Two-photon positron-K-electron annihilation

V. G. Gorshkov, A. I. Mikhaĭlov, and S. G. Sherman

Leningrad Institute of Nuclear Physics, USSR Academy of Sciences
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We evaluate the cross-section for the two-quantum positron annihilation by atomic K electrons for small αZ and arbitrary values of the momentum transfer q to the nucleus. We obtain the angular distribution of the emerging photons and various simple limiting formulae. We calculate the shift in the maximum of the line shape of the annihilation photon caused by the binding of the electron to the nucleus.

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1. INTRODUCTION

When a positron beam is scattered by atoms both single-photon and two-photon annihilation (TPA) processes for the positrons caused by the atomic electrons are possible. The annihilation process with the emission of a single γ quantum is possible only for a bound electron. The annihilation with the emission of two photons proceeds both for bound and for free electrons. The single-photon annihilation process is characterized by a cross-section of the order of $r_0^2\alpha^4Z^5$ ($r_0=\alpha/m$ is the classical electron radius, $\alpha=1/137$ the fine-structure constant, and Z the nuclear charge). The TPA cross section is of the order of r_0^2, i.e., for small Z this process is appreciably more probable. However, for heavy atoms the probabilities for the two processes are comparable.

The single-photon annihilation was studied in detail in[1,2]. Dirac[3] and Bethe[4] obtained the cross section for the annihilation of a positron by a free electron. There were no calculations for the cross-section of TPA of positrons by bound electrons. Recently[5] the results have been published of an experimental study of this process: the double angular distribution $d^2\sigma/d\Omega_1\,d\Omega_2$ for the TPA of positrons with energies of 300 keV by the K shell of silver ($Z=47$) was measured using the method of triple coincidences of the two annihilation γ quanta and the single x-ray quantum which is emitted by the atom when the vacancy in the K shell is filled.

In the present paper we evaluate different differential cross sections for the TPA by the K shell of an atom with an arbitrary momentum transfer $\mathbf{q}$ to the nucleus.

The TPA process with arbitrary momentum transfers to the nucleus is in the first approximation in the Coulomb field described by the three Feynman graphs shown in Fig. 1 plus the three diagrams with the photon lines interchanged.

The TPA cross section is maximal for small momentum transfers to the nucleus $q\sim\eta$ ($\eta=m\alpha Z$, m is the electron mass). In that case the process is described by diagram a alone. Up to terms of order α^2Z^2 the total contribution from the region of all q is determined by that graph and is the same as the cross section for the annihilation by a free electron. In the region $q\sim m$ all three diagrams turn out to be of the same order of magnitude. When the transfer of momentum to the nucleus is large the process must proceed at small distances from the nucleus $r\sim 1/q\sim 1/m$ where the wavefunction is proportional to $(\alpha Z)^{3/2}$. As for such q the annihilation by a free electron is kinematically impossible, there arises an additional small factor αZ in diagram a due to the transfer of a momentum q to the nucleus through the wavefunction of the bound electron. In the diagrams b and c the transfer of momentum to the nucleus takes place through a Coulomb photon which also leads to an additional factor αZ. As a result we get for $q\sim m$ for the cross section a quantity of the order of $r_0^2(\alpha Z)^5$. However, for large q there exists a region where the cross section is $\sim r_0^2(\alpha Z)^4$. This is the region of resonant behavior of the diagram c, caused by the production of a positron with a small virtual component $\sim\eta$ when it is scattered by the Coulomb field of the nucleus and the subsequent annihilation of this positron by an electron. The physical nature of such a resonant behavior was discussed in detail in[6,7].

2. CROSS SECTION OF THE PROCESS

TPA of positrons by an atom is a cross-symmetrical reaction channel for the scattering of photons by bound electrons with ionization of the atom. The annihilation amplitude and cross section can be obtained from the corresponding quantities for the Compton scattering of photons by K electrons[6] when we replace the electron energy and momentum $\varepsilon, \mathbf{p}$ by $-\varepsilon$, $-\mathbf{p}$ and the photon energy and momentum $\omega_1, \mathbf{k}_1$ by $-\omega_1$, $-\mathbf{k}_1$. It is necessary to make also a replacement of the electron phase volume d^3p by the phase volume d^3k_1 of the γ quantum and the flux $j=1$ of the incident photons by the positron flux $j=p/\varepsilon$.

After such a substitution the amplitude A of the TPA process takes the form[1)]

$$A=r_0(4\pi)^2N\eta a^{-1}\bar{u}_{-\mathbf{p}}Fu_0,\quad N=\eta^3/\pi,\quad p\gg\eta; \tag{1}$$

$$F=L_1\hat{e}_2(\hat{K}_2-\hat{P}+m)\hat{e}_1(\hat{Q}+m)+L_2\hat{e}_2(\hat{K}_2-\hat{P}+m)\gamma^0(-\hat{K}_1+m\gamma^0+m)\hat{e}_1 +L_3\gamma^0(-\hat{K}+m\gamma^0+m)\hat{e}_2(-\hat{K}_1+m\gamma^0+m)\hat{e}_1+(K_2\rightleftarrows K_1,\ e_2\rightleftarrows e_1), \tag{1a}$$

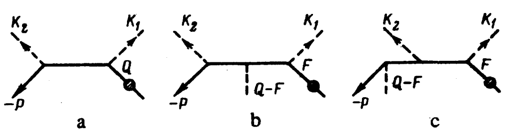

FIG. 1.

