The Pomeranchuk singularity in nonabelian gauge theories

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An integral equation is derived for the t-channel partial wave amplitudes in the investigation of the multi-Regge form of the $2\to 2+n$ amplitude. For a t-channel state with isospin $T=1$ the solution of this equation is a Regge pole. The analytic properties of the isospin $T=0$, 2 partial wave amplitudes are investigated near the threshold for the production of two or three particles. It is shown that in the j-plane there are moving poles and cuts. For the $T=0$ vacuum channel it was found that the partial wave amplitude has a fixed square-root type branch point to the right of $j=1$.

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1. INTRODUCTION

The most attractive models of strong interactions are at present models based on the gauge vector fields of the Yang-Mills[1] type. In distinction from quantum electrodynamics,[2] in these models the interaction vanishes at short distances, leading to an approximately scale-invariant behavior of the hadronic structure functions.[3] The infrared instability of the theory at large distances seems to be the mechanism which confines the quarks within the hadron.[4] The Yang-Mills theory is renormalizable. Moreover, this property is retained in the massive theory which arises from the massless one via the Higgs-Kibble mechanism.[5] For some of the models obtained in this manner factorization relations hold for the Born amplitudes, a necessary condition for the reggeization of vector bosons and spinor particles.[6]

In our preceding papers[7,8] the hypothesis that the Yang-Mills fields reggeize was confirmed to eighth order of perturbation theory (cf. also[9]). It was discovered that the inelastic amplitudes have a multiregge behavior.[8] This gave rise to the hope that in the nonabelian case, in distinction from quantum electrodynamics,[10] the total cross sections will not exceed the Froissart bound as the energy grows.[11] In our preceding note[12] we have shown that in the leading logarithmic approximation, in spite of the multiregge form of the inelastic amplitudes, the total cross sections increase with energy according to a power law. In the present paper we consider questions related to the Pomeranchuk singularity in nonabelian gauge theories in more detail.

In the following section we shall derive a multiregge equation for partial waves with different quantum numbers in the t-channel and show its self-consistency. In Sec. 3 we investigate the analytic properties in t of the partial-wave amplitudes and the moving singularities in the j-plane. In Sec. 4 we consider the leading singularity in the j-plane for the vacuum channel.

2. A MULTIREGGE EQUATION FOR THE t-CHANNEL PARTIAL WAVES

Below we shall consider the simplest model,[13] based on an isotriplet of Yang-Mills vector fields $\mathbf{V}_\mu$ of mass m, the latter being the result of the appearance of a nonvanishing vacuum expectation value of an isodoublet complex field. In this model there is a scalar field φ necessary for the renormalizability of the theory. The calculation of the asymptotic behavior of the scattering amplitudes for large energies $s^{1/2}$ is carried out in the leading logarithmic approximation:

$$g^2 \ln\frac{s}{m^2} \sim 1, \quad g^2 \ll 1, \quad s=(p_A+p_B)^2 \gg m^2, \quad -t\sim m^2. \tag{1}$$

In a preceding paper[8] we have shown that for an inelastic process in the multiregge kinematics (cf. Fig. 1)

$$s_k=(p_{D_{k-1}}+p_{D_k})^2 \gg m^2, \quad -t_i=-q_i^2\sim m^2, \quad \prod_{i=1}^{n+1} s_i = s\prod_{i=1}^{n}(m_i^2+\mathbf{p}_{D_i}^{\perp 2}), \tag{2}$$

which yields the main contribution to the absorptive part of the s-channel elastic amplitude; the corresponding inelastic amplitude has the factorized form (cf. Eq. (55) in[8]):

$$A_{2\to 2+n} = s\Gamma_{AD_0}^{i_1}\frac{(s_1/m^2)^{\alpha(t_1)}}{t_1-m^2}\gamma_{i_1 i_2}^{D_1}(q_1,q_2)\frac{(s_2/m^2)^{\alpha(t_2)}}{t_2-m^2} \ldots \gamma_{i_n i_{n+1}}^{D_n}(q_n,q_{n+1})\frac{(s_{n+1}/m^2)^{\alpha(t_{n+1})}}{t_{n+1}-m^2}\Gamma_{BD_{n+1}}^{i_{n+1}}, \tag{3}$$

where

$$j=1+\alpha(t)=1+\frac{g^2}{(2\pi)^3}(t-m^2)\int\frac{d^2k}{[(k-q)^2-m^2][k^2-m^2]}, \tag{4}$$
$$(t=q^2,\ q=q_\perp,\ k=k_\perp,\ k^2=-\mathbf{k}_\perp^2)$$

is the Regge pole trajectory. For $t=m^2$ this pole traverses the point $j=1$, corresponding to the spin of the original Yang-Mills boson, and this means the reggeization of the latter.

The vertices γ_{ij}^D are

$$\gamma_{ij}^D = mg\delta_{ij} \tag{5}$$

for the emission of scalar particles, and

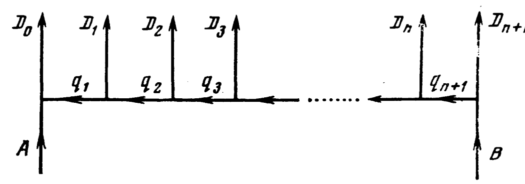

FIG. 1.

