_n p^{-n-1} \tag{I.2}$$

where the coefficients a_n are given by

$$a_n=\frac{1}{n!}\int_0^\infty y^n\,dy\left(\int_{-\infty}^{\infty}D_y{}^n\varphi(x,0)\,e^{-ixy}\,dx\right). \tag{I.3}$$

Proof. The inner integral in (I.1) is equal to

$$I(y,p)=\int_{-\infty}^{\infty}e^{-itx}\varphi(x,\varepsilon)\,dx,$$

where $\varepsilon=t/p$. According to Taylor's formula

$$\varphi(x,\varepsilon)=\sum_{n=0}^{N}\frac{\varepsilon^n}{n!}D_y{}^n\varphi(x,0)+R_N(x,\varepsilon),$$

where

$$R_N(x,\varepsilon)=\frac{1}{N!}\int_0^\varepsilon(\varepsilon-\tau)^N D_\tau{}^{N+1}\varphi(x,\tau)\,d\tau.$$

Substituting this expansion in (I.1), we get

$$\Phi(p)=\sum_{n=0}^{N}a_n p^{-n-1}+F_N(p).$$

The residual term is

$$F_N(p)=\int_0^\infty I_N(t,p)\,dt.$$

Here

$$I_N=\int_{-\infty}^{\infty}e^{-itx}R_N(x,\varepsilon)\,dx=\frac{1}{N!}\int_0^\varepsilon(\varepsilon-\tau)^N\left(\int_{-\infty}^{\infty}e^{-itx}\psi(x,\tau)\,dx\right)d\tau,$$

where $\psi(x,\tau)=D_\tau^{N+1}\varphi(x,\tau)$. The positions of the integrals can be interchanged because the function $\varphi(x,\tau)$ is finite. For any $k>0$ the estimate

$$\left| \int_{-\infty}^{\infty} e^{-itx}\psi(x,\tau)\,dx \right| \leqslant C_k(1+|t|)^{-k}$$

is valid uniformly over $\tau \in [0, \infty)$, by virtue of the finiteness and smoothness of the function $\varphi(x, \tau)$. Consequently

$$|I_N| \leqslant \frac{C_k}{N!}(1+|t|)^{-k}\int_0^{\varepsilon} (\varepsilon-\tau)^N\,d\tau = C_k'(1+|t|)^{-k}\varepsilon^{N+1},$$

$$|R_N| \leqslant p^{-N-1}C_k' \int_0^{\infty} t^{N+1}(1+|t|)^{-k}\,dt \leqslant C_k'' p^{-N-1}.$$

This proves the lemma.

Let us calculate the coefficients a_n. We have (see[11])

$$\int_0^{\infty} e^{-itx}t^n\,dt = i^{n+1}n!x^{-n-1} + i^n\pi\delta^{(n)}(x).$$

Consequently,

$$a_n = i^{n+1}(x^{-n-1}, D_y{}^n\varphi(x,0)) + \frac{i^n\pi}{n!}D_x{}^nD_y{}^n\varphi(0,0). \qquad \text{(I.4)}$$

Here x^{-n-1} is a generalized function; at $n=3$, in particular, we have

$$(x^{-4}, \psi) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\psi(x)+\psi(-x)-2\psi(0)-x^2\psi''(0)}{x^4}\,dx \qquad \text{(I.5)}$$

for any finite smooth function ψ. We note that the main contribution to the asymptotic form of the integral $\Phi(p)$ is made by the entire x axis (more accurately, by that part of the axis on which $\varphi \neq 0$), and not only by the stationary point $(0, 0)$.

We calculate now the asymptotic form of the integral (23). We introduce an auxiliary even function $\eta(x) \in C_0^{\infty}(-1, 1)$, $\eta(x) \equiv 1$ at small $|x|$. Then

$$F(p) = F_0(p) + F_1(p),$$

where $F_0(p)$ is of the form (I.1), and

$$\varphi(x, y) = x^2\eta(x)\exp[-y^3(1-x^2)].$$

From the lemma and from formula (I.4) it follows that the expansion (I.2) with $a_n = 0$ at $n = 0$, 1, and 2 holds for $F_0(p)$, and

$$a_3 = 6\int_{-\infty}^{\infty} [(x^2-1)\eta(x)+1]x^{-2}\,dx. \qquad \text{(I.6)}$$