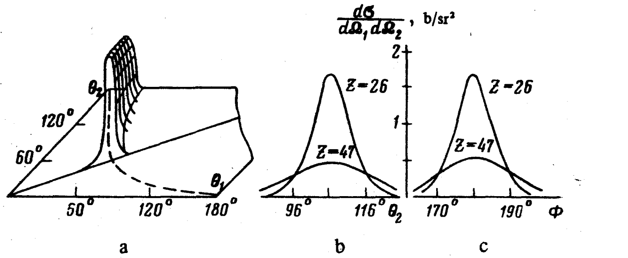

FIG. 2. Double angular distribution of the two-photon annihilation $d\sigma/d\Omega_1 d\Omega_2$ for $T_{e^+} = 300$ keV under the conditions of[5]. a) The contours for $\phi = \varphi_2 - \varphi_1 = 180°$; we show the part where $\theta_2 \geqslant \theta_1$; the dashed lines show the connection between θ_2 and θ_1 for the free kinematics. b) The distribution for $\theta_1 = 30°$, $\phi = 180°$. c) The distribution for $\theta_1 = 30°$ and $\theta_2 = 106°$.

$$L_4 = \frac{1}{aa_2}, \quad L_5 = -\frac{1}{a_1 a_2}, \quad L_6 = \frac{1}{a_1 b}, \tag{1b}$$

$$a = q^2 + \eta^2, \quad a_1 = -2m\omega_1, \quad a_2 = -2(\varepsilon\omega_2 - \mathbf{pk}_2), \quad b = \varkappa^2 - (p + i\eta)^2. \tag{1c}$$

Here e_i, $K_i = (\omega_i, \mathbf{k}_i)$ are the photon polarization and momentum four-vectors, $P = (\varepsilon, \mathbf{p})$ the positron four-momentum, $Q = K_1 + K_2 - P = (m, \mathbf{q})$, $\mathbf{q}$ the momentum transferred to the nucleus, $K = K_1 + K_2 = (\omega_1 + \omega_2, \boldsymbol{\varkappa})$ the total energy and momentum of the two photons, $\hat{V} = \gamma^0 V_0 - \boldsymbol{\gamma} \cdot \mathbf{V}$.

The energy conservation law has the form $m + \varepsilon = \omega_1 + \omega_2$. We use everywhere a system of units in which $\hbar = c = 1$.

The differential cross section of the process, averaged over the initial and summed over the final polarizations of the photons ν and the electrons λ, is given by the following expression:

$$d\sigma = \frac{1}{4mp}\frac{1}{4}\sum_{\nu\lambda}|A|^2 \frac{d^3k_1}{2\omega_1(2\pi)^3}\frac{d^3k_2}{2\omega_2(2\pi)^3} 2\pi\delta(\varepsilon + m - \omega_1 - \omega_2) = \frac{2r_0^2}{\pi^2}\frac{\eta^5}{a^2 p} J\, d\Gamma, \tag{2}$$

where

$$J = \frac{1}{16m}\sum_{\nu\lambda}|\bar{u}_{-p} F u_0|^2 = \frac{1}{16m}\sum_{\nu\lambda} \mathrm{Sp}\, F(\gamma_0 + 1)\gamma_0 F^+ \gamma_0(\hat{p} - m),$$

$$d\Gamma = \frac{d^3k_1\, d^3k_2}{\omega_1\omega_2}\delta(\varepsilon + m - \omega_1 - \omega_2). \tag{3}$$

The expression for J is obtained from Eq. (19) from[6] by the substitutions $\varepsilon, \mathbf{p} \to -\varepsilon, -\mathbf{p}$ and $\omega_1, \mathbf{k}_1 \to -\omega_1, -\mathbf{k}_1$.

TPA by a bound electron is a process in which two particles (the positron and the atom) change into three (two photons and an ion). This process is therefore characterized by five independent variables: ω_1, ω_2, θ_1, θ_2, and $\phi = \varphi_2 - \varphi_1$, where ω, θ, and φ are the energy, and the polar and azimuthal angles of the photons relative to the direction of the positron momentum $\mathbf{p}$. Of most experimental interest is the double angular distribution of the two-photon annihilation (DADTA) over the angles θ_1, θ_2, and ϕ, which can be obtained by integrating (2) over ω_1.

The DADTA is maximal in the region of angles determined by the free kinematics $q = 0$. The condition $q = 0$ fixes three variables and the cross-section for the process becomes dependent on only two variables where $\phi = 180°$, while the angles θ_1 and θ_2 turn out to be connected by the equation $\cos\theta_2 = f(\cos\theta_1) = (p - \varepsilon\cos\theta_1)/(\varepsilon - p\cos\theta_1)$.

In Fig. 2a we show the contours of the DADTA with respect to θ_1 and θ_2 for $\phi = 180°$; the ridge lies along the curve $\cos\theta_2 = f(\cos\theta_1)$. Figure 2b shows the cross section as function of θ_2 for $\theta_1 = 30°$ and $\phi = 180°$, and Fig. 3b the cross section as function of ϕ for the angles $\theta_1 = 30°$ and $\theta_2 = 106°$, which satisfy the condition $q = 0$ for a positron energy of 300 keV.

The DADTA due to the K shell of silver was measured in[5] for a positron kinetic energy of 300 keV for the angles $\theta_1 = 30°$, $\theta_2 = 100°$, and $\phi = 180°$. This point is close to the region of the maximum in Fig. 2. The theoretical value of the cross-section in that range of angles equals 0.5 b/sr^2 which corresponds to the magnitude of the cross-section for the annihilation process due to free electrons. The experimental value obtained in[5] equaled 7.7 ± 6.4 mb/sr^2 which is two orders of magnitude smaller than the theoretical value. There are no other experimental data for this process.

We now consider the cross section for the process in different ranges of the kinematic variables which are of physical interest.

3. REGION OF SMALL MOMENTUM TRANSFERS

The region where the momentum transferred to the nucleus is small $q \sim \eta$ makes the main contribution to the cross-section and is therefore of most interest. We obtain in that region the cross-section with a relative accuracy of the order of αZ. To do this we must take into account in the amplitude terms linear in q/m and αZ. These terms make it possible to evaluate the main part of the magnitude of the shift in the quasi-free peak corresponding to the annihilation process due to a free electron.

