Large-amplitude collective oscillations and the double parametric resonance of magnons

V. V. Zautkin, V. S. L'vov, B. I. Orel, and S. S. Starobinets

V. V. Kuibyshev Far-Easten Polytechnic Institute
(Submitted June 24, 1976)
Zh. Eksp. Teor. Fiz. 72, 272–284 (January 1977)

The effect of a radio-frequency (RF) magnetic field ($\Omega \sim 10^6$ HZ) on the parametric excitation of spin waves in yttrium iron garnet has been investigated theoretically and experimentally by the parallel-pumping technique ($\omega_p \sim 10^{10}$ Hz). The theoretical and experimental dependences of the parallel pumping threshold on the frequency, Ω, and amplitude, H_m, of the RF field coincide. It is shown that the RF field, in accord with theory, increases the imaginary part of the nonlinear susceptibility and, as a rule, decreases the real part. A new phenomenon, called double parametric resonance and consisting in the parametric excitation, in a system of SHF magnons ($\omega_{\mathbf{k}} \sim 10^{10}$ Hz) parametrically excited with the aid of a RF magnetic field ($\Omega \approx 10^6$ Hz), of collective oscillations of frequency $\Omega/2$ when the amplitude of the RF field exceeds some threshold value $H_{m\text{thr}}$ (of the order of 0.2 Oe), has been theoretically predicted and experimentally observed. Qualitative agreement between the theoretical and experimental dependences of $H_{m\text{thr}}$ on Ω is demonstrated.

PACS numbers: 76.70.–r

Collective excitations in a system of parametrically pumped magnons were discovered and investigated in experiments performed by the present authors.[1,2] These excitations, which are normal oscillations of the length of the magnetization vector, can be excited by a radio-frequency (RF) magnetic field of small amplitude at some resonance frequency, Ω, coinciding with the natural frequency, Ω_0, of the collective mode. Also well studied are the auto-oscillations of the magnetization, which arise as a result of the development of instabilities in the "soft" modes of the collective oscillations against a background of the steady state of parametrically excited magnons (see, for example, [3]).

In the present paper we investigate theoretically and experimentally the nonlinear properties of the collective oscillations of magnons excited by a large-amplitude RF field. As shown below (§2), the intense oscillations exert an appreciable influence on the steady state of the system, leading to changes in the mean excitation level, the magnon distribution in k space, and the nonlinear susceptibilities χ' and χ''. It should also be noted that similar phenomena are observed in the excitation of intense magnetization auto-oscillations.

Naturally, before going on to investigate the above-threshold magnon state in the presence of a strong RF field, it is necessary to study the influence of this field on the threshold for parametric excitation. This question was first considered by Suhl,[4] who showed that upon the periodic modulation of the magnetizing field (or of the pump frequency) the parametric-instability threshold increases as a result of the violation on the average of the condition for parametric resonance. According to Suhl, upon the proper choice of the modulation frequency and amplitude, the instability disappears—its threshold becomes infinitely large. Suhl's idea was experimentally verified by Hartwick, Peressini, and Weiss.[5] According to these authors, they did not obtain complete agreement with the results of the theory; in particular, they could not increase the threshold above 10 dB.

In §1, we develop a theory of parametric excitation of magnons in the presence of a RF field and present detailed experimental results which support the theory.

In the last section (§3), we report the observation of a new phenomenon—a double parametric magnon resonance, which consists in the following. At a sufficiently large amplitude of the RF field of frequency $\Omega \approx \Omega_0 \sim 10^6$ Hz there arise collective oscillations of frequency $\Omega/2$ in the system of parametrically pumped microwave magnons of frequency $\omega_{\mathbf{k}} = \omega_p/2 \sim 10^{10}$ Hz. The excitation of the oscillations has a strongly pronounced threshold character. The threshold amplitude of the RF field (~0.2 Oe) and the range of frequencies Ω in which the effect is observed agree with the results of the theoretical computations of the parametric excitation of the collective magnon oscillations carried out here.

§1. THE INSTABILITY THRESHOLD

Let us consider the parametric excitation of magnons in a magnetic field of the following form:

$$H = H_0 + H_m(t) + h\cos\omega_p t, \tag{1}$$

where H_0 is a constant magnetic field, $H_m(t)$ is a modulating field, and $h\cos\omega_p t$ is the microwave pump.

The computation of the threshold for parallel pumping in the field (1) with arbitrary modulation law, $H_m(t)$, is a rather difficult problem. Therefore, let us first consider the simplest modulation variant, i.e., modulation with the aid of rectangular periodic pulses (a "meander" type of regime with a period of 2τ). The equations of motion for the normal magnon amplitudes in the linear approximation have the form[3]:

$$\begin{aligned}&\frac{dc_{\mathbf{k}}}{dt} + \gamma_{\mathbf{k}}c_{\mathbf{k}} + i(\Delta_{\mathbf{k}} + 2U_{\mathbf{k}}H_m)c_{\mathbf{k}} + ihV_{\mathbf{k}}c_{-\mathbf{k}}^* = 0 \quad (0<t<\tau),\\&\frac{dc_{\mathbf{k}}}{dt} + \gamma_{\mathbf{k}}c_{\mathbf{k}} + i(\Delta_{\mathbf{k}} - 2U_{\mathbf{k}}H_m)c_{\mathbf{k}} + ihV_{\mathbf{k}}c_{-\mathbf{k}}^* = 0 \quad (\tau<t<2\tau).\end{aligned} \tag{2}$$

Here and below the notation is the same as the notation used in[3]:

 Copyright © 1977 American Institute of Physics

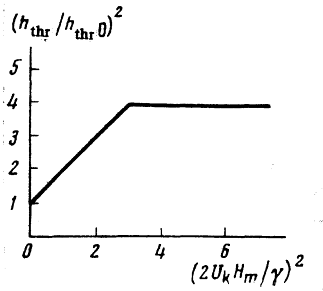

FIG. 1. Dependence of the threshold for parallel pumping on the amplitude of a modulating field of rectangular shape. In the first region the detuning $\Delta_k=0$; in the second section $\Delta_k=\pm 2U_k H_m$.

$$\Delta_k=\omega_k-\omega_p/2,\quad U_k=gA_k/2\omega_k,\quad V_k=-gB_k/2\omega_k,\quad \omega_k^2=A_k^2-|B_k|^2.$$

Assuming $c_k,\ c_{-k}\sim e^{\nu t}$, we find from (2) that the increment ν in different time intervals is equal to

$$\begin{aligned}\nu_{0-\tau}&=-\gamma\pm[|hV_k|^2-(\Delta_k+2U_kH_m)^2]^{1/2},\\ \nu_{\tau-2\tau}&-\gamma\pm[|hV_k|^2-(\Delta_k-2U_kH_m)^2]^{1/2}.\end{aligned}\tag{3}$$

Let us consider the case $\tau\gg\gamma^{-1}$, when the damped solutions, corresponding to the negative sign in front of the square root in (3), can be neglected. Then the total increment of the growth over the pulse-repetition interval is equal to

$$\begin{aligned}\nu_{0-2\tau}=\nu_{0-\tau}+\nu_{\tau-2\tau}=&-2\gamma_k+[|hV_k|^2-(\Delta_k+2U_kH_m)^2]^{1/2}\\&+[|hV_k|^2-(\Delta_k-2U_kH_m)^2]^{1/2}.\end{aligned}$$

From the condition $\mathrm{Re}\,\nu=0$ we obtain the instability threshold:

$$\begin{aligned}|hV_k|^2_{thr}&=\gamma_k^2+(2U_kH_m)^2+\Delta_k^2(1+4U_k^2H_m^2/\gamma_k^2)\\ &\qquad|\Delta_k\pm 2U_kH_m|<|hV_k|.\\ |hV_k|^2_{thr}&=4\gamma_k^2+(\Delta_k\mp 2U_kH_m)^2,\qquad |\Delta_k\pm 2U_kH_m|>|hV_k|>|\Delta_k\mp 2U_kH_m|.\end{aligned}\tag{4}$$

It can be seen that at small modulation amplitudes, when the first of the inequalities (4) is fulfilled, the minimum instability threshold is attained at $\Delta_k=0$, i.e., for magnons whose wave vectors lie in **k** space on the resonance surface $\omega_k=\omega/2$. The corresponding threshold is equal to

$$|hV_k|^2_{thr}=\gamma_k^2+(2U_kH_m)^2,\ \Delta_k=0.\tag{5}$$

At large modulation amplitudes there arise on the surfaces $\Delta_k=\pm 2U_kH_m$, starting from $4|U_kH_m|>|hV_k|$, additional local threshold minima symmetrically located relative to the resonance surface. The magnitude of the threshold on these surfaces is constant and is equal to:

$$|hV_k|_{thr}=2\gamma_k,\quad \Delta_k=\pm 2U_kH_m.\tag{6}$$

The complete picture of the behavior of the threshold as a function of the modulation amplitude H_m is shown in Fig. 1. The critical modulation amplitude

$$H_{m\,cr}=3^{1/2}\gamma/2U_k$$

separates the two parallel-pumping regimes: the weak-modulation regime ($H_m<H_{m\,cr}$) and the strong-modulation regime ($H>H_{m\,cr}$). The weak modulation is characterized by zero detuning ($\Delta_k=0$) and a square-law dependence of the threshold on the modulation amplitude. In the case of strong modulation there arises detuning whose magnitude increases in proportion to the amplitude of the modulating field.

Another characteristic effect of strong modulation is the freezing of the threshold at $H_m>H_{m\,cr}$, an effect which is, generally speaking, peculiar only to the rectangular type of modulation. Thus, it is not difficult to show that, for sinusoidal and sawtoothed modulations, the quantity h_{thr} increases without restriction, although slowly, with increasing H_m. For $H_m\gg H_{m\,cr}$, we have for sinusoidal modulation

$$|hV_k|_{thr}=\frac{\pi}{2}\gamma^{1/2}(U_kH_m)^{1/2},\quad \Delta_k=\pm 2U_kH_m,\tag{7}$$

and for sawtoothed modulation

$$|hV_k|_{thr}=(\gamma U_kH_m)^{1/2},\quad \Delta_k=0,\ \pm 2U_kH_m.\tag{8}$$

Thus far we have considered low-frequency modulation with a period less than the magnon relaxation time. Let us now discard this restriction and consider parallel pumping of magnons in a magnetic field modulated according to a sinusoidal law with arbitrary frequency Ω:

$$H(t)={}^1/_2H_me^{-i\Omega t}+\text{c.c.}$$

The linearized equations of motion have the form

$$\frac{dc_k}{dt}+\gamma_kc_k+i[\Delta_k+U_k(H_me^{-i\Omega t}+H_m^*e^{i\Omega t})]c_k+ihV_kc_{-k}^*=0.\tag{9}$$

Restricting ourselves to small amplitudes of H_m, we write

$$c_k=c_k^{(0)}+\alpha_{\Omega,k}e^{i\Omega t}+\alpha_{-\Omega,k}e^{-i\Omega t}.\tag{10}$$

The steady-state solution to Eqs. (9) corresponds to the instability threshold. The substitution of (10) into (9) yields a system of equations for $c_k^{(0)}$ and $\alpha_{\pm\Omega,k}$, from which we find to second order in α the relation

$$|hV_k|^2_{thr}=(\gamma_k^2+\Delta_k^2)\left(1+\frac{4U_k^2H_m^2}{\Omega^2+4\gamma_k^2}\right)^2,$$

determining the threshold field. As was to be expected, the minimum threshold for the weak-modulation regime is attained at $\Delta_k=0$, and is equal to

$$|hV_k|_{thr}=\gamma_k\left(1+\frac{4U_k^2H_m^2}{\Omega^2+4\gamma_k^2}\right),\quad \Delta_k=0.\tag{11}$$

At low frequencies the formula (11) is equivalent to the exact formula (5), obtained above for the rectangular type of modulation. This, apparently, means that the formula (11), which was derived in the framework of perturbation theory, is applicable with a high degree of accuracy right up to values of $H_m\approx H_{m\,cr}$. The reason for this is that the small parameter of the theory at high modulation frequencies is $2U_kH_m/\Omega$, while at low frequencies it is Ω/γ_k.

