Contribution to the theory of photon spectra in a semiconductor laser

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The spectrum of the supercondensate photons in a semiconductor laser is obtained at a large excess above threshold. The supercondensate photons are the result of an interaction between the condensate photons (via the electron system). This phenomenon is analogous to the "pushing out" of particles from a non-ideal Bose condensate. In contrast to the case of an equilibrium non-ideal Bose gas, however, the number of supercondensate photons increases with time. The instability of the photon condensate in the laser is due to the fact that the processes of production of supercondensate photons from the condensate are real.

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It is suggested in a number of recent papers (see, e.g., [1]) that a definite analogy exists between phase transitions in thermodynamic-equilibrium systems, on the one hand, and the transition that occurs in lasers when the generation threshold is exceeded, on the other. In fact, the electromagnetic field generated in a laser above threshold is a Bose condensate of photons with frequency $\Omega \neq 0$. The Bose-condensate photons interact with one another via electrons that move between bands under the influence of the field.

The interacting-photon condensate in a laser is thus analogous in some respects to the interacting-boson system considered by Bogolyubov.[2]

Starting with this analogy, we can expect the interaction between the photons to lead to the appearance of above-condensate photons, with frequencies different from the frequency Ω of the Bose-condensate photons. To highlight this effect, it is natural to investigate the spectrum of the photons in the laser at a large excess above threshold, when the field is strong and the damping processes are inessential.

We consider in this case the limiting case of a lossless semiconductor laser. The photon lifetime in the resonator is assumed to be infinite, and the pump current infinitesimally small. A certain limiting field $\mathbf{E}_0$ is then established.[3]

Our principal result consists in the following: Allowance for the interaction between the condensate photons leads to the appearance of supercondensate photons with frequencies not equal to Ω. This phenomenon is analogous to the effect of "pushing out" the particles from a non-ideal Bose condensate.[2] There is, however, an essential difference, in that the number of supercondensate photons increases with time. This means that the photon condensate in a lossless laser is unstable. The reason for the instability is that the production of supercondensate photons from the condensate in the laser is a real process, since the energy and momentum conservation laws are satisfied for them, whereas in a nonideal equilibrium gas the production processes are virtual.

It is also interesting to note that in the equilibrium case the Bose condensation of phonons is not accompanied by the effect of crowding out of the phonons from the Bose condensate (with zero energy), by virtue of the symmetry properties of the crystal. This result is natural, since Bose condensation of phonons corresponds to a restructuring of the crystal, and in the new phase only zero-point oscillations exist at $T=0$.

1. HAMILTONIAN OF SYSTEM

We consider a traveling-wave semiconductor laser at $T=0$, in which an electromagnetic field

$$\mathbf{E}=\mathbf{E}_0\cos(\Omega t-\mathbf{k}_0\mathbf{r}). \tag{1}$$

is established. The system Hamiltonian, which takes into account the interaction of the electrons with the electromagnetic field, is of the form

$$H=\sum_{\mathbf{p}}(a_{1\mathbf{p}}^+a_{1\mathbf{p}}-a_{2\mathbf{p}}^+a_{2\mathbf{p}})\xi_{\mathbf{p}}+\sum_{\mathbf{k}}(\omega_{\mathbf{k}}^0-\Omega)c_{\mathbf{k}}^+c_{\mathbf{k}}+\sum_{\mathbf{p},\mathbf{k}}(a_{2\mathbf{p}+\mathbf{k}}^+a_{1\mathbf{p}}c_{\mathbf{k}}^++a_{1\mathbf{p}}^+a_{2\mathbf{p}+\mathbf{k}}c_{\mathbf{k}})M_{\mathbf{k}}+H_{\text{int}},$$

$$\xi_{\mathbf{p}}=\frac{\mathbf{p}^2}{2m}+\frac{E_g}{2}-\frac{\Omega}{2},\quad \omega_{\mathbf{k}}^0=c|\mathbf{k}|,\quad M_{\mathbf{k}}=\frac{ev_{cv}}{c}\left(\frac{2\pi}{\omega_{\mathbf{k}}^0}\right)^{1/2},\quad \hbar=1. \tag{2}$$