When $q \leqslant \eta$ the contribution I_a from diagram a of Fig. 1 is proportional to the quantity

$$\eta a^{-2} = \eta(q^2 + \eta^2)^{-2} \sim \eta^{-3}.$$

Diagram b of Fig. 1 which contains the Coulomb correction to the electron Green function is proportional to

$$\alpha Z \omega_1^{-1}(q^2 + \eta^2)^{-1} \sim (m\omega_1\eta)^{-1} \sim (\alpha Z)^2 I_a$$

and can be neglected. The diagram of Fig. 1c is proportional to the quantity

$$\eta(ab)^{-1} \sim \eta(q^2 + \eta^2)^{-1}(pq - ip\eta)^{-1} \sim p^{-1}\eta^{-2} \sim \alpha Z I_a.$$

It is necessary to take into account in the diagram of Fig. 1c also terms of order $\xi\eta a^{-2} \sim \alpha Z I_a$ ($\xi = \alpha Z\varepsilon/p$) omitted in (1) when $q \gg \eta$. Terms of the same order are contained in diagram a and stem from the relativistic correction to the wavefunction of the bound electron. The amplitude of the process is thus given by the sum of the diagrams a and c in the region of small $q \sim \eta$, up to terms of order αZ inclusive.

Using the equations

$$(\hat{Q}+m)u_0 \approx 2m(u_{\mathbf{q}}+O(q^2/m^2)),$$
$$\bar{u}_{-\mathbf{p}}\gamma_0(\hat{K}+m\gamma_0+m) = -2\varepsilon(\bar{u}_{-\mathbf{p}}+O(q/m)),$$

we can appreciably simplify Eq. (1) for the amplitude:

$$\bar{u}_{-\mathbf{p}}Fu_0 = 2m\left(L_1 - \frac{\varepsilon}{m}L_3'\right)\bar{u}_{-\mathbf{p}}\hat{e}_2(\hat{K}_2-\hat{P}+m)\hat{e}_1 u_{\mathbf{q}} = \frac{2m}{a}\left(1-\frac{\varepsilon}{m}L\right)U_{\mathbf{q}}; \tag{4}$$

$$L_3' = \frac{1}{a_1 b} + \frac{1}{a_1 a}\frac{i\eta}{p}\ln\left(\frac{2p}{i\lambda}\frac{a}{b}\right), \qquad L = aa_1L_3', \tag{4a}$$

$$U_{\mathbf{q}} = \frac{\bar{u}_{-\mathbf{p}}\hat{e}_2(\hat{Q}-\hat{K}_1+m)\hat{e}_1 u_{\mathbf{q}}}{(Q-K_1)^2-m^2} + (1 \rightleftarrows 2), \tag{4b}$$

where $u_{\mathbf{q}}$ and $U_{\mathbf{q}}$ are the Dirac bispinor and the amplitude of the scattering by a free electron with initial momentum $\mathbf{q}$.

The term in L_3' with the infrared divergence arose from the expansion of the infrared phase factor in the positron wavefunction.[8] This term disappears from the cross section:

$$|\bar{u}_{-\mathbf{p}}Fu_0|^2 = \frac{4m^2}{a^2}\left(1-2\frac{\varepsilon}{m}\operatorname{Re}L\right)|U_{\mathbf{q}}|^2, \tag{5}$$

$$\operatorname{Re}L = \frac{\eta}{p}\operatorname{arctg}\frac{\varkappa^2-p^2+\eta^2}{2p\eta} + \frac{(\varkappa^2-p^2+\eta^2)a}{|b|^2}$$
$$\approx \frac{\eta}{p}\operatorname{arctg}\frac{\mathbf{nq}}{\eta} + \frac{\mathbf{nq}}{2p}\frac{q^2+\eta^2}{(\mathbf{nq})^2+\eta^2}, \qquad \mathbf{n} = \frac{\mathbf{p}}{p}. \tag{5a}$$

The square of the amplitude $|U_{\mathbf{q}}|^2$ of the process involving a free electron with initial momentum $\mathbf{q}$ depends on the invariant variables $QK_1 = m\omega_1 - \mathbf{q}\cdot\mathbf{k}_1$ and $QK_2 = m\omega_2 - \mathbf{q}\cdot\mathbf{k}_2$. Expanding $|U_{\mathbf{q}}|^2$ in terms of $q/m \sim \alpha Z$ we obtain the following expression for the cross section of the process up to terms linear in αZ when $q \sim \eta$ (we have dropped terms proportional to $(\alpha Z)^2$, $\alpha Zq/m$, $(q/m)^2$):[2)]

$$d\sigma = r_0^2\frac{8\eta^5}{\pi^2 a^4}\frac{m}{p}\left\{f_0\left[1-\xi\left(2\operatorname{arctg}\frac{\mathbf{nq}}{\eta}+\frac{\mathbf{nq}}{\eta}\frac{q^2+\eta^2}{(\mathbf{nq})^2+\eta^2}\right)\right]\right.$$
$$\left. - f_1\frac{\mathbf{qk}_1}{m\omega_1} - f_2\frac{\mathbf{qk}_2}{m\omega_2}\right\}d\Gamma, \tag{6}$$

$$d\Gamma = \frac{d^3k_1\,d^3k_2}{\omega_1\omega_2}\delta(\omega_1+\omega_2-\varepsilon-m) = \omega_1\omega_2 d\omega_1\,d\Omega_1\,d\Omega_2, \tag{6a}$$

$$f_0 = \frac{1}{2}\left[\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}+2\left(\frac{m}{\omega_1}+\frac{m}{\omega_2}\right)-\left(\frac{m}{\omega_1}+\frac{m}{\omega_2}\right)^2\right], \tag{7}$$

$$f_1 = \omega_1\frac{\partial f_0}{\partial\omega_1} = \frac{1}{2}\left[\frac{\omega_1}{\omega_2}-\frac{\omega_2}{\omega_1}+\frac{2m}{\omega_1}\left(\frac{m}{\omega_1}+\frac{m}{\omega_2}-1\right)\right], \tag{7a}$$

$$f_2 = \omega_2\frac{\partial f_0}{\partial\omega_2} = f_1(\omega_1 \rightleftarrows \omega_2). \tag{7b}$$