In the most general case of modulation with an arbitrary frequency and arbitrary amplitude a comparatively simple analytic expression for the instability threshold

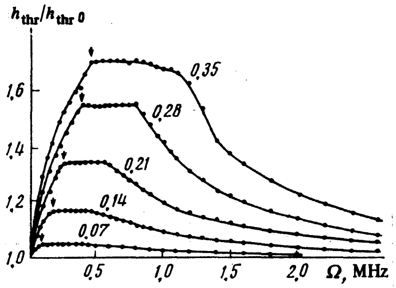

FIG. 2. Dependence of the threshold for parallel pumping on the frequency of the RF field. The numbers on the curves are the amplitudes of the RF field in oersteds. The constant magnetic field $H = H_c - 100$ Oe. (H_c is the field corresponding to $k = 0$). The arrows indicate the right-hand boundary of the low-frequency region in which the formula (11) is not applicable.

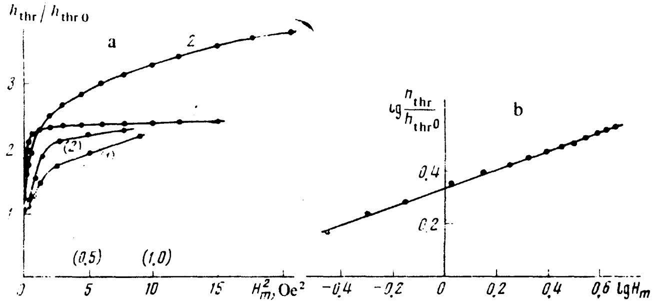

FIG. 4. a) Dependence of the SHF threshold field on H_m^2 for a rectangular (curve 1) and a sinusoidal (curve 2) shape. The numbers in brackets correspond to the initial sections of the curves with the H_m^2 scale increased by a factor of ten. $H = H_c - 100$ Oe. b) Dependence of the SHF threshold on the amplitude of the sinusoidal modulating field H_m in log–log coordinates. $H = H_c - 100$ Oe.

is obtainable only for rectangular modulation. Let us give, without derivation, the equation determining h_{thr}:

$$\operatorname{ch}(2\gamma_k\tau) = \left[\operatorname{ch}\mu_+\tau\operatorname{ch}\mu_-\tau + \frac{|hV_k|^2_{thr} - \Delta_+\Delta_-}{\mu_+\mu_-}\operatorname{sh}\mu_+\tau\operatorname{sh}\mu_-\tau\right], \quad (12)$$

$$\Delta_\pm = \Delta_k \pm 2U_kH_m, \quad \mu_\pm^2 = |hV_k|^2_{thr} - \Delta_\pm^2.$$

In the case of low frequencies (large τ) it is simple to derive from this the formulas (5) and (6). At high frequencies ($\tau \to 0$) we have

$$|hV_k|_{thr} = \gamma(1 + 4U_k^2H_m^2\tau^2/3)^{1/2}.$$

Let us now consider the experimental data on the investigation of the effect of sinusoidal modulation of the magnetic field on the threshold for parallel pumping. The measurements were carried out, using the standard procedure in the pulsed regime (the length of the pump pulses was 500 μsec) at a pump frequency of 9400 MHz. The samples were $Y_3Fe_5O_{12}$ single crystals with a ferromagnetic resonance curve of width $2\Delta H \approx 0.4$ Oe. In Fig. 2 we show the dependences of the threshold for parallel pumping on the frequency of the modulating field. In Fig. 3 we depict these dependences in those coordinates for which the formula (11) predicts a straight line:

$$(h_{thr}/h_{thr\,0} - 1)^{-1} = (\Omega^2 + 4\gamma^2)/4(U_kH_m)^2. \quad (13)$$

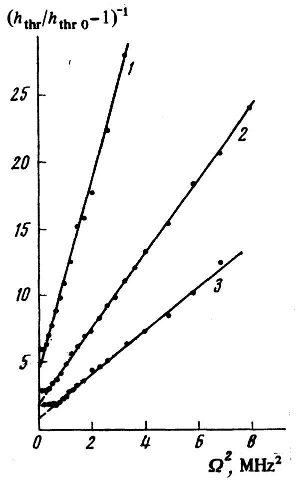

FIG. 3. Dependence of the threshold for parallel pumping on the frequency of the RF field ($H = H_c - 100$ Oe). The curves 1, 2, and 3 correspond to $H_m = 0.14$, 0.21, and 0.28 Oe. In these coordinates the theory (formula (11)) predicts a straight line.

It can be seen that the experimental points fall quite well on a straight line; the slopes of these straight lines and the intercepts the lines make on the ordinate axis coincide within the limits of the experimental errors (10%) with the theory. Deviation begins in the region, lying below a certain frequency Ω_{lim}, to which the theory developed above is inapplicable. We, in fact, assumed that the RF field modulates the magnon natural frequency ω_k, not moving, however, the magnon packet in **k** space. It is clear that this condition is not observed at sufficiently low frequencies, and the packet will follow the changes in ω_k. The condition, $\omega_k = \omega_p/2$, for parametric resonance is then automatically fulfilled at each moment of time, and the modulation of the magnetic field ceases to have an effect on the threshold.