Here $a_{2\mathbf{p}}^+$ and $a_{1\mathbf{p}}^+$ are the electron creation operators in the valence and conduction bands, respectively, $c_{\mathbf{k}}^+$ is the photon creation operator, v_{cv} is the interband-transition matrix element, and H_{int} is a term that takes into account the interaction of the electrons with the phonons and electrons. In the absence of losses, this term leads to a renormalization of the gap λ[4,5] (see below), which will assume to take place. We note that in (2) we have carried out the unitary transformation

$$U=\exp\left\{\frac{i\Omega t}{2}\left[\sum_{\mathbf{p}}(a_{1\mathbf{p}}^+a_{1\mathbf{p}}-a_{2\mathbf{p}}^+a_{2\mathbf{p}})+\sum_{\mathbf{k}}c_{\mathbf{k}}^+c_{\mathbf{k}}\right]\right\}.$$

We divide the photon operators into two parts

$$c_{\mathbf{k}}=c_{\mathbf{k}0}\delta_{\mathbf{k}\mathbf{k}_0}+c_{\mathbf{k}}',$$

where $c_{\mathbf{k}0}$ corresponds to condensate photons with wave vector $\mathbf{k}_0$, and $c_{\mathbf{k}}'$ to the above-condensate photons. The Hamiltonian (2) can then be represented in the form

$$H=H_0+H_1,$$

$$H_0=\sum_{\mathbf{p}}\xi_{\mathbf{p}}(a_{1\mathbf{p}}^+a_{1\mathbf{p}}-a_{2\mathbf{p}}^+a_{2\mathbf{p}})+\Omega_{\mathbf{k}0}c_{\mathbf{k}0}^+c_{\mathbf{k}0}+\sum_{\mathbf{p}}M_{\mathbf{k}0}(a_{2\mathbf{p}+\mathbf{k}0}^+a_{1\mathbf{p}}c_{\mathbf{k}0}^++\text{h.c.}), \tag{3}$$

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$$H_1=\sum_{\mathbf{k}}\Omega_{\mathbf{k}}c_{\mathbf{k}}'^{+}c_{\mathbf{k}}'+\sum_{\mathbf{pk}}M_{\mathbf{k}}(a^{+}_{2\mathbf{p}+\mathbf{k}}a_{1\mathbf{p}}c_{\mathbf{k}}'^{+}+\text{h.c.}),$$

$$\Omega_{\mathbf{k}}=\omega_{\mathbf{k}}^0-\Omega. \quad (4)$$

2. DESCRIPTION OF PHOTON BOSE CONDENSATE

We assume that the number of supercondensate photons is small in comparison with the number of photons in the condensate. To describe the condensate of the photons and electrons we shall therefore take H_0 into account. Averaging the equation of motion for the operator $c_{\mathbf{k}0}$, we obtain

$$\langle c_{\mathbf{k}_0}\rangle\Omega_{\mathbf{k}_0}=M_{\mathbf{k}_0}\sum_{\mathbf{p}\omega}G_{12}(\mathbf{p},\mathbf{p}+\mathbf{k}_0,\omega), \quad (5)$$

where G_{12} is the electron Green's function and describes the transition between the bands:

$$G_{12}(\mathbf{p},\mathbf{p}+\mathbf{k}_0,t-t')=\langle T\{a_{1\mathbf{p}}(t)a^{+}_{2\mathbf{p}+\mathbf{k}_0}(t')\}\rangle.$$

To find G_{12} we must write down the system of equations for the Green's functions G_{12} and G_{22}:

$$G_{22}(\mathbf{p},t-t')=-i\langle T\{a_{2\mathbf{p}}(t)a_{2\mathbf{p}}^{+}(t')\}\rangle,$$

which is analogous to the system obtained in[4]:

$$(\omega-\xi_{\mathbf{p}})G_{12}(\mathbf{p},\mathbf{p}+\mathbf{k}_0,\omega)=iM_{\mathbf{k}_0}\langle c_{\mathbf{k}_0}\rangle G_{22}(\mathbf{p}+\mathbf{k}_0,\omega), \quad (6)$$

$$(\omega+\xi_{\mathbf{p}+\mathbf{k}_0})G_{22}(\mathbf{p}+\mathbf{k}_0,\omega)+iM_{\mathbf{k}_0}\langle c_{\mathbf{k}_0}^{+}\rangle G_{12}(\mathbf{p},\mathbf{p}+\mathbf{k}_0,\omega)=1. \quad (7)$$