Replacing d^3k_2 in (6a) by d^3q and integrating over d^3q[6] we get in our approximation an expression which is the same as the TPA cross section due to a free electron:

$$d\sigma = d\sigma_{2\gamma}^0(\omega_1) = r_0^2 f_0\frac{\omega_1^2}{p(\varepsilon+m)}d\Omega_1 = 2\pi r_0^2 f_0\frac{m}{p^2}d\omega_1, \tag{8}$$

where f_0 is defined by Eq. (7) with the values

$$\omega_2 = \varepsilon+m-\omega_1, \qquad \omega_1 = \omega_{10} = \frac{m(\varepsilon+m)}{\varepsilon+m-p\cos\theta_1}. \tag{9}$$

Integrating (6) over $d\Omega_2$ we get the distribution over the energy and solid angle of one of the photons:

$$\frac{d\sigma}{d\omega_1\,d\Omega_1} = \frac{8}{3\pi}\frac{r_0^2}{\eta}\frac{m\omega_1}{p\omega_2}\frac{f_0}{(1+x^2)^3}[1+\alpha ZxF(\omega_1,t_1)], \tag{10}$$

$$F(\omega_1,t_1) = \chi(\omega_1)+\frac{\varepsilon}{p}\frac{\Phi(x,t)}{f_0}, \tag{10a}$$

$$\chi(\omega_1) = \frac{f_1}{f_0}\left(1-\frac{m}{\omega_1}-\frac{m}{\omega_2}\right)+\frac{f_2}{f_0}-\frac{m}{\omega_2}, \qquad \Phi(x,t) = \sum_{i=1}^{3}\Phi_i(x,t), \tag{10b}$$

where

$$\Phi_1(x,t) = {}^3/_4 ct\gamma^4[x^6(5-3t^2)+x^4(13-21t^2+14t^4)$$
$$+x^2(8-26t^2+23t^4+t^6)-t^2(8-9t^2-t^4)],$$

$$\Phi_2(x,t) = 2ct\gamma\{1+h-0{,}75c^2s^4\gamma^4(7x^4-12x^2t^2+t^4)$$
$$+x^2s^4\gamma^2(1+3h)\}x^{-1}\operatorname{arctg}x,$$

$$\Phi_3(x,t) = -s^2\gamma\{1+h+0{,}375c^2s^4\gamma^4[x^4-6x^2t^2+t^4$$
$$-3cs^{-2}(x^4-10x^2t^2+5t^4)]-c^2t^2\gamma^2(1+3h)\}\ln[(1+t)/(1-t)].$$

The functions f_0, f_1, f_2 are given by (7), (7a), (7b),

$$h = 0.5cs^2\gamma^2(x^2-t^2), \qquad t = \mathbf{p}\boldsymbol{\varkappa}_1/p\varkappa_1 = (p-\omega_1t_1)/\varkappa_1,$$
$$c = 1+x^2, \qquad \gamma = (x^2+t^2)^{-1}, \qquad s^2 = 1-t^2,$$
$$x = (\varkappa_1-\omega_2)/\eta, \qquad \boldsymbol{\varkappa}_1 = \mathbf{p}-\mathbf{k}_1, \qquad \varkappa_1 = |\boldsymbol{\varkappa}_1| = (p^2+\omega_1^2-2p\omega_1t_1)^{1/2},$$
$$t_1 = \mathbf{pk}_1/p\omega_1 = \cos\theta_1.$$

The value $x=0$ corresponds to $\omega_1=\omega_{10}$ of (9). Equation (10) is valid for $x \ll (\alpha Z)^{-1}$, i.e., in the vicinity of the curve $\omega_1=\omega_{10}$ and is the contour of the annihilation photon line. The width of the line is determined by the factor $(1+x^2)^{-3}$ and has the magnitude $\delta_x \sim 1$. The correction terms $\sim \alpha Zx$ do not contribute to the total cross-section, but shift the line maximum relative to $x=0$ ($\omega_1=\omega_{10}$) by an amount $\Delta x \sim \alpha Z$ ($\Delta\omega_1 \sim m(\alpha Z)^2$). One checks easily that terms of order $(\alpha Z)^2$ which are omitted in (10) shift the maximum of the line by an amount $\Delta x \sim (\alpha Z)^3$ ($\Delta\omega_1 \sim m(\alpha Z)^4$). Equation (10) thus enables us to obtain the shift in the maximum of the line contour of the annihilation photon with a relative accuracy of the order $(\alpha Z)^2$.