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$$\gamma_{ij}^{D}(q_1,q_2)=ig\varepsilon_{dij}\left[-(q_1+q_2)^{\mu}-p_A^{\mu}\left(\frac{2p_Bp_D}{p_Ap_B}-\frac{m^2-q_1^2}{p_Ap_D}\right)+p_B^{\mu}\left(\frac{2p_Ap_D}{p_Ap_B}-\frac{m^2-q_2^2}{p_Bp_D}\right)\right]e_{\mu}(\lambda_D) \qquad (6)$$

for the emission of vector particles D with polarization $\lambda_D=1, 2, 3$. The value $\lambda_D=3$ corresponds to a longitudinally polarized vector particle. The letters, i,j,d, label the isospin indices.

The vertices Γ^{i}_{AD} have a different form, depending on the type of scattered particle. For isospin $T=\frac{1}{2}$ quarks we have

$$\Gamma_{AD}^{i}=-{}^1/_2\sqrt{2}g(\tau^i)_{da}\delta_{r_Dr_A}. \qquad (7)$$

For the scattering of vector particles

$$\Gamma_{AD}^{i}=ig\sqrt{2}\varepsilon_{ida}a_{\lambda_A}\delta_{\lambda_A\lambda_D}, \qquad (8)$$

where $a_{\lambda_A}=1$ for $\lambda_A=1, 2$ and $a_{\lambda_A}=\frac{1}{2}$ for $\lambda_A=3$, and for the transition of a vector particle into a scalar particle

$$\Gamma_{AD}^{i}={}^1/_2\sqrt{2}g\delta_{\lambda_A3}\delta_{ai}. \qquad (9)$$

We note that some relations between the different vertex functions (7)–(9) follow from the group properties of the initial massless Yang-Mills theory.[7]

Making use of the fact that the contribution to the absorptive part of the elastic amplitude coming from the $(2+n)$-particle s-channel intermediate state can be calculated to logarithmic accuracy (1) according to the formula

$$\mathrm{Im}_s A_{2\to2+n\to2'}=\pi\sum_{D_i}\left(\frac{1}{2(2\pi)^3}\right)^{n+1}\frac{1}{s}\int\prod_{i=1}^{n+1}ds_i\,d^2k_i \times\delta\left(s\prod_{i=1}^{n}(m^2+\mathbf{p}_{D_i}^{\perp 2})-\prod_{i=1}^{n+1}s_i\right)A_{2\to2+n}(k_i)A_{2+n\to2'}(q-k_i), \qquad (10)$$

where the integration is over the region (2), we obtain, after using dispersion relations in s, the elastic amplitude in the following form (cf.[8] for more details):

$$A_{AB}^{A'B'}(s,q)=\Gamma_{AA'}A^{(0)}(s,q)\Gamma_{BB'}+\Gamma_{AA'}^{i}A^{(1)}(s,q)\Gamma_{BB'}^{i}+\Gamma_{AA'}^{ij}A^{(2)}(s,q)\Gamma_{BB'}^{ij}; \qquad (11)$$

here $A^{(T)}(s,q)$ is the scattering amplitude with definite isospin T in the t-channel. These amplitudes can be represented in a multiregge form (cf. Eq. (60) in[8]):

$$A^{(T)}(s,q)=\frac{s}{4i}\int_{\delta-i\infty}^{\delta+i\infty}d\omega\left(\frac{s}{m^2}\right)^{\omega}\frac{e^{-i\pi\omega}-(-1)^T}{\sin\pi\omega}F_{\omega}^{(T)}(q^2), \qquad (12)$$

where

$$F_{\omega}^{(T)}(q^2)=A_T^{-1}+\sum_{n=0}^{\infty}\left(\frac{q^2}{(2\pi)^3}\right)^{n+1}\int K^{(T)}(q_1,q_2)\ldots K^{(T)}(q_n,q_{n+1}) \times\prod_{i=1}^{n+1}\frac{d^2q_i}{(q_i^2-m^2)[(q_i-q)^2-m^2][\omega-\alpha(q_i^2)-\alpha((q_i-q)^2)]}. \qquad (13)$$

Here $A_T=C_T(t-(3/2)m^2)+\frac{1}{2}m^2$, $C_T=2-\frac{1}{2}T(T+1)$, and the scattering amplitude for reggeons on each other, $K^{(T)}(q_1,q_2)$ has the form

$$K^{(T)}(q_1,q_2)=A_T-C_T\frac{(q_1^2-m^2)[(q-q_2)^2-m^2]+(q_2^2-m^2)[(q-q_1)^2-m^2]}{(q_1-q_2)^2-m^2}. \qquad (14)$$

The vertices in Eq. (11) for vector (V), scalar (S), and spinor (F) particles (cf. also (7)–(9)) are:

$$\Gamma_{SS'}=-{}^1/_2\sqrt{3}ig,\quad \Gamma_{VV'}=-{}^2/_3\sqrt{3}ig\delta_{vv'}\delta_{\lambda_V\lambda_{V'}}(2a_{\lambda_V}^2+{}^1/_4\delta_{\lambda_V3}),$$
$$\Gamma_{SV}=\Gamma_{VS}=0,\quad \Gamma_{VV'}^{ij}=-2igC_{vv'}^{ij}\delta_{\lambda_V\lambda_{V'}}({}^1/_4\delta_{\lambda_V3}-a_{\lambda_V}^2), \qquad (15)$$
$$\Gamma_{SS'}^{ij}=\Gamma_{SV}^{ij}=\Gamma_{VS}^{ij}=\Gamma_{FF'}^{ij}=0,\quad \Gamma_{FF'}=-{}^1/_2\sqrt{3}ig\delta_{\varphi\varphi'}\delta_{r_Fr_{F'}},$$

where

$$C_{vv'}^{ij}={}^1/_2(\delta_v^i\delta_{v'}^j+\delta_v^j\delta_{v'}^i-{}^2/_3\delta_{vv'}\delta^{ij}).$$