The solution of (6) is

$$G_{12}(\mathbf{p},\mathbf{p}+\mathbf{k}_0,\omega)=\frac{i\lambda}{\mathrm{Det}},\quad G_{22}(\mathbf{p}+\mathbf{k}_0,\omega)=\frac{\omega-\xi_{\mathbf{p}}}{\mathrm{Det}},$$

$$\mathrm{Det}=(\omega-\xi_{\mathbf{p}})(\omega+\xi_{\mathbf{p}})-\lambda^2, \quad (8)$$

$$\lambda=M_{\mathbf{k}_0}\langle c_{\mathbf{k}_0}\rangle\equiv(ev_{cv}\mathbf{E}_0)/2\Omega. \quad (9)$$

It should be noted that the system (6) for G_{12} and G_{22} is closed if the field in the laser is a traveling wave. Substituting G_{12} in (5), we obtain

$$\Omega_{\mathbf{k}_0}=\frac{1}{2}M_{\mathbf{k}_0}^2\sum_{\mathbf{p}}\left[\left(\frac{\xi_{\mathbf{p}}+\xi_{\mathbf{p}+\mathbf{k}_0}}{2}\right)^2+\lambda^2\right]^{-1/2}. \quad (10)$$

In this equation, the field λ is regarded as a parameter and must be determined from the imaginary part of $\langle c_{\mathbf{k}0}\rangle$ with allowance for the photon damping in the resonator. In the case of a lossless laser (when the photon lifetime tends to infinity), the field reaches its limiting value, equal to half the photon frequency[3] ω_{ph}:

$$\lambda=\omega_{\mathrm{ph}}/2.$$

The frequency Ω is obtained from the condition that the laser gain be a maximum at the lasing threshold. The values of λ and Ω are thus given, and $\mathbf{k}_0$ is determined from Eq. (10).

The results that follow hold also for the case when the photon Bose condensate constitutes an electromagnetic field incident on the semiconductor. The frequency Ω and the field amplitude $\mathbf{E}_0$ are governed in this situation by the external source.

3. SPECTRUM OF COLLECTIVE EXCITATIONS (OF SUPERCONDENSATE PHOTONS)

The properties of the supercondensate particles are conveniently described with the aid of the ordinary and anomalous Green's functions D and $\hat{D}$, respectively[6]:

$$D(\mathbf{k},t-t')=-i\langle T\{c_{\mathbf{k}}'(t)c_{\mathbf{k}}'^{+}(t')\}\rangle,$$

$$\hat{D}(\mathbf{k},\mathbf{k}',t-t')=-i\langle T\{c_{\mathbf{k}}'^{+}(t)c_{\mathbf{k}'}'^{+}(t')\}\rangle.$$

Using the equations of motion for the operators

$$i\partial c_{\mathbf{k}}/\partial t=[c_{\mathbf{k}}H],$$

we easily obtain an equation for D:

$$\left(i\frac{\partial}{\partial t}-\Omega_{\mathbf{k}}\right)D(\mathbf{k},t-t')=-i\sum_{\mathbf{p}}M_{\mathbf{k}}\langle T\{a^{+}_{2\mathbf{p}+\mathbf{k}}(t)a_{1\mathbf{p}}(t)c_{\mathbf{k}}'^{+}(t')\}\rangle.$$

To find the self-energy part $\langle\cdots\rangle$ we use the device employed in[4], viz., we change over in $\langle\cdots\rangle$ to the interaction representation with Hamiltonian H_0:

$$\langle a^+ac^+\rangle\to\langle a_0^+a_0c_0^+S\rangle,\quad S=\exp\left[-i\int_{-\infty}^{\infty}H_1(t_1)dt_1\right].$$

Wick's theorem holds for the operators a_0^+ and c_0^+ and we can use the usual diagram technique. Neglecting the renormalization of the vertex part, in view of the weakness of the electron–photon interaction, we obtain the system of equations

$$(\omega-\Omega_{\mathbf{k}}-\Sigma_{11}(\mathbf{k},-\omega))D(\mathbf{k},\omega)-\Sigma_{20}(\mathbf{k},-\omega)\hat{D}(2\mathbf{k}_0-\mathbf{k},\mathbf{k},\omega)=1, \quad (11)$$

$$(\omega+\Omega_{2\mathbf{k}_0-\mathbf{k}}+\Sigma_{11}(2\mathbf{k}_0-\mathbf{k},\omega))\hat{D}(2\mathbf{k}_0-\mathbf{k},\mathbf{k},\omega)+\Sigma_{02}(-\mathbf{k},\omega)D(\mathbf{k},\omega)=0, \quad (12)$$

where

$$\Sigma_{11}(\mathbf{k},-\omega)=-iM_{\mathbf{k}}^2\sum_{\mathbf{p}\omega_1}G_{11}(\mathbf{p},\omega_1)G_{22}(\mathbf{p}+\mathbf{k},\omega_1-\omega),$$