It is easily seen that the next nonvanishing coefficient in the expansion (I.2) is a_6. Thus,

$$F_0(p) = a_3p^{-4} + O(p^{-7}), \qquad \text{(I.7)}$$

where a_3 takes the form (I.6). The remaining integral $F_1(p)$ is of the form

$$F_1(p) = \int_{-1}^{1} dx \left(\int_0^{\infty} e^{-ipxy}\psi(x, y)\,dy \right),$$

where

$$\psi(x, y) = x^2(1-\eta(x))\exp[-y^3(1-x^2)].$$

Integrating by parts with respect to dy, we obtain the asymptotic expansion

$$F_1(p) \sim \sum_{n=0}^{\infty} b_np^{-n-1},$$

where

$$b_n = i^{n+1}\int_{-1}^{1} x^{-n-1}D_y{}^n\psi(x, 0).$$

In particular, $b_n = 0$ at $n = 0$, 1, 2

$$b_3 = 6\int_{-1}^{1} (x^2-1)(1-\eta(x))x^{-2}\,dx,$$

and the next nonzero coefficient is b_6; since

$$F(p) = (a_3+b_3)p^{-4} + O(p^{-7}),$$

we obtain ultimately

$$F(p) = 24p^{-4} + O(p^{-7}), \quad (p \to +\infty).$$

APPENDIX II

Let us calculate the collision integral $I^{(2)}(\mathbf{v})$. After averaging over the angle variables in velocity space, we can represent $I^{(2)}(\mathbf{v})$ in the form

$$I^{(2)}(\mathrm{v}) = \left(\frac{e}{m}\right)^4\left(\frac{\partial}{\partial v}+\frac{2}{v}\right)\int I_{\mathbf{k}}I_{\mathbf{k}_1}\psi(\tau, \tau_1, \tau_2, k, k_1, v)\,d\tau\,d\tau_1\,d\tau_2\,dk\,dk_1, \qquad \text{(II.1)}$$

where $(x = \cos\theta)$,

$$\psi = k_v\int e^{-i\mathbf{k}\mathbf{v}\tau}\left(\mathbf{k}_1\frac{\partial}{\partial \mathbf{v}}\right)e^{-i(\mathbf{k}-\mathbf{k}_1)\mathbf{v}\tau_2}\left(\mathbf{k}\frac{\partial}{\partial \mathbf{v}}\right)e^{i\mathbf{k}_1\mathbf{v}\tau_1} \times \left(\mathbf{k}_1\frac{\partial\Phi}{\partial \mathbf{v}}\right)\frac{dx\,dx_1\,d\varphi\,d\varphi_1\,dx_0\,d\varphi_0}{4\pi}. \qquad \text{(II.2)}$$

Here θ_0, φ_0; θ, φ; θ_1, φ_1 are the angle variables in the spaces of $\mathbf{v}$, $\mathbf{k}$, and $\mathbf{k}_1$, respectively. We express ψ in the form

$$\psi = \int \left(L\frac{\partial^3\Phi}{\partial v^3} + M\frac{\partial^2\Phi}{\partial v^2} + N\frac{\partial\Phi}{\partial v}\right)k^2k_1{}^2\frac{dx_0\,d\varphi_0}{4\pi}, \qquad \text{(II.3)}$$

$$L = \int A^2A_1{}^2dx\,dx_1\,d\varphi\,d\varphi_1 e^{ik_1v(\tau_1+\tau_2)A_1 - ikvA(\tau+\tau_2)}, \qquad \text{(II.4)}$$

$$M = \int \Big\{ik_1\tau_1AA_1{}^2A_2 + \frac{1}{v}[2AA_1(BB_1+CC_1\sin^2\theta_0) + A^2(B_1{}^2+C_1{}^2\sin^2\theta_0)] + ik_1(\tau_1+\tau_2)A^2A_1 - ik\tau_2A^2A_1A_2\Big\}dx\,dx_1\,d\varphi\,d\varphi_1e^{ik_1vA_1(\tau_1+\tau_2)-ikvA(\tau+\tau_2)}, \qquad \text{(II.5)}$$