Let us estimate the characteristic frequency at which the center of gravity of the packet begins to follow the changes in the field. The characteristic time of growth of the magnons from the thermal amplitude N_T to the amplitude N is equal to:

$$\tau_1 = (hV - \gamma)^{-1}\ln(N/N_T),$$

and the time during which the natural magnon frequency changes by the quantity 2γ—the packet width—is clearly equal to $\tau_2 = \gamma\pi/\Omega UH_m$. For $\tau_2 \gtrsim \tau_1$ the motion of the packet will be able to keep up with the changes in the magnetic field. From the condition $\tau_1 = \tau_2$ we find the frequency limit:

$$\Omega_{lim} \approx \pi UH_m/2\ln(N/N_T). \quad (14)$$

The quantity N entering into the formula (14) is some minimum number of magnons that can still be recorded by the experimental setup (usually $N \sim 10^{16} - 10^{17}$ cm^{-3}). At room temperature the number of thermal magnons in the frequency band $\Delta\omega_k = \gamma \sim 10^6$ sec^{-1} near $\omega_k = \omega_p/2$ is equal to $N_T \sim 10^9$ cm^{-3}, i.e., $\ln(N/N_T) \sim 10$. Taking $H_m = 0.1$ Oe, we find from the formula (14) that $\Omega_{lim} \approx 0.02$ MHz, which, considering the approximate nature of the formula, is in quite good agreement with the experimental result (see Fig. 2, in which Ω_{lim} is indicated by an arrow). Experiment also indicates that Ω_{lim} is proportional to H_m, as follows from (14).

In Fig. 4a we show the dependences of the threshold field on the modulation amplitude H_m for signals of rectangular (meander) and sinusoidal shapes. For $H_m \lesssim 0.3$ Oe the dependence of h_{thr} on H_m^2 is linear, in accord with the formulas (5) and (11). At $H_m > 0.3$ Oe a transition into the strong-modulation regime occurs. The theory in this case predicts the excitation of two magnon packets with wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ ($\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} = 2U_{\mathbf{k}}H_m$). As a result of the dependence $\gamma(\mathbf{k})$, the excitation thresholds for the packets $\mathbf{k}_1$ and $\mathbf{k}_2$ should differ somewhat from each other. Indeed, at $H_m \gtrsim 0.3$ Oe we can observe in some frequency region two successive thresholds, the higher of them appearing considerably more abruptly: the instability rise time corresponding to it is much shorter.

In the strong modulation regime the dependence $h_{thr}(H_m)$ for the meander (the curve 1) reaches a plateau, in accord with theory. The limiting value of h_{thr} depends on Ω, and tends to the value $2h_{thr\,0}$ with decreasing Ω. For a sinusoidal signal the threshold monotonically increases. In Fig. 4b the dependence $h_{thr}(H_m)$ is shown on a log–log scale. It can be approximated by the function $h_{thr} = cH^{\nu}$, with $\nu \approx 0.35$, which agrees with the formula (7). In our experiments the maximum increase in the threshold attained a value of 3.8 at $H_m = 5$ Oe. At considerably higher H_m, h_{thr} ceases to increase. The point is that the formula (7) is applicable when $\Omega > \Omega_{lim}$. According to (14), the frequency limit increases in proportion to H_m, and for a sufficiently high H_m the inverse inequality $\Omega_{lim} > \Omega$ is fulfilled at any frequency. In this case the theory should be modified to take account of the effect of displacement of the packet during the modulation of the field. The maximum increase in the threshold is attained at $\Omega \approx 2\gamma$. According to our estimates, $\max[(hV)_{thr}/\gamma] \approx 4-5$. A somewhat higher increase in the threshold is attained in the case of sawtoothed modulation of the magnetic field. The saturation of the threshold at large amplitudes of H_m limits the possibility of using magnetic-field modulation as a means of suppressing the parametric instabilities of magnons.

§2. THE STEADY STATE BEYOND THE THRESHOLD

Let us briefly describe the physical processes which limit the development of the parametric instability of magnons and which lead the system into the steady state. Parametric pumping, as is well known, "pairs" the magnons, extracting from the thermal background magnon pairs with wave vectors $\mathbf{k}$ and $-\mathbf{k}$ for which the instability increment is a maximum. These pairs are in resonance with the pump ($\omega_{\mathbf{k}} = \omega_p/2$) and have with respect to it the optimal phase $\psi_{\mathbf{k}} \equiv \varphi_{\mathbf{k}} + \varphi_{-\mathbf{k}} = \pi/2$. Because of thermal fluctuations, up to the excitation threshold the pair phases $\psi_{\mathbf{k}}$ vary in time in random fashion, so that the mean value $\langle \exp(i\psi_{\mathbf{k}})\rangle \approx 0$. However, there is established beyond the threshold a rigid phase correlation of the pairs: $\langle \exp(i\psi_{\mathbf{k}})\rangle = \langle c_{\mathbf{k}}c_{\mathbf{k}}^*\rangle/\langle c_{\mathbf{k}}c_{-\mathbf{k}}\rangle$, in spite of the fact that the individual magnon phases remain stochastic as before. In such an ordered state the magnon pairs exert on each other a strong influence describable by the interaction Hamiltonian of the S theory[3]:

$$\mathscr{H}_{int} = \sum_{\mathbf{kk}'} T_{kk'}|c_{\mathbf{k}}|^2|c_{\mathbf{k}'}|^2 + \frac{1}{2}\sum_{\mathbf{kk}'} S_{\mathbf{kk}'}c_{\mathbf{k}}^*c_{-\mathbf{k}}^*c_{\mathbf{k}'}c_{-\mathbf{k}'}. \qquad (15)$$