We note that the first term in (13) leads to the Born term in the amplitude $A_{AB}^{A'B'}(s,q)$ for $T=1$. In the case $T=0, 2$ the signature factor in (12) does not contain a pole at $\omega=0$, i.e., the first term in (13) yields a negligibly small contribution in the even signature. Elimination of this term from $F_{\omega}^{(T)}$ is necessary in order that t-channel unitarity hold (cf. the following section). The amplitude $A_{AB}^{A'B'}(s,q)$ was calculated in[9] to eighth order of perturbation theory. The results of Lo and Cheng coincide with the expansion to eighth order of our expressions (11)–(13); the results of McCoy and Wu for $T=0, 2$ also agree with ours, after corrections of misprints.

As is easily verified, the amplitude $F_{\omega}^{(T)}(q^2)$ can be expressed in terms of the off-shell amplitude $F_{\omega}^{(T)}(k, q-k)$ according to the formula

$$F_{\omega}^{(T)}(q^2)=F_{\omega}^{(T)}(k,q-k)\,|_{k^2=(q-k)^2=m^2}, \qquad (16)$$

where the amplitude $F_{\omega}^{(T)}(k, q-k)$ satisfies the Bethe-Salpeter equation

$$[\omega-\alpha(k^2)-\alpha((q-k)^2)]F_{\omega}^{(T)}(k,q-k)=\frac{\omega}{A_T}+\frac{g^2}{(2\pi)^3}\int d^2k'\frac{K^{(T)}(k,k')}{(k'^2-m^2)[(q-k')^2-m^2]}F_{\omega}^{(T)}(k',q-k'). \qquad (17)$$

The solution of this equation for $T=1$

$$F_{\omega}^{(1)}(k,q-k)=\frac{\omega}{(t-m^2)(\omega-\alpha(q^2))},\quad t=q^2, \qquad (18)$$

shows that the branch points in the j-plane which appear in individual diagrams according to the Amati-Fubini-Stanghellini mechanism[14] cancel in the sum for $T=1$, so that as a result we have only a pole for $\omega=\alpha(q^2)$, corresponding to a reggeized vector meson. Thus, the multiregge equation leads in this case to the same singularity in the j-plane as that which determines the high-energy behavior of the initial inelastic amplitudes (3): we have a peculiar "bootstrap."[15]

In the case $T=0$ and $T=2$ there is no cancellation of the branch cuts related to the exchange of two reggeized vector mesons, as we will see in the following section.

3. MOVING SINGULARITIES IN THE j-PLANE PARTIAL WAVES

We introduce the new functions $f_{\omega}^{T}(t)$ according to the definition

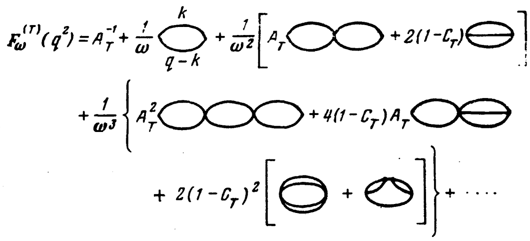

FIG. 2.

$$f_\omega^T(t)=\frac{8g^2\omega^{-1}}{t-4m^2}A_T^2F_\omega^{(T)}(t). \tag{19}$$

These functions coincide with the partial wave amplitudes $T^{(T)}_{1-1,1-1}$ for nonsense–nonsense transitions for the t-channel scattering of two vector mesons.[6]

From the equation[13] with the use of the explicit form of the kernel $K^{(T)}(q_1, q_2)$ (cf. (14)) it follows that in the region $4m^2<t<9m^2$ the partial waves we have introduced in (19) satisfy the unitarity condition in the form

$$\operatorname{Im} f_\omega^T(t)=\frac{1}{32\pi}\left(\frac{t-4m^2}{t}\right)^{1/2}|f_\omega^T(t)|^2. \tag{20}$$

As is well known,[16] it follows from the equation (20) that the partial waves $f_\omega^T(t)$ must have Regge poles, whereas near the fixed singularities at $\omega=\omega_0$ the function $f_\omega^T(t)$ cannot be singular (the limit of $f_\omega^T(t)$ for $\omega\to\omega_0$ must exist). As will be shown in the following section, the leading singularity in the vacuum channel with $T=0$ is a fixed square-root branch point for $\omega_0=j-1=2\pi^{-2}g^2\ln 2$. We then obtain from the relation (20) that

$$f_\omega^0(t)|_{\omega\to\omega_0}=C(t)+C_1(t)(\omega-\omega_0)^{1/2}. \tag{21}$$

It follows from Eqs. (19) and (20) that for $t\to 4m^2$ the leading singularity in all three channels $T=0, 1, 2$ is the Regge pole

$$f_\omega^T(t)|_{t\to4m^2}=\frac{64\pi m B_T(t-4m^2)^{-1}}{\omega-B_T(t-4m^2)^{-1/2}},\quad B_T=\left(\frac{1}{2}+\frac{5}{2}C_T\right)\frac{g^2}{8\pi}m. \tag{22}$$

