Transition radiation emitted by an ultrarelativistic particle crossing a curved interface between media

M. I. Ryazanov and I. S. Tilinin

Moscow Engineering Physics Institute
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It is shown that the transition radiation emitted by ultrarelativistic particles depends on the curvature of the interface crossed by the charge. The effect of focusing of transition radiation of a charge crossing a conducting surface in the shape of a paraboloid of revolution is investigated. The intensity of the transition radiation is found at the focus of the paraboloid and at large distances. The possibility of using the focusing of transition radiation to detect ultrarelativistic particles is discussed.

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1. INTRODUCTION

Transition radiation emitted by a uniformly moving charge crossing the interface between two media was predicted in 1946 by Ginzburg and Frank[1] and since then this phenomenon has been investigated in detail by many authors.[2–10] However, the investigation was restricted to plane interfaces between the two media. Only Askar'yan[13] has investigated the transition radiation emitted by a nonrelativistic charge crossing a spherical surface. Meanwhile, an uncommon dependence of the transition radiation on the curvature of the interface arises for ultrarelativistic particles, as can be seen from the following considerations. Transition radiation can be regarded as the appearance of a reflected wave and a refracted wave when the self-field of a uniformly moving charge passes through the interface. As is well known, the self-field of an ultrarelativistic charge with energy $\varepsilon \gg m$ ($c=1$) is concentrated in a thin disk-shaped region with small longitudinal and macroscopically large transverse dimensions (at a frequency ω, the transverse size of the field region is $\sim \varepsilon/m\omega$). If the radius of curvature of the interface $f \sim (\varepsilon/m\omega)$, the self-field of the charge intersects different points of the interface between the two media at different moments of time and, consequently, the transition radiation from different points of the interface arrives at the observation point with different phases. Taking into account the possibility of using transition radiation for the detection of ultrarelativistic particles, one can use the interference of the waves at the observation point for a maximal amplification of the signal by an appropriate choice for the shape of the interface. In order to amplify the signal it is also advisable to choose a perfectly reflecting interface, i.e., the interface between the vacuum and a perfect conductor.

The optimal shape of the interface is determined from the condition that the transition radiation from every point on the surface arrive at the observation point with the same phase. If the difference between the velocity of the charge and that of light is neglected, the surface of a paraboloid of revolution having its axis along the particle velocity and its focus at the point where the detector is located will satisfy these conditions. In fact, in this case all phase relations for the transition radiation remain the same as for the reflected waves produced by a short pulse of light incident in the same direction. It should be noted that it follows from the conditions $f \sim (\varepsilon/m\omega)$ and $\varepsilon \gg m$ that the wavelength λ of the radiation is small in comparison with the focal distance f and the parabolic reflecting surface ensures focusing of the radiation.

2. THE CURRENT DENSITY ON THE INTERFACE

The physical cause of the onset of transition radiation is the current density produced in the surface layer of matter by the charge's self-fields $\mathbf{E}_0(\mathbf{r}, \omega)$ and $\mathbf{H}_0(\mathbf{r}, \omega)$.

In the case of a perfect conductor this current is only generated in a thin skin layer at the interface and can be regarded as a purely surface current. In this case for the determination of the surface current density it is convenient to use the boundary conditions at the interface between vacuum and a perfect conductor

$$[\mathbf{n}(\mathbf{r}) \times \mathbf{H}(\mathbf{r}, \omega)] = 4\pi \mathbf{j}(\mathbf{r}, \omega), \tag{2.1}$$

where $\mathbf{n}(\mathbf{r})$ denotes the unit vector normal to the interface at the point $\mathbf{r}$, and $\mathbf{H}(\mathbf{r}, \omega)$ denotes the total field in vacuum near the point $\mathbf{r}$, consisting of the charge's self-field $\mathbf{H}_0(\mathbf{r}, \omega)$ and the field of the transition radiation $\mathbf{H}_1(\mathbf{r}, \omega)$. Following Fock,[11] we eliminate the field $\mathbf{H}_1(\mathbf{r}, \omega)$ from Eq. (2.1), and express the current density in terms of only the charge's self-field $\mathbf{H}_0(\mathbf{r}, \omega)$. This yields the following integral equation for the current density