Actually, such an interaction implies that each pair is in a self-consistent field consisting of the external pump $hV_{\mathbf{k}}$ and the effective field of the system of pairs, $\sum_{\mathbf{k}'} S_{\mathbf{kk}'}c_{\mathbf{k}'}c_{-\mathbf{k}'}$. In the steady state there occurs phase matching of the pairs with the total pump $hV_{\mathbf{k}} + \sum_{\mathbf{k}'} S_{\mathbf{kk}'}c_{\mathbf{k}'}c_{-\mathbf{k}'}$, and a mismatch with the external pump inevitably arises; the energy flux into the system decreases to a level at which the energy inflow will be equal to the dissipation. From the condition of energy balance we obtain the following expressions for the integrated (averaged over all the directions of $\mathbf{k}$) intensity and phase of the packet[3]:

$$N = \sum_{\mathbf{k}} |c_{\mathbf{k}}|^2 = \frac{(|hV|^2 - \gamma^2)^{1/2}}{|S|}, \quad \sin\psi = \frac{\gamma}{hV}, \quad \cos\psi = -\frac{SN}{hV}. \qquad (16)$$

In general outline, the character of the processes leading to the limitation of the parametric instability of the magnons is preserved during the modulation of the magnetic field. However, besides the considered processes of phase correlation, there arise a number of additional effects that can significantly change the values of the steady-state amplitude N and phase ψ. Under the action of the modulating RF field, there get excited collective magnon oscillations that interact nonlinearly with the initial parametric magnons and the RF field, as a result of which the attenuation of the magnons increases and their interaction with each other changes. Responsible for the additional damping are interaction processes of the type $H_m c\alpha^*$, which are processes of fusion of a magnon with a RF photon with the generation of a collective mode α. It is obvious that these processes lead to a correction to γ proportional to the square of the modulating-field amplitude. Other interaction processes of the type $c^*c^*\alpha\alpha$ provide an additional magnon–magnon coupling channel through the collective oscillations; these processes can be interpreted as renormalization of the four-magnon interaction coefficient $S_{\mathbf{kk}'}$.

Let us write the equation of motion of the magnons with allowance for their interaction with each other, (15), and the interaction with the RF field:

$$\frac{dc}{dt} + \gamma c + i(\Delta_{\mathbf{k}} + 2T|c|^2 + 2UH_m(t))c + i(hV + Sc^2)c^* = 0. \qquad (17)$$

It is obtained as a result of the averaging of the equations of motion for the amplitudes $c_{\mathbf{k}}$ and $c_{-\mathbf{k}}^*$ over all the directions of the wave vector $\mathbf{k}$. We can do this without loss of generality, since we are interested in the response of the magnons to a uniform RF field and we do not consider here the inhomogeneous collective oscillation modes.

Setting

$$H_m(t) = \tfrac{1}{2}(H_m e^{-i\Omega t} + H_m^* e^{i\Omega t}),$$

we seek the solution to (17) in the form

$$c = c_0 + \alpha_+ e^{i\Omega t} + \alpha_- e^{-i\Omega t}. \qquad (18)$$

Here we restrict ourselves to the case of small amplitudes of the collective oscillations: $|\alpha_\pm| \ll |c_0|$. Substituting (18) into Eq. (17), we obtain a system of algebraic equations for the amplitudes $\alpha_\pm$ and c_0, from which we have to second order in α:

$$\alpha_\pm = UH_m^\pm \frac{\pm\Omega - 2SN}{D_\pm} c_0, \qquad (19)$$
$$H_m^+ = H_m^*, \quad H_m^- = H_m, \quad D_\pm = \Omega_0^2 - \Omega^2 \pm 2i\gamma\Omega,$$
$$\Omega_0 = 2N\sqrt{S(2T+S)}, \quad N = |c_0|^2.$$

The equation for the steady-state amplitude c_0 has the form

$$\gamma\left(1+4|UH_m|^2\frac{\Omega^2}{|D|^2}\right) + i\left[\Delta_{k_0} + \left(2T+S+2|UH_m|^2\frac{S\Omega_0^2+(2T+3S)\Omega^2}{|D|^2}\right)N + hVe^{i\psi}\right]=0. \qquad (20)$$

The position of the surface in **k** space on which the magnons are excited in the steady state is found from the condition of stability of the solution to (20) against small perturbations δc lying outside the stationary surface—the so-called condition for "external stability," in the terminology of[3]. Let us write down the steady-state equation of motion for the perturbation:

$$[\gamma + i(\Delta_k + 2T|c_0|^2 + U(H_m e^{-i\Omega t} + H_m^* e^{i\Omega t}))]\delta c + i(hV + Sc_0^2)\delta c^* = 0. \qquad (21)$$

Let us seek the solution to Eq. (21) in a form similar to (18):

$$\delta c = \delta c_0 + \delta\alpha_+ e^{i\Omega t} + \delta\alpha_- e^{-i\Omega t}.$$

After the substitution of the solution into (21), we obtain a system of equations for $\delta\alpha_\pm$ and δc_0, from the condition of consistency of which we find two surfaces in k space: $\Delta_k = \Delta_{k_0}$ and $\Delta_k = \Delta_{k_1}$ between which lies the instability region. It is not difficult to understand that the external stability of the steady state will be guaranteed only under the condition that $\Delta_{k_0} = \Delta_{k_1}$. Let us omit the tedious calculations and give the final answer:

$$\Delta_{k_0} = -2\tilde{T}N,$$
$$\tilde{T} = T - \frac{4|UH_m|^2}{\Omega^2+4\gamma^2} \qquad (22)$$
$$\times \frac{T(\Omega^2+4\gamma^2)(\Omega^2+4S^2N^2)+4S[\Omega^2(\gamma^2+2S^2N^2)+4\gamma^2(SN)^2]}{|D|^2}.$$