$$N = \int \Big\{-k_1{}^2\tau_1(\tau_1+\tau_2)AA_1A_2 + kk_1\tau_1\tau_2AA_1A_2{}^2 + \frac{ik_1(\tau_1+\tau_2)}{v}A(BB_1+CC_1\sin^2\theta_0) - \frac{ik\tau_2}{v}A(BB_1+CC_1\sin^2\theta_0)A_2$$
$$+ \frac{ik_1\tau_1}{v}AA_2B_1{}^2 + \frac{ik_1\tau_1}{v}AA_2\sin^2\theta_1\sin^2(\varphi_1-\varphi_0) - \frac{1}{v^2}AA_1(BB_1+CC_1\sin^2\theta_0) - \frac{1}{v^2}AB_1(AB_1+A_1B)$$
$$- \frac{1}{v^2}AC_1[\sin\theta\sin\theta_1\sin^2\theta_0\sin(\varphi+\varphi_1-2\varphi_0) + \sin\theta\cos\theta_1\sin\theta_0\cos\theta_0\sin(\varphi-\varphi_0) + \cos\theta\sin\theta_1\sin\theta_0\cos\theta_0$$
$$\times \sin(\varphi_1-\varphi_0)]\Big\}e^{ik_1v(\tau_1+\tau_2)A_1-ikvA(\tau+\tau_2)}\,dx\,dx_1\,d\varphi\,d\varphi_1, \qquad \text{(II.6)}$$

where

$$\begin{gathered}A=\sin\theta\sin\theta_0\cos(\varphi-\varphi_0)+\cos\theta\cos\theta_0=\mathbf{kv}/kv,\\ A_1=\sin\theta_1\sin\theta_0\cos(\varphi_1-\varphi_0)+\cos\theta_1\cos\theta_0=\mathbf{k}_1\mathbf{v}/k_1v,\\ A_2=\sin\theta\sin\theta_1\cos(\varphi-\varphi_1)+\cos\theta\cos\theta_1=\mathbf{kk}_1/kk_1,\\ B=\frac{\partial A}{\partial\theta_0},\quad B_1=\frac{\partial A_1}{\partial\theta_0},\quad C\sin^2\theta_0=\frac{\partial A}{\partial\varphi_0},\quad C_1\sin^2\theta_0=\frac{\partial A_1}{\partial\varphi_0}.\end{gathered}\tag{II.7}$$

The integration with respect to φ and φ_1 is carried out with account taken of the relations

$$\begin{gathered}\int_0^{2\pi}e^{ia\cos\varphi}d\varphi=2\pi J_0(a),\qquad\int_0^{2\pi}e^{ia\cos\varphi}\cos\varphi\,d\varphi=2\pi iJ_1(a),\\ \int_0^{2\pi}e^{ia\cos\varphi}\sin\varphi\,d\varphi=0,\qquad\int_0^{2\pi}e^{ia\cos\varphi}\cos\varphi\sin\varphi\,d\varphi=0,\\ \int_0^{2\pi}e^{ia\cos\varphi}\cos^2\varphi\,d\varphi=\pi[J_0(a)-J_2(a)],\\ \int_0^{2\pi}e^{ia\cos\varphi}\cos^3\varphi\,d\varphi=\frac{\pi i}{2}[3J_1(a)-J_3(a)].\end{gathered}\tag{II.8}$$

The integration with respect to x and x_1 is carried out by differentiating with respect to the fictitious parameters β and γ the relation

$$\int_{-1}^{1}dx\,e^{i\gamma ax}J_0(\beta b(1-x^2)^{1/2})=2\frac{\sin(a^2\gamma^2+b^2\beta^2)^{1/2}}{(a^2\gamma^2+b^2\beta^2)^{1/2}}.\tag{II.9}$$

The results of the integration with respect to x, x_1, φ, and φ_1 are

$$L=16\pi^2\frac{d^2}{da^2}\left(\frac{\sin a}{a}\right)\frac{d^2}{db^2}\left(\frac{\sin b}{b}\right),\tag{II.10a}$$