As t decreases for $T=0$ this pole moves to the left and in the s-channel physical region ($t<0$) it will hide under the cut $\omega=\omega_0$ (cf. (21)), no longer influencing the asymptotic behavior of the scattering amplitude (12). The result (22) can also be obtained directly from (13), if one collects the terms which are most singular for $t\to 4m^2$ (for this one has to omit in (14) the last term). We note in addition that Eq. (22) makes sense only in the region where the expansion (13) is valid, i.e., for $\omega\sim g^2\ll 1$. In the region $\omega\sim 1$ the unitarity condition (20) must be changed,[16] which leads to the "freezing" of the Regge pole (22) for $t\to 4m^2$, somewhere near the point $\omega=1$.

It is easy to see by means of the expansion of (13) into a perturbation theory series (Fig. 2), that for $T=0$ and $T=2$ the partial waves $f_\omega^T(t)$ have in t thresholds for $t=(nm)^2$ of arbitrarily high order $n=2, 3, 4, \ldots$. This seems to be the explanation of the appearance of a fixed square-root branch point in the j-plane, if one considers this problem from the point of view of analyticity and unitarity in the t-channel. Indeed, the mechanism of appearance of a fixed singularity related to the nondecrease at large t of the singular part of the discontinuity of the partial-wave amplitude on the left-hand cut[17] which occurs in quantum electrodynamics,[10] will be absent in this case, since the partial wave (13) has only singularities for positive t. As is well known, the slope of the Regge trajectory corresponding to the bound state of n particles decreases with the increase of n. This mechanism could lead for $n\to\infty$ to a fixed singularity.[1)]

We consider in more detail the unitarity condition at the three-particle threshold. It follows from Eqs. (13) and (19) that the discontinuity of the partial-wave amplitude $f_\omega^T(t)$ produced by the three-particle intermediate state in the t-channel can be represented in the following form (cf. Eq. (20)):

$$\begin{aligned}\operatorname{Im}_3 f_\omega^T(t)=&-\frac{1}{2}\int\delta^{(2)}\left(q-\sum k_i\right)\prod_{i=1}^{3}2\pi\delta(k_i^2-m^2)\theta(k_i^0)\,d^2k_i\\ &\times f_{2\to3}(k_1+k_2,k_3)f^*_{2\to3}(k_1+k_2,k_3)+\frac{1}{2}C_T\int\delta^{(2)}\left(q-\sum k_i\right)\\ &\times\prod_{i=1}^{3}2\pi\delta(k_i^2-m^2)\theta(k_i^0)\,d^2k_i f_{2\to3}(k_1+k_2,k_3)f^*_{2\to3}(k_1+k_3,k_2),\end{aligned} \tag{23}$$

where the t-channel transition amplitude of two particles into three, $f_{2\to3}$, can be expressed in terms of the off-shell two-particle amplitude in the following manner (cf. (19)):

$$f_{2\to3}(k_1+k_2,k_3)=\frac{4}{(2\pi)^3}\left(\frac{g}{\sqrt{\omega}}\right)^3\frac{A_T^2}{(t-4m^2)^{1/2}}F_\omega^{(T)}(k_1+k_2,k_3)|_{k_i^2=m^2}. \tag{24}$$

In Eq. (24) the function $F_\omega^{(T)}$ represents a solution of Eq. (17) analytically continued into the region $k^2>0$, $(q-k)^2>0$, $q^2>0$ (this continuation requires a pseudo-euclidean rotation in the integrals (17)).

The first term in Eq. (23) corresponds to the diagram a in Fig. 3, where the particles with momenta k_1 and k_2 go over into reggeons of isospin $T=1$ and negative signature, and the second term corresponds to the interference graph b.

We note that on account of the fact that for negative signature and $T=1$ the function $F^{(T)}(k_1+k_2, k_3)|_{k_3^2=m^2}$ does not depend on $(k_1+k_2)^2$ (cf. Eq. (18)), the two terms in Eqs. (23) cancel mutually (since $C_T|_{T=1}=1$), so that the partial wave with negative signature has only two-particle singularities in the t-channel.

In the case of positive signature ($T=0, 2$) the cancellation of the imaginary part at the three-particle threshold does not occur. The first term in Eq. (23) is the contribution to the imaginary part from the three-particle intermediate state in the t-channel, which appears when one cuts the $T=1$ reggeon from the particles (cf. Fig. 3, a). This contribution is of opposite sign com-

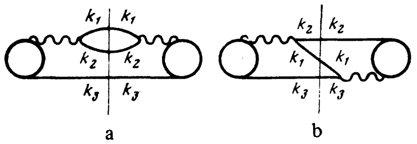

FIG. 3.