Substituting the steady-state detuning Δ_{k_0}, determined by the formula (22), into Eq. (20), we represent the latter in the form

$$hVe^{i\psi} + \tilde{S}N = i\Gamma, \qquad (23)$$

where

$$\Gamma = \gamma\left(1+\frac{4|UH_m|^2\Omega^2}{|D|^2}\right),$$
$$\tilde{S} = S\left[1+2\frac{|H_mU|^2(3\Omega^2+4\gamma^2)(\Omega^2-4S^2N^2)}{(\Omega^2+4\gamma^2)|D|^2}\right]. \qquad (24)$$

Equation (23) coincides in form with the equation for the steady state in the S theory[3] if the coefficients γ and S in the latter are assumed to be renormalized in accordance with (24). Therefore, in the presence of a RF field we should in place of (16) write

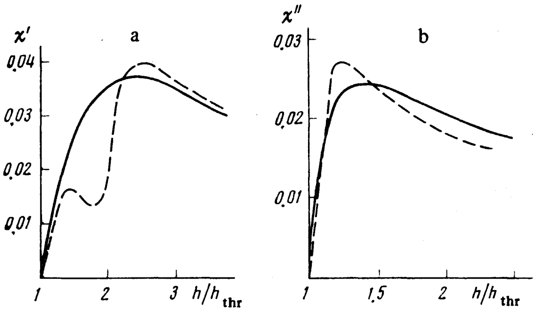

FIG. 5. a) Theoretical dependence of χ' on the microwave-pump amplitude h for $H_m = 0$ (the solid curve) and $UH_m/2\gamma = 0.3$ (the dashed curve); $2T = S$, $\Omega/2\gamma = 2$ ($H_m = 0.15$ Oe, $\Omega = 1.6$ MHz). b) Theoretical dependence of χ'' on h for $H_m = 0$ (the solid curve) and $UH_m/2\gamma = 0.3$ (the dashed curve); $2T = S$; $\Omega/2\gamma = 1$ ($H_m = 0.15$ Oe, $\Omega = 0.8$ MHz).

$$N = \frac{\sqrt{|hV|^2-\Gamma^2}}{|\tilde{S}|}, \quad hV\sin\psi = \Gamma, \quad \cos\psi = -\frac{\tilde{S}N}{hV}. \qquad (25)$$

The nonlinear susceptibility χ'', averaged over the a time interval exceeding $1/\Omega$, is determined by the expression:

$$\chi'' = 2W_+/\omega_p h^2,$$

where W_+ is the microwave-field energy flux flowing into the sample per second. Notice that in the presence of a RF field, W_+ does not coincide with the total power absorbed in the sample. Defining W_+ according to the formula

$$W_+ = -\frac{\partial}{\partial t}\left[\frac{1}{2}hV(c^*)^2 + \text{c.c.}\right] = \omega_p hV\,\mathrm{Im}(c_0^2 + 2\alpha_+\alpha_-)$$

and using (19) and (23), we obtain

$$\chi'' = \frac{2\Gamma N}{h^2}\left[1+2|H_mU|^2\frac{4S^2N^2-\Omega^2}{|D|^2}\right]. \qquad (26)$$

Similarly, for the real part of the susceptibility we obtain the expression:

$$\chi' = \frac{2V}{h}\mathrm{Re}(c_0^2+2\alpha_+\alpha_-) = \frac{2\tilde{S}N^2}{h^2}\left[1+2|H_mU|^2\frac{4S^2N^2-\Omega^2}{|D|^2}\right]. \qquad (27)$$

In Fig. 5 we show the dependence of the nonlinear microwave susceptibilities, χ' and χ'', on the excess SHF power over the threshold. $\Delta\chi'$ and $\Delta\chi''$ were calculated on an electronic computer from the formulas (26), (24), and (11) for different values of the amplitude and frequency of the RF field. As can be seen from Fig. 5, the response of the system to the modulating RF field is quite sensitive to the supercriticality $\zeta = h/h_{thr}$ and the modulation frequency Ω. The maximum response, as was to be expected, is observed near the collective-oscillation resonance ($\Omega = \Omega_0$).

There exists on the curves $\Delta\chi'(\zeta)$ a characteristic point $\Omega = 2SN(\zeta)$ at which $\Delta\chi' = 0$. Both frequencies increase with increasing supercriticality:

$$\Omega_0 = 4\gamma^2\frac{2T+S}{S}(\zeta^2-1), \quad (2SN)^2 = 4\gamma^2(\zeta^2-1). \qquad (28)$$

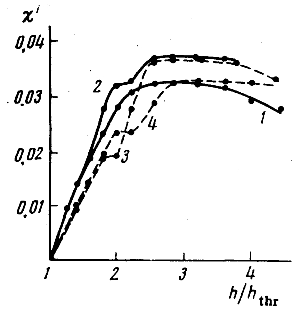

FIG. 6. Experimental dependences of χ' on the supercriticality for $H = H_c - 100$ Oe. The curve 1 corresponds to $H_m = 0$ and the curves 2, 3, and 4 correspond to $H_m = 0.14$ Oe; $\Omega = 0.8$, 1.6, and 2.4 MHz.

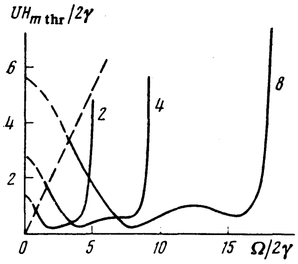

FIG. 8. Theoretical dependences (the formula (31)) of the instability threshold, $H_{m\,\mathrm{thr}}$, for the collective oscillations on the frequency, Ω, of the RF field. The numbers on the curves correspond to the values of the natural frequency ($\Omega_0/2\gamma$) of the collective oscillations. The region of applicability of the theory is the region below the dashed line.