$$\mathbf{j}(\mathbf{r}, \omega) = \frac{1}{2\pi}[\mathbf{n}(\mathbf{r}) \times \mathbf{H}_0(\mathbf{r}, \omega)] - \frac{1}{2\pi}\int dS \left[\mathbf{n}(\mathbf{r}) \times \left[\mathbf{j}'(\mathbf{r}', \omega) \times \nabla \frac{\exp(i\omega|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\right]\right], \tag{2.2}$$

where the integration is over the interface on which the points $\mathbf{r}$ and $\mathbf{r}'$ are located. In the usual theory of diffraction by convex surfaces, where Eq. (2.2) is frequently used, the integral term in (2.2) is small in the case $f\omega \gg 1$, where f is the characteristic radius of curvature, and a rather good approximation is given by the well known equation

$$\mathbf{j}(\mathbf{r}, \omega) = \frac{1}{2\pi}[\mathbf{n}(\mathbf{r}) \times \mathbf{H}_0(\mathbf{r}, \omega)] = 2\mathbf{j}_{ex}(\mathbf{r}, \omega) \tag{2.3}$$

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(the surface current is simply equal to twice the external current[11,12]). This corresponds to solution of the integral equation (2.2) by the method of iterations. The validity of the first approximation is essentially due to the fact that diffraction by a convex surface is caused by single scattering.

When applying (2.2) to the reflection of a plane wave from a concave paraboloid, on which a plane wave is incident parallel to its axis, it must be borne in mind that in this case each light ray is reflected from the surface of the paraboloid, passes through the focus and is reflected a second time from the paraboloid. One can say the same about the transition radiation of an ultrarelativistic particle moving parallel to the axis of a paraboloid of revolution ($z=-\rho^2/4f+f$) in the positive direction of the z axis. The first reflection of the self-field leads to the formation of a transverse wave. This wave passes through the focus, is reflected a second time, and emerges from the paraboloid.

It follows from this that the first approximation (2.3) is insufficient for the solution of the problem of a concave paraboloid, and it is necessary to take the second approximation into account, replacing (2.3) by

$$\mathbf{j}(\mathbf{r},\omega)=\frac{1}{2\pi}[\mathbf{n}(\mathbf{r})\mathbf{H}_0(\mathbf{r},\omega)]$$
$$-\frac{1}{(2\pi)^2}\int dS\left[\mathbf{n}(\mathbf{r})\left[[\mathbf{n}(\mathbf{r}')\mathbf{H}_0(\mathbf{r}',\omega)]\nabla\frac{\exp(i\omega|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\right]\right]. \quad (2.4)$$

For a paraboloid of revolution having a symmetry axis z along the particle velocity and a focus at the point $z=0$, it is convenient to evaluate the integral and to estimate the errors of the approximation in the parabolic coordinates ξ, η, and φ:

$$x=\rho\cos\varphi,\quad y=\rho\sin\varphi,\quad z=\xi-\eta,\quad (\rho^2=4\xi\eta).$$

In these coordinates Eq. (2.2) takes on the surface $\xi=f$ the form

$$j_\eta(\eta,\varphi)=\frac{1}{2\pi}H_0(\eta,\varphi)$$
$$+\frac{f}{\pi}\iint d\eta'\,d\varphi'\,j_\eta(\eta',\varphi')\left(\frac{f}{f+\eta}\right)^{1/2}e^{i\omega R}F(\eta',\varphi'), \quad (2.5)$$

where

$$R^2=(\eta-\eta')^2+4f[\eta+\eta'-2(\eta\eta')^{1/2}\cos(\varphi-\varphi')],$$
$$F(\eta',\varphi')=\frac{1-i\omega R}{R^3}\{\eta+\eta'-2(\eta\eta')^{1/2}\cos(\varphi-\varphi')\}.$$

Solving Eq. (2.5) by successive approximations, we put

$$j_\eta=j_\eta^{(1)}+j_\eta^{(2)}+j_\eta^{(3)},$$

where

$$j_\eta^{(1)}(\eta,\varphi)=\frac{1}{2\pi}H_0(\eta,\varphi), \quad (2.3')$$

$$j_\eta^{(2)}(\eta,\varphi)=\frac{f}{\pi}\int d\eta'\int d\varphi' H_0(\eta',\varphi')\left(\frac{f}{f+\eta}\right)^{1/2}F(\eta',\varphi')e^{i\omega R}. \quad (2.6)$$