If we fix ω_1 in the region allowable by the free kinematics we can evaluate the shift in the line maximum as function of the variable x. For the evaluation of the derivative it is sufficient to put $x=0$, $t_1=t_{10}$ in the function $F(\omega_1,t_1)$ in (10). For the position of the maximum we get:

$$\Delta x = x_{max} = \frac{\alpha Z}{6}F(\omega_1,t_1) = \frac{\alpha Z}{6}\left(\chi(\omega_1)+\frac{\varepsilon}{p}\frac{\Phi(0,t_0)}{f_0}\right), \tag{11}$$

where

$$\Phi(0,t) = \frac{15}{2t^5}\left(-1+\frac{7}{6}t^2+\frac{3}{10}t^4+\frac{2-3t^2+t^6}{4t}\ln\frac{1+t}{1-t}\right) \approx \frac{18}{7}t+O(t^2),$$
$$t_0 = \frac{p-\varepsilon t_{10}}{\varepsilon-pt_{10}}, \qquad t_{10} = \frac{p^2+\omega_1^2-\omega_2^2}{2\omega_1 p}. \tag{11a}$$

Using the definition of x we find the shift of the maximum in the distribution over the angular variable t_1:

$$\Delta t_1 = t_{1\,max}-t_{10} = -\frac{\omega_2}{\omega_1}\frac{\eta}{p}x_{max}, \qquad {}^1/_2(\varepsilon+m-p) < \omega_1 < {}^1/_2(\varepsilon+m+p). \tag{12}$$

Equation (12) determines the angle at which the majority of the photons with a fixed energy ω_1 fly away.

The shift in the maximum in the energy ω_1 for a fixed angle θ turns out to be of the order of the binding energy of the electron $\eta^2/2m$ so that it is necessary to take the latter into account in the energy conservation law:

$$\omega_1+\omega_2 = \varepsilon+m-\eta^2/2m. \tag{13}$$

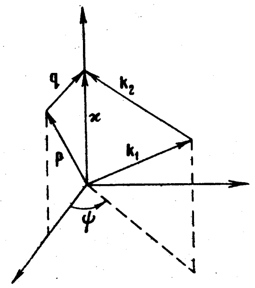

FIG. 3.

Putting $\omega_1 = \omega_{10}$ into $F(\omega_1, t_1)$ in (10) and evaluating the derivative of (10) for a fixed angle t_1 we get:

$$\Delta\omega_1 = \omega_{1\,max} - \omega_{10} = \frac{(\alpha Z)^2}{6}\frac{\omega_1\omega_2}{\varepsilon+m}\left[\chi(\omega_1) + \frac{\varepsilon}{p}\frac{\Phi(0,t)}{f_0} + \frac{\omega_2 f_1 - \omega_1 f_2}{(\varepsilon+m)f_0} - 2\right]\Bigg|_{\omega_1=\omega_{10}}, \qquad t=(p-\varepsilon t_1)/(\varepsilon - p t_1), \tag{14}$$

where the functions f_i, $\chi(\omega_1)$, and $\Phi(0,t)$ were defined in (7), (7a), (7b), (10b), and (11a). In the non-relativistic region $\eta \ll p \ll m$ (14) becomes

$$\Delta\omega_1 = \frac{\eta^2}{12p}\left[\Phi(0,t) - 3\frac{p}{m} + O\left(\frac{p^2}{m^2}\right)\right]. \tag{15}$$

4. REGION OF THE RESONANT BEHAVIOR OF THE CROSS SECTION

For the case where the momentum transferred to the nucleus is large $q \gg (m\eta)^{1/2}$ there exists a region of Coulomb resonance where the differential cross section $d\sigma/d^3k_1 d\Omega_2$ is of the order $r_0^2(\alpha Z)^3$. In the non-resonant region the cross section for such q is of the order $r_0^2(\alpha Z)^5$. The region of the Coulomb resonance is connected with the presence of a factor in the shape of a pole b^{-1} in diagram c of Fig. 1 for the amplitude (1). When $|\varkappa - p| \sim \eta$ and $q \gg (m\eta)^{1/2}$ this diagram is of the order η^{-1} and determines the cross-section (in the region $q \sim \eta$ the contribution from the diagram c of Fig. 1 is smaller by a factor αZ than that from diagram a):

$$F = L_3\gamma^0(\hat{M} - \hat{K} + m)\hat{e}_2(\hat{M} - \hat{K}_1 + m)\hat{e}_1 + (1 \rightleftarrows 2); \qquad M = (m, 0, 0, 0), \quad L_3 = (a_1 b)^{-1}. \tag{16}$$

Putting $\varkappa = p$ everywhere except in the resonance denominator b and using the relation

$$\sum_\lambda u_{-p}^\lambda \bar{u}_{-p}^\lambda = \hat{P} - m,$$

we transform the expression in the first brackets of Eq. (16) to the form

$$\hat{M} - \hat{K} + m = -\varepsilon\gamma^0 + \varkappa\gamma + m = -\hat{P}' + m = -\sum_\lambda u_{-p'}^\lambda \bar{u}_{-p'}^\lambda, \tag{17}$$

where $P' = (\varepsilon, \varkappa)$, $|\varkappa| = p$.

Using (17) we can write the amplitude for the process (16) in the form of a product of the amplitude of the elastic scattering of a positron by the Coulomb field A_c^λ and the amplitude of the TPA of the scattered positron by a free electron $A_{2\gamma}^\lambda$ (both amplitudes are connected with the polarization of the intermediate positron):

$$\begin{gathered} A = -Nb^{-1}\sum_\lambda A_c^\lambda A_{2\gamma}^\lambda, \\ A_c^\lambda = 4\pi\alpha Z q^{-2}\bar{u}_{-p}\gamma^0 u_{-p'}^\lambda, \qquad \mathbf{q} = \boldsymbol{\varkappa} - \mathbf{p}, \\ A_{2\gamma}^\lambda = 4\pi\alpha\left[\bar{u}_{-p'}^\lambda\hat{e}_2\frac{\hat{V}+m}{V^2-m^2}\hat{e}_1 u_0 + (1 \rightleftarrows 2)\right], \\ V = M - K_1, \quad V^2 - m^2 = -2m\omega_1. \end{gathered} \tag{18}$$