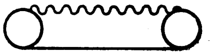

FIG. 4.

pared to the contribution (20) of the two-particle intermediate state, which is directly related to the fact of reggeization of the vector meson. Indeed, the partial-wave corresponding to a graph with the exchange of a state consisting of a reggeon and a particle (cf. Fig. 4) is proportional to an integral over k containing a factor $A = -(k^2 - m^2 + i\varepsilon)^{-1}(\omega - \alpha(k^2))^{-1}$. If the vector meson reggeizes $\alpha(k^2)|_{k^2=m^2} = 0$, so that the contribution to the two-particle absorptive part is proportional to the integral of $\mathrm{Im}_1 A = \pi\omega^{-1}\delta(k^2 - m^2)$, and the contribution of the three-particle intermediate state is proportional to the integral of $\mathrm{Im}_2 A|_{k^2>4m^2} = -(k^2 - m^2)^{-1}|\omega - \alpha(k^2)|^{-2}\,\mathrm{Im}\,\alpha$ and is a negative quantity on account of the two-particle unitarity relation (20). The negative sign of the imaginary part of the partial wave with positive signature at the three-particle threshold for real $j \neq 2n$ is possible on account of the fact that for an analytic continuation from physical values $j = 2n$ in the three-particle case, in distinction from the two-particle case (20), there appear in the right-hand side of the unitarity condition terms which are proportional to $\tan(\pi j/2)$, so that the result is no longer positive definite.[18]

We now consider the question of the character of the singularities of the partial-wave amplitude in the j-plane near the three-particle threshold $t = 9m^2 + 6mE_0 \to 9m^2$, where E_0 is the energy of the particles in the center-of-mass frame. After integrating over the "time" components of the two-vectors k_i, Eq. (23) can be rewritten in the nonrelativistic limit in the form

$$\mathrm{Im}_3 f_\omega^T(t)|_{t\to 9m^2} = -\frac{1}{2}\left(\frac{\pi}{m}\right)^3 \int dk_1\,dk_2\,dk_3\,\delta\left(q - \sum k_i\right)$$
$$\times\delta\left(E_0 - \frac{1}{2m}\sum k_i^2 + \frac{1}{6}\frac{q^2}{m}\right)[f_{2\to3}(k_1+k_2,k_3)f^*_{2\to3}(k_1+k_2,k_3)$$
$$-C_T f_{2\to3}(k_1+k_2,k_3)f^*_{2\to3}(k_1+k_3,k_2)], \qquad (25)$$

where q is the momentum of the center of mass of the three particles, whereas the equation for $f_{2\to3}$ is easily obtained in the same limit from Eq. (17), taking into account the relation (24), in the following form

$$\omega f_{2\to3}(k_1+k_2,k_3) - \frac{3g^2}{16\pi^2}\int dk'\,[f_{2\to3}(k_1+k_2,k_3) - C_T f_{2\to3}(k'+k_3,k_1+k_2-k')]$$
$$\times\left[\frac{k'^2}{2m} + \frac{(k_1+k_2-k')^2}{2m} + \frac{k_3^2}{2m} - E - i\varepsilon\right]^{-1} = \frac{g^3 P_T m}{2\sqrt{5}\,\omega\pi^3}, \qquad (26)$$
$$E = E_0 + q^2/6m, \quad P_T = {}^1/_2 + {}^{15}/_2 C_T,$$

where it was assumed that $\omega/g^2 \sim m(t - 9m^2)^{-1/2} \gg 1$. This makes it possible to omit the three-particle terms, which appear in the case when one retains in the kernel of the integral equation (17) the first term of Eq. (14). Equation (26) is a one-dimensional Faddeev equation for the three-particle scattering amplitude with a delta-like pair interaction.[19] Going over into the coordinate representation

$$f_{2\to3}(k_1+k_2,k_3)\Big/\left[\sum_{i=1}^{3} k_i^2(2m)^{-1} - E - i\varepsilon\right]$$
$$= \int dx_1\,dx_2\,dx_3 \exp\left(i\sum_{j=1}^{3} k_j x_j\right)\varphi(x_1,x_2;x_3), \qquad (27)$$

where $\varphi(x_1, x_2; x_3) = \varphi(x_2, x_1; x_3)$, we obtain in place of (26) the equation

$$\left[-\frac{1}{2m}\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2} - E\right]\varphi(x_1,x_2;x_3) - \frac{\lambda}{m}\delta(x_1-x_2)\,[\varphi(x_1,x_2;x_3)$$
$$-C_T\varphi(x_1,x_3;x_2)\,\Big] = \chi\delta(x_1)\delta(x_2)\delta(x_3). \qquad (28)$$

Here

$$\lambda = \frac{3g^2 m}{8\pi\omega}, \quad \chi = \left(\frac{g^2}{\omega}\right)^{3/2}\frac{P_T m}{2\sqrt{5}\,\pi^3}.$$

We consider the case $C_T = -2$, corresponding to the unphysical value $T = 3.4$, where it is possible to obtain an exact solution. For $C_T = -2$ the unitarity relation (25) contains the square of the absolute value of the function

$$f(k_1,k_2,k_3) = f_{2\to3}(k_1+k_2,k_3) + f_{2\to3}(k_1+k_3,k_2) + f_{2\to3}(k_2+k_3,k_1);$$

in the coordinate representation for the function $G(x_1, x_2, x_3)$ defined according to (27) with the substitution

$$f_{2\to3}(k_1+k_2,k_3) \to f(k_1,k_2,k_3), \quad \varphi(x_1,x_2;x_3) \to G(x_1,x_2,x_3),$$

we obtain

$$\left[-\frac{1}{2m}\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2} - E - \frac{\lambda}{m}\sum_{i<j}\delta(x_i - x_j)\right]G(x_1,x_2,x_3) = 3\chi\delta(x_1)\delta(x_2)\delta(x_3). \qquad (29)$$