$$\Sigma_{20}(\mathbf{k},-\omega)=iM_{\mathbf{k}}M_{2\mathbf{k}_0-\mathbf{k}}\sum_{\mathbf{p}\omega_1}G_{12}(\mathbf{p},\mathbf{p}+\mathbf{k}_0,\omega_1)G_{12}(\mathbf{p}+\mathbf{k}-\mathbf{k}_0,\mathbf{p}+\mathbf{k},\omega_1-\omega),$$

$$\Sigma_{02}(-\mathbf{k},\omega)=\Sigma_{20}(2\mathbf{k}_0-\mathbf{k},\omega). \quad (13)$$

The anomalous function $\hat{D}$ describes the production of two photons with momenta $\mathbf{k}$ and $2\mathbf{k}_0-\mathbf{k}$ from the condensate. The solution of the system (11), (12) is

$$D(\mathbf{k},\omega)=(\omega+\Omega_{2\mathbf{k}_0-\mathbf{k}}+\Sigma_{11}(2\mathbf{k}_0-\mathbf{k},\omega))\Pi^{-1}(\mathbf{k},\mathbf{k}_0,\omega), \quad (14)$$

$$\Pi(\mathbf{k},\mathbf{k}_0,\omega)=(\omega-\Omega_{\mathbf{k}}-\Sigma_{11}(\mathbf{k},-\omega))(\omega+\Omega_{2\mathbf{k}_0-\mathbf{k}}+\Sigma_{11}(2\mathbf{k}_0-\mathbf{k},\omega))+\Sigma_{20}(-\mathbf{k},\omega)\Sigma_{02}(-\mathbf{k},\omega).$$

The excitation spectrum is given by the equation

$$\Pi(\mathbf{k},\mathbf{k}_0,\omega)=0. \quad (15)$$

The functions $\Sigma_{11}(\mathbf{k}_0,0)$ and $\Sigma_{20}(\mathbf{k}_0,0)$ satisfy the important relation

$$\Sigma_{20}(\mathbf{k}_0,0)-\Sigma_{11}(\mathbf{k}_0,0)=\Omega_{\mathbf{k}_0}, \quad (16)$$

which can be easily derived with the aid of (8) and (5). Expression (16) plays the role of the condition imposed on the chemical potential in Belyaev's theory,[6] namely that the number of particles be given.

We confine ourselves to an investigation of the spectrum (15) in the region of low frequencies $\omega \ll \lambda$ and at $\mathbf{k}$ close to $\mathbf{k}_0$. In these regions of ω and $\mathbf{k}$, the self-energy part $\Sigma_{20}(\mathbf{k}, -\omega)$ can be represented in the form

$$\Sigma_{20}(\mathbf{k}, -\omega) \approx \Sigma_{20}(\mathbf{k}_0, 0)\left\{1+\frac{1}{6\lambda^2}\left[\omega-\frac{\mathbf{k}_0(\mathbf{k}-\mathbf{k}_0)}{2m}\right]^2-\frac{v_0^2(\mathbf{k}-\mathbf{k}_0)^2}{18\lambda^2}\right\},$$

$$\Sigma_{20}(\mathbf{k}_0, 0)=\frac{p_0 m M_{\mathbf{k}_0}^2}{2\pi^2}, \quad m v_0^2=\Omega-E_g,$$

where v_0 is the velocity on the Fermi surface. It is easily seen that the terms with $\mathbf{k}$ make a small contribution compared with ω. Indeed, bearing in mind the fact that $\omega \approx c|\mathbf{k}-\mathbf{k}_0|$, we obtain for the ratios of the second and third terms to the first

$$k_0/mc \ll 1, \quad v_0/c \ll 1$$

respectively. This gives grounds for setting the vector $\mathbf{k}$ equal to $\mathbf{k}_0$ in Σ:

$$\Sigma_{20}(\mathbf{k}_0, -\omega) \approx \Sigma_{20}(\mathbf{k}_0, 0)\left(1+\frac{\omega^2}{6\lambda^2}\right),$$
$$\Sigma_{11}(\mathbf{k}_0, -\omega) \approx \Sigma_{11}(\mathbf{k}_0, 0)-\frac{\omega^2}{3\lambda^2}\Sigma_{20}(\mathbf{k}_0, 0). \quad (17)$$