Using the explicit form of the bispinors and of the matrix γ^0 we easily get the equation

$$\sum_{\lambda_1}(\bar{u}_{-p}^{\lambda_1}\gamma^0 u_{-p'}^\lambda)(\bar{u}_{-p'}^{\lambda'}\gamma^0 u_{-p}^{\lambda_1}) = \mathrm{Sp}(\hat{P} - m)\gamma^0 u_{-p'}^\lambda \bar{u}_{-p'}^{\lambda'}\gamma^0$$

$$= 2(\varepsilon^2 + m^2 + \mathbf{p}\boldsymbol{\varkappa})\delta_{\lambda\lambda'} = 4\varepsilon^2(1 - q^2/4\varepsilon^2)\delta_{\lambda\lambda'},$$

where we have summed over the polarization of the initial positron. Using this equation we can easily sum over the remaining polarizations when evaluating the square of the amplitude:

$$\begin{gathered} \frac{1}{4}\sum|A|^2 = N^2|b|^{-2}\sum|A_c|^2 \cdot \frac{1}{4}\sum|A_{2\gamma}|^2, \\ \sum|A_c|^2 = 4(4\pi\alpha Z)^2 q^{-4}\varepsilon^2(1 - q^2/4\varepsilon^2), \\ \frac{1}{4}\sum|A_{2\gamma}|^2 = 4(4\pi\alpha)^2 f_0; \end{gathered} \tag{19}$$

f_0 is defined by Eq. (7).

When studying the region of the Coulomb resonance and of large momentum transfers to the nucleus it is convenient to change to such a set of independent variables that it includes q and $\varkappa$. The phase volume (3) contains differentials of five independent variables. After averaging over the polarizations one of the azimuthal angles corresponds to the rotation of the fixed system of vectors and its differential can be replaced by 2π. We choose as the other variables q, $\varkappa$, ω_1, and ψ, where ψ is the angle between the planes determined by the vectors $\mathbf{p}$, $\mathbf{q}$, $\boldsymbol{\varkappa}$ and $\mathbf{k}_1$, $\mathbf{k}_2$, and $\boldsymbol{\varkappa}$. The phase volume $d\Gamma$ takes in the new variables the form

$$d\Gamma = (2\pi q/p)\,d\varkappa\,dq\,d\omega_1\,d\psi. \tag{20}$$

One can easily check that this expression is correct by choosing the coordinate axes in the way shown in Fig. 3.

It can be seen from Eqs. (18) and (19) that the dependence of the cross section on the azimuthal angle ψ disappears in the resonance region and we replace therefore its differential by 2π:

$$d\Gamma = (2\pi)^2(q/p)\,d\varkappa\,dq\,d\omega_1. \tag{20a}$$

Substituting (19) and (20a) into (2) we get the following expression for the cross section in the Coulomb resonance region:

$$d\sigma(\varkappa, q, \omega_1) = R(\varkappa)\frac{d\varkappa}{2\pi}d\sigma_c(q)\,d\sigma_{2\gamma}^0(\omega_1)\left[1 + O\left(\frac{\varkappa - p}{p}\right)\right], \tag{21}$$

where

$$R(\varkappa) = N^2/[(\varkappa - p)^2 + \eta^2], \quad N^2 = \eta^3/\pi;$$

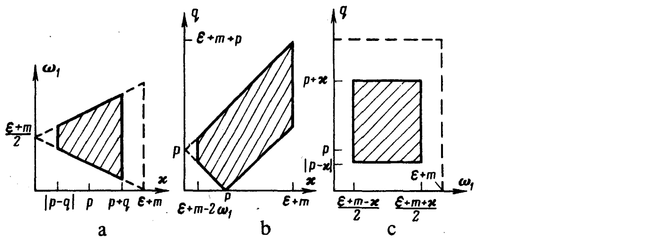

FIG. 4. The physical regions for the variables ω_1, $\varkappa$, and q. The projections of the whole region onto the plane are bounded by the dashed lines. The cross section of the physical region with the plane corresponding to a fixed value of the third variable is hatched.

$d\sigma_c(q)$ and $d\sigma^0_{2\gamma}(\omega_1)$ are respectively the cross sections for the scattering of the positron by the Coulomb field of the nucleus and for the TPA of the positron by a free electron:

$$d\sigma_c(q)=\left(\frac{d\sigma_c}{d\Omega}\right)\frac{2\pi q\,dq}{p^2},$$
$$\frac{d\sigma_c}{d\Omega}=\frac{(2\alpha Z\varepsilon)^2}{q^4}\left(1-\frac{q^2}{4\varepsilon^2}\right),$$

while $d\sigma^0_{2\gamma}(\omega_1)$ is given by Eq. (8).

One can observe the Coulomb resonance only for large momentum transfers to the final atom $q \gg (m\eta)^{1/2}$. Resonance occurs when $|\varkappa - p| \sim \eta$. If the momentum transfer $q\sim\eta$ only the region $|\varkappa - \mathbf{p}|\sim\eta$ ($|p-q| \leqslant \varkappa \leqslant p+q$) is kinematically possible. Moreover, when $q\sim\eta$ the main contribution, as was noted in Sec. 3, comes from the impulse approximation determined by diagram a of Fig. 1 which parametrically exceeds the contribution from the "resonance" diagram c. Small q correspond to the region where $\mathbf{p}\approx\boldsymbol{\varkappa}$, i.e., the total momentum of the final photons $\boldsymbol{\varkappa}$ is the same as the positron momentum $\mathbf{p}$ both in direction and in magnitude. The Coulomb resonance region ($q\gg(m\eta)^{1/2}$, $\varkappa\approx p$) corresponds to the case where $\boldsymbol{\varkappa}$ and $\mathbf{p}$ have the same magnitude but different directions.