This is an equation for the Green's function of three identical pointlike spinless particles. A complete set of wave functions of the homogeneous equation (29) is known,[19] in terms of which $G(x_1, x_2, x_3)$ can be expressed in the standard form. We first consider the repulsive case $\lambda < 0$ (in terms of the variable ω this means $\omega < 0$). In this case the set of functions $\psi(x_1 x_2 x_3 | k_1 k_2 k_3)$ with real k_i is complete.[19] Use of dispersion relations for the reconstruction of $f_\omega^T(t)$ in terms of its imaginary part yields for $C_T = -2$:

$$f_\omega^T(t)|_{C_T=-2} = \frac{\chi}{2\pi}\left(\frac{\pi}{m}\right)^3\int dk_1\,dk_2\,dk_3\,\delta\left(q - \sum_i k_i\right)f(k_1,k_2,k_3)$$
$$\times\left[E + i\varepsilon - \frac{1}{2m}\sum_i k_i^2\right]^{-1} = -2\pi\chi\left(\frac{\pi}{m}\right)^3\int dz\,G(z,z,z)\,e^{iqz}$$
$$= \frac{3g^6 P_T^2}{40\pi^4\omega^3 m}\int dk_1\,dk_2\,dk_3\left[E + i\varepsilon - \frac{1}{2m}\sum_i k_i^2\right]^{-1}$$
$$\times\delta\left(q - \sum_i k_i\right)\prod_{i<j}\left(1 + \frac{\lambda^2}{k_{ij}^2}\right)^{-1}, \qquad (30)$$

where $E = E_0 + q^2/6m = (t - 9m^2 + q^2)/6m$; $k_{ij} = k_i - k_j$. In integrating over k_i in Eq. (30) a cutoff at momenta of the order of m is understood. It is obvious that in the region below the threshold ($t < 9m^2$), for $\lambda < 0$ ($\omega < 0$) the expression (30) has no singularity in λ. In the case of attraction ($\lambda > 0$, $\omega > 0$) the result can be obtained either by using the complete set of functions for that case,[19] or by analytically continuing the expression (30):

$$f_\omega^T(t)|_{C_T=-2}=\frac{3g^2P_T^2}{40\pi^4\omega^3 m}\left\{\int dk_1\,dk_2\,dk_3\,\delta\left(q-\sum k_i\right)\right.$$
$$\times\left[E+i\varepsilon-\frac{1}{2m}\sum_i k_i^2\right]^{-1}\left[\prod_{i<j}\left(1+\frac{\lambda^2}{k_{ij}^2}\right)\right]^{-1}$$
$$\left.+2\pi\lambda\int dp\frac{4p^2+\lambda^2}{(E_0+i\varepsilon-p^2/3m+\lambda^2/4m)(4p^2+9\lambda^2)}+\frac{8\pi^2\lambda^2}{3(E_0+\lambda^2/m)}\right\}, \quad (31)$$

i.e., we see that in the ω-plane there is a Mandelstam branch point for $\omega=3g^2m/8\pi(-4mE_0)^{1/2}$, corresponding in the language of the quantum-mechanical problem (29) to the production threshold of a particle and a bound state of two particles with a binding energy $-\lambda^2/4m$, and a pole for $\omega=3g^2m/8\pi(-mE)^{1/2}$, corresponding to a bound state of three particles with a binding energy $-\lambda^2/m$.

Although we have not succeeded in solving the system (28) for arbitrary T, we think that the analytic properties of the partial-wave amplitudes $f_\omega^T(t)$ in the ω-plane will be the same as for $C_T=-2$, i.e., the moving singularities of the partial waves (19) in the vacuum channel and in the channel $T=2$ are Mandelstam branch points and possible Regge poles.

The character of the moving singularities of the partial waves $f_\omega^T(t)$ in the ω-plane can be illustrated on the example of the wave with $C_T=0$ (this corresponds to the unphysical isospin value $T=1.6$). For this case the exact equation (17) is easily solved:

$$F_\omega^{(T)}(k,q-k)|_{C_T=0}=\frac{2}{m^2}[\omega-\alpha(k^2)-\alpha((q-k)^2)]^{-1}$$
$$\times\left\{1-\frac{g^2m^2}{2(2\pi)^3}\int\frac{d^2k_1}{(k_1^2-m^2)[(q-k_1)^2-m^2][\omega-\alpha(k_1^2)-\alpha((q-k_1)^2)]}\right\}^{-1}. \quad (32)$$

The existence of moving poles and cuts is obvious from (32).