Substituting (17) in (15) and taking (16) into account, we obtain an expression for the spectrum (we assume $\mathbf{k}$ to be parallel to $\mathbf{k}_0$):

$$\omega_{\mathbf{k}}=\frac{1}{1+b}\left[c(k-k_0) \pm i(bc^2(k-k_0)^2)^{1/2}\right],$$
$$b=\Sigma_{20}^2(\mathbf{k}_0, 0)/\lambda^2. \quad (18)$$

It is seen from (18) that $\omega_{\mathbf{k}}$ contains an imaginary part with two signs. This means that the state of the condensate is unstable, and that the number of supercondensate photons increases with time. The reason is the following: The processes of production of supercondensate photons with momenta $\mathbf{k}$ and $2\mathbf{k}_0-\mathbf{k}$ as a result of the loss of two condensate photons with $\mathbf{k}_0$ turn out to be real, since the energy and momentum conservation laws are satisfied:

$$\omega_{\mathbf{k}1}+\omega_{\mathbf{k}2}=\frac{1 \pm i\sqrt{b}}{1+b}(k_1+k_2-2k_0)=0, \quad k_1+k_2=2k_0$$

(we recall that the photon frequency is reckoned from Ω). The process of photon production from the condensate is equivalent to parametric conversion of two photons of like frequency into two photons that differ in frequency. It appears that an instability of this type can appear in a system of interacting Bose-condensed excitons and photons.[7]

It is of interest to compare the spectrum (18) with the excitation spectrum of a non-ideal Bose gas,[2] assuming that the condensation takes place in a state with $\mathbf{k}_0 \neq 0$:

$$\omega_{\mathbf{k}}={}^1/_2(E_{2\mathbf{k}_0-\mathbf{k}}-E_{\mathbf{k}}) \pm {}^1/_2[(E_{2\mathbf{k}_0-\mathbf{k}}+E_{\mathbf{k}})^2+8a(E_{2\mathbf{k}_0-\mathbf{k}}+E_{\mathbf{k}})]^{1/2},$$
$$a=\Sigma_{20}(\mathbf{k}_0, 0)=\Sigma_{11}(\mathbf{k}_0, 0), \quad E_{\mathbf{k}}=\mathbf{k}^2/2m.$$

In this case the imaginary part is equal to zero, i.e., the state with $k_0 \neq 0$ is stable. A similar result is obtained for photons

$$\omega_{\mathbf{k}}={}^1/_2(\mathring{\omega}_{2\mathbf{k}_0-\mathbf{k}}-\mathring{\omega}_{\mathbf{k}}) \pm {}^1/_2[(\mathring{\omega}_{2\mathbf{k}_0-\mathbf{k}}+\mathring{\omega}_{\mathbf{k}})^2+8a(\mathring{\omega}_{2\mathbf{k}_0-\mathbf{k}}+\mathring{\omega}_{\mathbf{k}})]^{1/2},$$

if we put (as is done in[2])

$$\Sigma_{11}(\omega)=\Sigma_{20}(\omega)=\Sigma(0)$$

and neglect the frequency dependence of Σ. The instability is thus due to the disequilibrium of the system and to the delay of the interaction between the photons.

CONCLUSION

It may seem at first glance that the instability obtained above for a photon Bose condensate in a state with momentum $\mathbf{k}_0$ is connected with a choice of a non-optimal state.

A similar situation takes place in superconductors with internal magnetic field, when the homogeneous state of the Bose condensate of Cooper pairs with a zero net momentum turns out to be unstable to production of a Bose condensate with a nonzero net pair momentum.[8,9]

The condition (10), however, corresponds precisely to the vanishing of the derivative of the free energy with respect to $\mathbf{k}_0$ in[8,9], i.e., it corresponds to the choice of the optimal state. Therefore the instability of the photon Bose condensate in a lossless laser is a fundamental fact.

It is natural to expect that allowance for the photon loss in the resonator will lead to stabilization of the condensate and of the supercondensate particles, and hence to a certain broadening of the laser line. It appears, however that stabilization is possible also in a lossless laser, if account is taken of the reaction of the supercondensate photons on the condensate. This should result in the formation of several condensates.

The appearance of several condensates means the onset of additional modes of the laser electromagnetic field. The modes are produced in pairs on both sides of the fundamental mode (the initial condensate) and at equal energy distances from this mode (smaller gap in the electron spectrum). It is interesting to note that in experiment the modes appear in semiconductor lasers in pairs, and this leads to a characteristic "ridge" of modes.[10]

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