$$\begin{aligned}M=&16\pi^2k_1\tau_1\frac{d^3}{da^3}\left(\frac{\sin a}{a}\right)\frac{d^2}{db^2}\left(\frac{\sin b}{b}\right)+\frac{32}{v}\pi^2\frac{1}{a}\frac{d}{da}\left(\frac{\sin a}{a}\right)\frac{d^2}{db^2}\left(\frac{\sin b}{b}\right)\\ &+16\pi^2k\tau_2\frac{d^2}{da^2}\left(\frac{\sin a}{a}\right)\frac{d^3}{db^3}\left(\frac{\sin b}{b}\right)-16\pi^2k_1(\tau_1+\tau_2)\frac{d}{da}\\ &\times\left(\frac{\sin a}{a}\right)\frac{d^2}{db^2}\left(\frac{\sin b}{b}\right),\end{aligned}\tag{II.10b}$$

$$\begin{aligned}N=&-16\pi^2k_1^2\tau_1(\tau_1+\tau_2)\frac{d^2}{da^2}\left(\frac{\sin a}{a}\right)\frac{d^2}{db^2}\left(\frac{\sin b}{b}\right)\\ &+16\pi^2kk_1\tau_1\tau_2\frac{d^3}{da^3}\left(\frac{\sin a}{a}\right)\frac{d^3}{db^3}\left(\frac{\sin b}{b}\right)-\frac{32}{v^2}\frac{\pi^2}{a}\frac{d}{da}\left(\frac{\sin a}{a}\right)\frac{d^2}{db^2}\left(\frac{\sin b}{b}\right)\\ &+32\pi^2kk_1\tau_1\tau_2\left(\frac{\sin a}{a^2}+\frac{3\cos a}{a^3}-\frac{3\sin a}{a^4}\right)\left(\frac{\sin b}{b^2}+\frac{3\cos b}{b^3}-\frac{3\sin b}{b^4}\right)\\ &+32\pi^2\frac{k\tau_2}{v}\left(-\frac{\cos a}{a^2}+\frac{\sin a}{a^3}\right)\left(\frac{\sin b}{b^2}+\frac{3\cos b}{b^3}-\frac{3\sin b}{b^4}\right)\\ &-32\pi^2\frac{k_1\tau_1}{v}\left(\frac{\sin a}{a^2}+\frac{3\cos a}{a^3}-\frac{3\sin a}{a^4}\right)\frac{d^2}{db^2}\left(\frac{\sin b}{b}\right),\end{aligned}\tag{II.10c}$$

Here

$$a=k_1v(\tau_1+\tau_2),\quad b=kv(\tau+\tau_2).$$

After integrating with respect to τ, τ_1, and τ_2 we obtain

$$\int L\,d\tau\,d\tau_1d\tau_2=\frac{8}{3}\pi^3\frac{1}{v^3}\cdot\begin{cases}1/k_3 & (k_1\leqslant k)\\ 1/k_1^3 & (k_1\geqslant k),\end{cases}\tag{II.11a}$$

$$\int M\,d\tau\,d\tau_1d\tau_2=-\frac{8}{3}\pi^3\frac{1}{v^4}\cdot\begin{cases}1/k^3 & (k_1\leqslant k)\\ 1/k_1^3 & (k_1\geqslant k),\end{cases}\tag{II.11b}$$

$$\int N\,d\tau\,d\tau_1d\tau_2=\frac{8}{3}\pi^3\frac{1}{v^5}\cdot\begin{cases}1/k^3 & (k_1\leqslant k)\\ 1/k_1^3 & (k_1\geqslant k).\end{cases}\tag{II.11c}$$

It follows ultimately from (II.1), (II.3), and (II.11) that

$$\begin{aligned}I^{(2)}(\mathbf{v})=&\frac{8}{3}\pi^3\left(\frac{e}{m}\right)^4\left(\frac{1}{v^3}\frac{\partial^4\Phi}{\partial v^4}-\frac{2}{v^4}\frac{\partial^3\Phi}{\partial v^3}+\frac{3}{v^5}\frac{\partial^2\Phi}{\partial v^2}-\frac{3}{v^6}\frac{\partial\Phi}{\partial v}\right)\\ &\times\left\{\int_{k_{min}}^{k_{max}}\frac{1}{k}I_{\mathbf{k}}dk\int_{k_{min}}^{k}k_1^2I_{\mathbf{k}_1}dk_1+\int_{k_{min}}^{k_{max}}k^2I_{\mathbf{k}}dk\int_{k}^{k_{max}}\frac{1}{k_1}I_{\mathbf{k}_1}dk_1\right\}.\end{aligned}\tag{II.12}$$

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Translated by J. G. Adashko