4. THE FIXED SINGULARITY OF THE PARTIAL-WAVE AMPLITUDE FOR THE VACUUM CHANNEL

For the investigation of the singularity in the j-plane of the partial wave with the quantum numbers of the vacuum in the t-channel ($T=0$) we consider the amplitude $F_\omega(k_1,k_2;q)$ with momenta $k_1,k_2,q-k_1,q-k_2$ off the mass shell, amplitude which satisfies the equation

$$[\omega-\alpha(k_1^2)-\alpha((q-k_1)^2)]F_\omega(k_1,k_2;q)=K^{(0)}(k_1,k_2)+\bar K(k_1,k_2)$$
$$+\frac{g^2}{(2\pi)^3}\int d^2k'\frac{K^{(0)}(k_1,k')}{(k'^2-m^2)[(q-k')^2-m^2]}F_\omega(k',k_2,q), \quad (33)$$

where

$$\bar K(k_1,k_2)=\frac{2(k_2^2-m^2)[(q-k_2)^2-m^2]}{(k_1-k_2)^2-m^2}\left[\frac{k_1^2-m^2}{(k_1-k_2)^2-m^2+(k_2^2-m^2)}\right.$$
$$\left.+\frac{(q-k_1)^2-m^2}{(k_1-k_2)^2+(q-k_2)^2-2m^2}\right], \quad (34)$$
$$\bar K(k_1,k_2)|_{k_2^2=(q-k_2)^2=m^2}=0,$$
$$\frac{g^2}{(2\pi)^3}\int d^2k_2\frac{\bar K(k_1,k_2)}{(k_2^2-m^2)[(q-k_2)^2-m^2]}=\alpha(k_1^2)+\alpha((q-k_1)^2).$$

Here $F_\omega(k_1,k_2,q)$ satisfies the relations

$$F_\omega^{(0)}(k_1,q-k_1)=\frac{1}{A_T}+\frac{g^2}{(2\pi)^3}\int d^2k_2\frac{F_\omega(k_1,k_2;q)A_T^{-1}}{(k_2^2-m^2)[(q-k_2)^2-m^2]}$$
$$F_\omega(k_1,k_2;q)|_{k_2^2=(q-k_2)^2=m^2}=\frac{A_T^2}{\omega}F_\omega^{(0)}(k_1,q-k_1). \quad (35)$$

The equation (33) can be rewritten in the form

$$\omega F_\omega(k_1,k_2;q)=K^{(0)}(k_1,k_2)+\bar K(k_1,k_2)$$
$$+\frac{g^2}{(2\pi)^3}\int\frac{d^2k'}{(k'^2-m^2)[(q-k')^2-m^2]}[K^{(0)}(k_1,k')F_\omega(k',k_2;q)$$
$$+\bar K(k_1,k')F_\omega(k_1,k_2;q)]. \quad (36)$$

We consider this equation for large $x_{1,2}=-k_{1,2}^2$. Neglecting m^2,q^2 and averaging over the directions of $\mathbf{k}_2$, we obtain

$$\omega F_\omega(x_1,x_2)=\frac{4x_1x_2}{(x_1^2+4x_2^2)^{1/2}}+\frac{g^2}{2\pi^2}x_1\int_0^\infty dx\left[\frac{F_\omega(x,x_2)}{x|x-x_1|}\right.$$
$$\left.-\left(\frac{1}{x|x-x_1|}-\frac{1}{x(x_1^2+4x^2)^{1/2}}\right)F_\omega(x_1,x_2)\right]. \quad (37)$$

We look for the solution in the form $F_\omega(x_1,x_2)=x_1L_\omega\times(x_1/x_2)$; for $L_\omega(z)$ we have the equation

$$\omega L_\omega(z)=\frac{4}{(4+z^2)^{1/2}}$$
$$+\frac{g^2}{2\pi^2}\int_0^\infty dz'\left[\frac{L_\omega(zz')}{|1-z'|}-\left(\frac{1}{z'|1-z'|}-\frac{1}{z'(1+4z'^2)^{1/2}}\right)L_\omega(z)\right]. \quad (38)$$

For the homogeneous equation (38) there exists a system of eigenfunctions $\eta_\nu(z)$, $-\infty<\nu<\infty$, which is complete in the interval $[0,\infty]$, corresponding to the eigenvalues $\omega=\chi(\nu)$:

$$\eta_\nu(z)=z^{i\nu-1/2}/(2\pi)^{1/2},$$
$$\chi(\nu)=\frac{g^2}{\pi^2}\mathrm{Re}\int_0^1\frac{dz}{1-z}(z^{i\nu-1/2}-1)=\frac{g^2}{2\pi^2}\sum_{n=0}^\infty\frac{n+1/2-2\nu^2}{(n+1)[(n+1/2)^2+\nu^2]}. \quad (39)$$

The solution of the inhomogeneous equation (38) can be expanded in terms of these functions:

$$L_\omega(z)=\int_{-\infty}^\infty d\nu\frac{\eta_\nu(z)}{\omega-\chi(\nu)}\int_0^\infty dz'\frac{4\eta_\nu^*(z')}{(4+z'^2)^{1/2}}. \quad (40)$$

The maximal value of $\chi(\nu)$ is attained for $\nu=0$ and equals $(2g^2\ln 2)/\pi^2$. The function $L_\omega(z)$ has, consequently, a cut in the ω plane for $\omega<\omega_0=(2g^2\ln 2)/\pi^2$. Since the position of the singularity of the partial wave amplitude does not depend on the degree of virtuality of the particle, we conclude that $f_\omega^{(0)}(t)$ has a fixed branch point at $\omega=\omega_0$.