It is clear from Eq. (21) that when $|\varkappa - p|\sim\eta$ the differential cross-section $d\sigma\sim r_0^2(\alpha Z)^3$, but the contribution from the resonance region is a quantity of order $r_0^2(\alpha Z)^4$, as the resonance has a width of order η. It is clear fron Eqs. (1) and (2) that the contribution from the non-resonance region for $q\gg(m\eta)^{1/2}$ is a quantity of order $r_0^2(\alpha Z)^5$. The resonance region therefore gives the main contribution to the cross section for large q:

$$d\sigma(q,\omega_1)=(\eta^2/2\pi)\,d\sigma_c(q)\,d\sigma_{2\gamma}^0(\omega_1). \tag{22}$$

The physical region is bounded in the variables $\varkappa$ and ω_1 by the conditions

$$|\omega_1-\omega_2|\leqslant\varkappa\leqslant\omega_1+\omega_2=m+\varepsilon,\qquad |p-q|\leqslant\varkappa\leqslant p+q$$

and shown in Fig. 4a. The resonance region is a narrow vertical strip $|\varkappa - p|\sim\eta$.

To observe Coulomb resonances it is necessary to know the momentum transferred to the nucleus. As it is impossible to measure directly the recoil momentum of the nucleus, it is necessary for its evaluation to have complete information about the momenta of the particles taking part in the reaction. If there are electrons amongst the final particles it is in practice only possible to measure their momentum by using a gaseous target, because of the small range of a charged particle, and this appreciably increases the time needed for performing the experiment. In this connection the TPA process differs conveniently from other processes as the final particles here are γ quanta, the range of which even in a solid target is relatively large. The momentum of the initial particle—the positron—on the other hand is given by the experimental conditions.

When resonance behavior of the cross section occurs it is most convenient to track the distribution in $\varkappa$ and q. In the region of large $q \gtrsim m$ this distribution changes by two orders of αZ: from a magnitude $\sim r_0^2(\alpha Z)^5$ for $|\varkappa - p|\gg\eta$ to a magnitude $\sim r_0^2(\alpha Z)^3$ for $|\varkappa-p|\sim\eta$. In the region $|\varkappa - p|\gg\eta$ the distribution of $d\sigma/dq\,d\varkappa$ is obtained from Eq. (2) with the phase volume $d\Gamma$ given by Eq. (20) and Eq. (19) of[6] by integrating over ψ (see Fig. 3) and ω_1. This integration was performed numerically. To obtain the above mentioned distribution in the region $|\varkappa - p|\sim\eta$ it is sufficient to integrate (21) over ω_1 from $(\varepsilon+m-p)/2$ to $(\varepsilon+m+p)/2$. As a result we get:

$$\frac{d\sigma}{d\varkappa\,dq}=R(\varkappa)\,\frac{q}{p^2}\,\frac{d\sigma_c(q)}{d\Omega}\,\sigma_{2\gamma}^0, \tag{23}$$

where $\sigma^0_{2\gamma}$ is the total cross section for TPA by a free electron:

$$\sigma_{2\gamma}^0=\pi r_0^2\frac{m}{p}\left\{\left[\frac{\varepsilon+m}{p}+\frac{2m\varepsilon}{p(\varepsilon+m)}\right]\times\ln\frac{\varepsilon+m+p}{\varepsilon+m-p}-\frac{2m}{\varepsilon+m}-1\right\}.$$

Summing the contributions from all q in the limits $2p\geqslant q\geqslant q_0\gg(m\eta)^{1/2}$ we find the distribution in $\varkappa$ near the resonance:

$$\frac{d\sigma}{d\varkappa}=\frac{R(\varkappa)}{2\pi}\,\sigma_c(q_0)\,\sigma_{2\gamma}^0, \tag{24}$$

where

$$\sigma_c(q_0)=\frac{4\pi\xi^2}{q_0^2}\left(1-\frac{q_0^2}{4p^2}-\frac{q_0^2}{2\varepsilon^2}\ln\frac{2p}{q_0}\right),\qquad \xi=\frac{\alpha Z\varepsilon}{p}.$$

We show in Fig. 5 the curves of the distributions (24) for different Z and $q_0=m$ obtained by integrating Eq. (2). In the resonance region these curves are described by Eq. (24).

5. CROSS SECTION OF THE ANNIHILATION PROCESS FOR SLOW POSITRONS

We shall regard positrons with a momentum $p\ll m$ as slow. In that region we can obtain a rather simple formula for the cross section if we restrict ourselves to small momentum transfers to the nucleus $q\sim\eta$. The main contribution to the amplitude of the process will come from the diagram a of Fig. 1, but only if we take

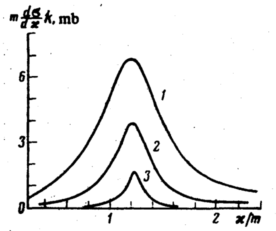

FIG. 5. Coulomb resonance in the two-photon annihilation of positrons with an energy of 300 keV. Curve 1 corresponds to $Z=47$, $k=1$. Curve 2 to $Z=26$, $k=3$. Curve 3 to $Z=13$, $k=10$. The smallest momentum transferred to the nucleus $q_0=m$; k is a scaling coefficient along the ordinate axis.

for the positron wavefunction the non-relativistic wavefunction instead of a plane wave. Diagram b of Fig. 1 which contains the correction to the free electron Green function is proportional to $(q^2+\eta^2)^{-1}$ and diagram a to the quantity $(q^2+\eta^2)^{-2}$. For small $q\sim\eta$ diagram b can thus be neglected (diagram c does not occur here; it is included in diagram a).