In Figs. 6 and 7 we show the experimental dependences of χ' and χ'' on the supercriticality for $H_m = 0$ and $H_m = 0.14$ Oe.

We can speak of qualitative agreement between the theory and experiment. The theoretical order of magnitude of the effect coincides with the experimental value; for example, for $H_m = 0.14$ Oe the RF corrections, $\Delta\chi'$ and $\Delta\chi''$, to the susceptibilities constitute 10–20% of $\chi_{\max}$. The expressions for $\Delta\chi'$ and $\Delta\chi''$ consist of a number of terms of different signs, but of the same order of magnitude, so that the net sign can be either plus or negative. In experiment the quantities $\Delta\chi'$ and $\Delta\chi''$ can also be of either sign, depending on the supercriticality. A more detailed comparison is made difficult by the presence of small-amplitude self-oscillations not taken into account in the theory.

§3. PARAMETRIC RESONANCE IN THE COLLECTIVE OSCILLATIONS

Besides the above-considered linear interaction of the collective magnon oscillations with the RF field, an interaction which leads to resonance at the frequency $\Omega = \Omega_0$, there exists a nonlinear interaction of the type $H_m\beta^*\beta^*$ that leads to parametric resonance in the collective modes in a RF field of frequency $\Omega = 2\Omega_0$. To the same effect, as is not difficult to understand, pertains the instability with respect to decay into two modes β with frequency $\Omega/2$ of the initial mode α, with frequency Ω, excited by the RF field. As a result of the action of both mechanisms, there arises, when a certain critical amplitude of the RF field is exceeded, the phenomenon of double parametric magnon resonance—the simultaneous excitation of microwave oscillations (of frequency $\omega_p/2$) and RF oscillations (of frequency $\Omega/2$).

To find the instability threshold for the collective oscillations, let us seek the solution to the equation of motion (17) in the form

$$c = c_0 + \alpha_+ e^{i\Omega t} + \alpha_- e^{-i\Omega t} + \beta_+ e^{i\Omega t/2} + \beta_- e^{-i\Omega t/2}.$$

Using the relation (23) for the ground state and linearizing (17) with respect to the small amplitudes α and β, we obtain the equation

$$\left[\frac{d}{dt} + \gamma + i2(T+S)N\right]\beta_+ + [-\gamma + i2TN]e^{-i\psi}\beta_-^* + i\omega_+\beta_- + iP_+e^{-i\psi}\beta_+^* = 0,$$
$$\omega_+ = (H_m U)^* + 2(2T+S)(c_0^*\alpha_+ + c_0\alpha_-^*), \quad \omega_- = \omega_+^*,$$
$$P_\pm = 2(2T+S)c_0\alpha_\pm e^{i\psi}. \tag{29}$$

Let us recall that the oscillation amplitudes $\alpha_\pm$ are given by the formulas (19).

The Eq. (29) can be written in a more symmetric form if from the variables $\beta_\pm$ we go over to the normal variables $\rho_\pm$ with the aid of the following u–v transformation:

$$\rho_+ = u\beta_+e^{i\psi/2} - v\beta_-^*e^{-i\psi/2}, \quad \rho_-^* = -v\beta_+e^{i\psi/2} + u\beta_-^*e^{-i\psi/2}.$$

Here

$$u = \left[\frac{2(T+S)N+\Omega_0}{2\Omega_0}\right]^{1/2}, \quad v = -\left[\frac{2(T+S)N-\Omega_0}{2\Omega_0}\right]^{1/2}.$$

Then for the variables $\rho_\pm$ we obtain the equation

$$\left[\frac{d}{dt} + \gamma + i\Omega_0\right]\rho_+ + iA_+\rho_+^* + iB_+\rho_- - \gamma\rho_-^* = 0,$$
$$A_\pm = 2uv\omega_\pm + u^2P_\pm + v^2P_\mp^*, \quad B_\pm = (u^2+v^2)\omega_\pm + uv(P_\pm + P_\mp^*). \tag{30}$$

Assuming $\rho_\pm(t) = \rho_\pm \exp(\pm\Omega t/2 + \nu t)$, and writing down the system of four equations for the amplitudes ρ_+, ρ_+^*,

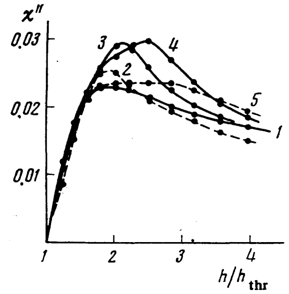

FIG. 7. Experimental dependences of χ'' on the supercriticality for $H = H_c - 100$ Oe. For the curve 1, $H_m = 0$. The curves 2, 3, 4, and 5 correspond to $H_m = 0.14$ Oe; $\Omega = 0.8$, 1.2, 1.6, and 3.2 MHz.

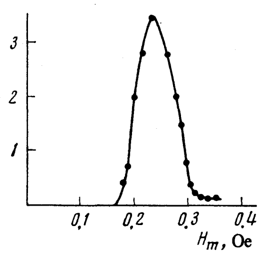

FIG. 9. Dependence of the amplitude (in arbitrary units) of the collective oscillations parametrically excited at the frequency $\Omega/2$ on the amplitude of the RF-field pump. $H = H_c - 100$ Oe, $\Omega = 1.7$ MHz.

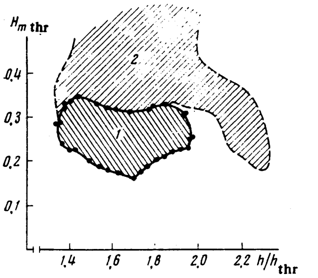

FIG. 10. The region of parametric instability of the collective oscillations. $H = H_c - 100$ Oe, $\Omega = 1.6$ MHz. 1) Region of strong signals of frequency $\Omega/2$; 2) weak-signal region.