One can arrive at this result from a somewhat different point of view. The appearance of the singularity of the partial-wave amplitude at the point $\omega=\omega_0$ means that the perturbation theory series with respect to $(g^2/\omega)^n$ diverges at that point for this wave. In order to understand the character of the singularity and of the divergence point it is necessary to know the asymptotic behavior of the terms of the series for $n\to\infty$. We shall iterate Eq. (17) in the case $T=0$:

$$F_\omega^{(0)}(k,q-k)=A_0^{-1}\sum_{n=1}^\infty\varphi_n(k,q);$$
$$\omega\varphi_n(k,q)=\frac{g^2}{(2\pi)^3}\int\frac{d^2k'}{(k'^2-m^2)[(q-k')^2-m^2]}[K^{(0)}(k,k')\varphi_{n-1}(k',q)$$
$$+\bar K(k,k')\varphi_{n-1}(k,q)],$$
$$\varphi_1(k,q)=1. \quad (41)$$

A similar series can be written for $F_\omega^{(0)}(t)$:

$$F_\omega^{(0)}(q^2)=\sum_{n=1}\varphi_n(q^2),\quad \varphi_1(q^2)=A_0^{-1}. \tag{42}$$

Here

$$\varphi_{2n}(q^2)=\int d^2k\frac{\varphi_n^2(k,q)}{(k^2-m^2)[(q-k)^2-m^2]}. \tag{43}$$

As n grows in the interval (43) the essential contribution comes from larger and larger values of $-k^2$. For large $-k^2$ one can neglect the q^2-dependence in (41); defining $\varphi_n(k,q)=\psi_n(\xi)(-k)^{1/2}$, where $\xi=\ln(-k^2/m^2)$, we obtain

$$\omega\psi_n(\xi)=\frac{g^2}{2\pi^2}\int d\xi'\Big[\frac{e^{(\xi'-\xi)/2}\psi_{n-1}(\xi')}{|1-e^{\xi'-\xi}|} -\Big(\frac{1}{|1-e^{\xi'-\xi}|}-\frac{1}{[1+4e^{2(\xi'-\xi)}]^{1/2}}\Big)\psi_{n-1}(\xi)\Big]. \tag{44}$$

By analogy with the diffusion equation (interpreting $\psi_n(\xi)$ as the probability of being situated on the nth step at the point ξ), one should expect that $\psi_n(\xi)$ will change over characteristic distances $\xi\sim n^{1/2}$; therefore in (44) one may expand $\psi_{n-1}(\xi')$ at the point ξ; as a result of this we obtain

$$\omega\psi_n(\xi)=\omega_0\psi_{n-1}(\xi)+C\frac{\partial^2}{\partial\xi^2}\psi_{n-1}(\xi), \tag{45}$$

where $\omega_0=(2g^2\ln 2)/\pi^2$, $C=7\pi^{-2}g^2\zeta(3)$, where $\zeta(n)$ is the Riemann zeta function. Going from the difference equation to a differential equation, we shall have

$$\omega\frac{\partial\psi(n,\xi)}{\partial n}=(\omega_0-\omega)\psi(n,\xi)+C\frac{\partial^2\psi(n,\xi)}{\partial\xi^2}. \tag{46}$$

Substituting the solution of this equation for large n

$$\psi_n(\xi)\sim\exp\Big(\frac{\omega_0-\omega}{\omega}n\Big)\frac{\xi}{n^{3/2}}\exp\Big(-\frac{\xi^2}{4nc}\Big) \tag{47}$$

into (43), where the q-dependence may now be neglected, we obtain

$$\varphi_{2n}(q^2)\sim\int_0^\infty d\xi\frac{\xi^2}{n^3}\exp\Big(2\frac{\omega_0-\omega}{\omega}n-\frac{\xi^2}{2nc}\Big)\sim n^{-3/2}\exp\Big[\frac{2(\omega_0-\omega)}{\omega}n\Big]. \tag{48}$$

Summation over n yields

$$F_\omega^{(0)}(t)\sim\int\frac{dn}{n^{3/2}}\exp\Big[(\omega_0-\omega)\frac{n}{\omega}\Big]\sim\sqrt{\omega-\omega_0}. \tag{49}$$

Thus, the leading singularity in the j-plane for the vacuum channel ($T=0$) is a fixed square-root branch point for $j=\omega_0+1>1$,

CONCLUSION

Starting from the multiregge equation for the partial waves with different quantum numbers in the t-channel, (13), obtained in the leading logarithmic approximation (1), we have shown that the only singularity in the j-plane for the $T=1$ partial wave is a Regge pole with the trajectory $j=1+\alpha(t)$, (18), whereas for $T=0$, 2 the j-plane structure is considerably more complicated. In this case an investigation of the partial-wave amplitudes near the two- and three-particle thresholds in the t-channel which we carried out in Sec. 3, shows the existence of Regge poles (22) and of Mandelstam cuts (31), (32). In Sec. 4 we have investigated the leading singularity in the physical region for the vacuum channel and have shown that it is a fixed square-root branch point for $j=\omega_0+1>1$, (49), meaning a power-law growth of the total cross sections with energy. In quantum electrodynamics the reason for the violation of the Froissart theorem is the growth of the cross sections for the production of an arbitrary number of e^+e^--pairs. Here, in spite of the decrease of each cross section for n-particle production with energy, the total cross sections increase, i.e., the growth of the number of open channels prevails. We remark here on the following circumstance. From the analysis carried out in Sec. 4 it can be seen that the determining contribution to the formation of a singularity at $\omega=\omega_0$ comes from high virtualities (i.e., deviations from the mass shell). One may therefore hope that in an asymptotically free theory the situation will change. If, however, in our equations we simply consider the coupling constant as dependent on k^2, there will not occur any cardinal changes; the character of the singularity changes, but there remains a singularity to the right of $j=1$.

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