The differential cross section calculated with the non-relativistic wavefunction is given by the expression

$$d\sigma=4\pi^{-2}r_0^2m\eta^2\xi^2N(\varkappa)M(\varkappa,\,q)\,[1+(\mathbf{n}_2\mathbf{n}_1)^2]\,d\Gamma, \tag{25}$$

where

$$N(\varkappa)=\frac{2\pi\xi}{e^{2\pi\xi}-1}\exp\left[2\xi\,\mathrm{arctg}\frac{2p\eta}{\varkappa^2-p^2+\eta^2}\right]$$
$$M(\varkappa,q)=\frac{p^2+\eta^2}{a^2}\left[\frac{1}{a^2}+\frac{1}{|b|^2}-\frac{2(p^2+\eta^2)}{a|b|^2}\right]+\frac{2\varkappa^2(p^2-\eta^2)}{a^2|b|^2}$$
$$a=q^2+\eta^2,\quad b=\varkappa^2-(p+i\eta)^2,\quad \xi=\eta/p,\quad \mathbf{n}_i=\mathbf{k}_i/\omega_i.$$

Using for $d\Gamma$ Eq. (6a) in (25) and integrating over $d\omega_1$ we can obtain the non-relativistic expression for the DADTA, but the integral over ω_1 can only be obtained numerically. Taking $d\Gamma$ in the form (20) we can integrate Eq. (25) analytically over ψ, q, and ω_1. The integral over ψ and q equals

$$\frac{d\sigma}{d\varkappa\,d\omega_1}=r_0^2\frac{2^7}{3}\frac{m\eta^5\varkappa^3(p^2+\eta^2+3\varkappa^2)}{|b|^6}N(\varkappa)\,[1+(\mathbf{n}_2\mathbf{n}_1)^2],$$
$$|b|^2=[(\varkappa-p)^2+\eta^2][(\varkappa+p)^2+\eta^2],\quad \varkappa=|\mathbf{k}_1+\mathbf{k}_2|.$$

Only the cosine of the angle between the photon momenta depends on ω_1

$$\mathbf{n}_2\mathbf{n}_1=1-(4m^2-\varkappa^2)/2\omega_2\omega_1\approx1-2m^2/\omega_2\omega_1.$$

The integration over ω_1 from $m-\frac{1}{2}\varkappa$ to $m+\frac{1}{2}\varkappa$, using the identity of the photons, gives

$$d\sigma/d\varkappa=r_0^2\cdot2^7\cdot3^{-1}m\eta^5\varkappa^4(p^2+\eta^2+3\varkappa^2)N(\varkappa)/|b|^6.$$

6. SMALL ω_2 REGION

We consider the region $\omega_2\ll\omega_1$ and $\xi=\alpha Z\varepsilon/p\ll1$. It follows from the energy conservation law and the definition of $\varkappa$ and q that

$$\omega_1\approx\varepsilon+m,\quad \varkappa\approx\omega_1,\quad q\geqslant\omega_1-p>m.$$

The main diagrams in this region are diagrams a and b of Fig. 1 which contain the pole $a_2^{-1}\propto\omega_2^{-1}$. The cross section of the process summed over the polarizations of the particles is in the form of a product of the probability dW_γ of the bremsstrahlung by the long-wave photon and the cross section $d\sigma_{1\gamma}$ for the single-photon annihilation of a positron by a K electron:

$$d\sigma_{2\gamma}(\mathbf{n}_1,\,\mathbf{n}_2,\,\omega_2)=d\sigma_{1\gamma}(\mathbf{n}_1)\,dW_\gamma(\mathbf{n}_2,\,\omega_2),$$
$$d\sigma_{1\gamma}(\mathbf{n}_1)=r_0^2Z(2\eta)^4\frac{pm^2(\varepsilon+m)}{q^8}[1-(\mathbf{n}\mathbf{n}_1)^2]\left[\frac{(\varepsilon+2m)q^2}{4m^3}-1\right]d\Omega_1, \tag{26}$$
$$dW_\gamma(\mathbf{n}_2,\omega_2)=\frac{\alpha}{(2\pi)^2}\frac{p^2-(\mathbf{p}\mathbf{n}_2)^2}{(\varepsilon-\mathbf{p}\mathbf{n}_2)^2}d\Omega_2\frac{d\omega_2}{\omega_2},\quad \mathbf{n}=\frac{\mathbf{p}}{p},\quad \mathbf{n}_i=\frac{\mathbf{k}_i}{\omega_i}.$$

Integrating (26) over the angles of departure $d\Omega_2$ and $d\Omega_1$ we get the cross section for TPA with the emission of a soft photon in the frequency range $d\omega_2$:

$$d\sigma_{2\gamma}(\omega_2)=\sigma_{1\gamma}dW_\gamma(\omega_2), \tag{27}$$

where

$$\sigma_{1\gamma}=2\pi r_0^2\alpha^4Z^5\frac{m^2}{p(\varepsilon+m)^2}\left[\frac{\varepsilon^2}{m^2}+\frac{2}{3}\frac{\varepsilon}{m}+\frac{4}{3}-\frac{\varepsilon+2m}{p}\ln\frac{\varepsilon+p}{m}\right],$$
$$dW_\gamma(\omega_2)=\frac{2\alpha}{\pi}\frac{d\omega_2}{\omega_2}\left(\frac{\varepsilon}{p}\ln\frac{\varepsilon+p}{m}-1\right),$$

$\sigma_{1\gamma}$ is the total cross section for single photon annihilation by a K electron, $dW_\gamma(\omega_2)$ the probability for the emission of a soft photon in the frequency interval $d\omega_2$. Equations (26) and (27) are valid when the energy of one of the photons is small: $\delta\leqslant\omega_2\ll m$, where δ is the resolving power of the apparatus.

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1)The minus sign was omitted in Eq. (16) for L_2 in[6].

2)In Eqs. (23) and (24) of[6] the term $-2\xi\arctan(\mathbf{n}\mathbf{q}/\eta)$ in the round brackets was omitted, which arises from the infrared term (4a). This term was taken into account when the shift in the maximum of the line (36) was evaluated in the preprint.[6]

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