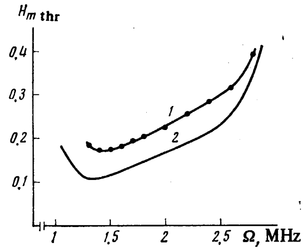

FIG. 11. The experimental (1) and theoretical (2) dependences of the collective-oscillation threshold on Ω. In the experiment $H = H_c - 100$ Oe, $\zeta = 1.6$ (4 dB). The theoretical curve corresponds to $\Omega_0 = 3.4\gamma$ ($2\gamma = 0.75$ MHz[2]).

ρ_-, ρ_-^*, we find the instability increment ν from the condition of equality to zero of the determinant:

$$\begin{vmatrix} \nu+\gamma+i(\Omega_0+\Omega/2), & iA_+, & iB_+, & -\gamma \\ -iA_+^*, & \nu+\gamma-i(\Omega_0+\Omega/2), & -\gamma, & -iB_+^* \\ iB_+^*, & -\gamma, & \nu+\gamma+i(\Omega_0-\Omega/2), & iA_- \\ -\gamma, & -iB_+, & -iA_-^* & \nu+\gamma-i(\Omega_0-\Omega/2) \end{vmatrix} = 0. \qquad (31)$$

The instability threshold ($\nu = 0$) for given values of Ω and Ω_0 was computed from (31) with the aid of an electronic computer. The results of these computations are presented in Fig. 8. It turns out that the minimum threshold is attained at the resonance $\Omega = \Omega_0$, and its magnitude weakly depends on Ω_0:

$$\min\{H_m\}_{\mathrm{thr}} \approx 0.4\gamma/U \approx \gamma/g.$$

The second threshold minimum at the frequency $\Omega = 2\Omega_0$ appears only at sufficiently high Q of the collective oscillations, when $\Omega_0 \gg \gamma$.

To observe experimentally the double parametric resonance, we applied to a spherical $Y_3Fe_5O_{12}$ sample three parallel magnetic fields: a constant magnetizing field H, a microwave pump $he^{i\omega t}$, and a RF pump $H_m e^{i\Omega t}$; the RF field was produced by a miniature coil located at the center of the microwave resonator. The measurements were carried out in the radio-frequency range from 0.1 to 10 MHz. The parametric oscillations at the frequency $\Omega/2$ arose at some critical value of the RF field. The signal of frequency $\Omega/2$, induced in the coil, was amplified by a selective receiver and recorded on an oscillograph. The appearance of this signal has a strongly pronounced threshold character—see Fig. 9. As H_m is increased, the signal almost disappears in liminal fashion: its amplitude decreases by a factor of more than ten.

In Fig. 10 we show the instability region for the collective oscillations in the coordinates H_m, h/h_{thr} for a fixed frequency Ω. As the frequency is varied, the curve bounding the region of large amplitudes shifts primarily along the horizontal toward the region of stronger microwave fields. From the series of these curves for different Ω, we constructed the dependences $H_{m\,\mathrm{thr}}(\Omega)$ for $h/h_{\mathrm{thr}} = \mathrm{const}$. One of such dependences is shown in Fig. 11, where we also show the corresponding theoretical curve computed from the condition (31). The shapes of the two curves are quite similar to each other, with the possible exception of the low-frequency region, where the signal is very weak and it is not possible to carry out reliable measurements of the threshold. Notice that our theory, constructed in the second-order approximation in H_m, predicts only one threshold, $H_{m\,\mathrm{thr}}$, at fixed frequency Ω and supercriticality h/h_{thr}. Meanwhile, we observe in experiment (Figs. 9 and 10) a second threshold: a threshold for the disappearance, as H_m is increased, of the intense collective oscillations. Physically, this threshold is connected with the dependence of the natural frequency of the collective oscillations on H_m^2, a dependence which leads to the violation of the condition for parametric resonance. However, its actual computation is quite difficult, since it requires allowance for higher terms in H_m^2 and may, in general, be impossible in the framework of perturbation theory in UH_m/γ.

In conclusion, let us note that we have considered spatially homogeneous collective oscillations with a wave vector $\varkappa \approx 0$. Besides them, there exists a wide spectrum of collective oscillations in a system of parametric magnons.[3] Individual sections of this spectrum can, in principle, be excited in a parametric manner with the aid of a homogeneous RF field at a frequency satisfying the condition $\Omega = \Omega_\varkappa + \Omega_{-\varkappa}$. These oscillations with $\varkappa \neq 0$ are poorly coupled to the resonator, and therefore the emission at half frequency will be strongly attenuated. It may be inferred that the weak radiation recorded by us at the frequency $\Omega/2$ (the region 2 in Fig. 10) precisely indicates the excitation of inhomogeneous collective oscillations. For their reliable indication, it is necessary to use some other method, e.g., the registration of the inverse effect of the excited oscillations on the pump.

[1]V. V. Zautkin, V. S. L'vov, and S. S. Starobinets, Zh. Eksp. Teor. Fiz. 63, 182 (1972) [Sov. Phys. JETP 36, 96 (1973)].
[2]B. I. Orel and S. S. Starobinets, Zh. Eksp. Teor. Fiz. 68, 317 (1975) [Sov. Phys. JETP 41, 154 (1975)].
[3]V. E. Zakharov, V. S. L'vov, and S. S. Starobinets, Usp. Fiz. Nauk 114, 609 (1974) [Sov. Phys. Usp. 17, 896 (1975)].
[4]H. Suhl, Phys. Rev. Lett. 6, 174 (1961).
[5]T. S. Hartwick, E. R. Peressini, and M. T. Weiss, Phys. Rev. Lett. 6, 176 (1961).

Translated by A. K. Agyei