One can use the method of stationary phase[10-12] to estimate this integral. It is not difficult to find that the phase will be stationary at the point $\varphi'=\varphi+\pi$, $\eta'=f^2/\eta$. Hence we obtain

$$j_\eta^{(2)}(\eta,\varphi)=\frac{i}{4\pi^2}\left(\frac{1}{\eta(\eta+f)}\right)^{1/2}\left(2f\left(\frac{f}{\eta}\right)^{1/2}\frac{m\omega}{\varepsilon v}\right)$$
$$\times K_1\left(2f\left(\frac{f}{\eta}\right)^{1/2}\frac{m\omega}{\varepsilon v}\right)\exp(3i\omega f+i\omega\eta) \quad (2.7)$$

(K_1 denotes the Macdonald function). The following restriction on the dimensions of the charge's self-field follows from the requirements for applicability of the stationary phase method:

$$(\omega f)^{1/2}\gg\varepsilon/m\gg(\omega f)^{1/4}. \quad (2.8)$$

Substituting (2.6) into the expression for $j_\eta^{(3)}$, we have

$$j_\eta^{(3)}(\eta,\varphi)=\frac{f}{\pi}\int d\eta'\int d\varphi' j_\eta^{(2)}(\eta',\varphi')e^{i\omega R}F(\eta',\varphi'), \quad (2.9)$$

from which it is not difficult to obtain the result that, in order of magnitude,

$$j_\eta^{(3)}=-\frac{1}{8i\omega(f\eta)^{1/2}}\left(\frac{f}{f+\eta}\right)^{1/2}j_\eta^{(2)}, \quad (2.10)$$

so that $j_\eta^{(3)}\ll j_\eta^{(2)}$ for $\lambda\ll f$ and one can neglect the quantity $j_\eta^{(3)}$. Thus, for $f\omega\gg1$ the total surface current is given by the sum of expressions (2.3) and (2.6). The electromagnetic field of the transition radiation may now be found with the aid of the usual formulas for the retarded potentials.

It should be noted that the obtained expressions (2.3′), (2.7), and (2.10) very much resemble the analogous relations in the case of diffraction of a plane wave by convex bodies at $\omega a\gg1$, where a is the characteristic radius of curvature.[12] This fact is quite understandable if it is taken into consideration that at $v\sim c$ the field of an ultrarelativistic particle is nearly transverse or, in other words, the combination of the particle's magnetic transverse magnetic and electric fields is almost equivalent to a wave packet of linearly polarized radiation. Since the approximation of geometrical optics is valid for $\omega a\gg1$, it is not difficult to see that the magnitude of the field at the focal point is completely determined by the surface current $j_\eta^{(1)}$ whereas the spectral and angular distribution of the transition radiation at infinity is determined solely by the current function $j_\eta^{(2)}$. It is also necessary to note that expression (2.7) takes into consideration the influence of the region $\eta\lesssim\lambda$ under the assumed restriction (2.8) on the dimensions of the particle's field.

3. THE FIELD OF THE TRANSITION RADIATION NEAR THE FOCUS OF A PARABOLOID

We consider first the case of particle motion along the axis of the paraboloid. To evaluate the transition radiation field at the focus we shall utilize the expression

$$\mathbf{E}(\mathbf{r},\omega)=\frac{i}{\omega}\int\{\nabla(\nabla)+\omega^2\}\mathbf{j}\frac{e^{i\omega R}}{R}d\mathbf{r}'. \quad (3.1)$$

Since it follows from the symmetry of the problem that only the component E_z, obviously directed against the particle's motion, differs from zero on the axis of the paraboloid, to within terms of order $(\omega f)^{-1}$ we obtain the following expression from Eq. (3.1):

$$E_z(0,\omega)=-4i\omega f\int\frac{\eta^{1/2}f}{(f+\eta)^{5/2}}j_\eta^{(1)}(\eta,\varphi)e^{i\omega f+i\omega\eta}d\eta\,d\varphi. \quad (3.2)$$

In Eq. (3.2) the current density $j_\eta^{(1)}$ is determined by Eq. (2.3′) and $f+\eta=R$ denotes the distance from the focal point to the point of integration on the surface of the paraboloid. We note that mathematically the focusing of the transition radiation is associated with the fact that the current density $j_\eta^{(1)}$ contains the rapidly oscillating factor $\exp\{(i\omega/v)(f-\eta)\}$, which cancels the corresponding factor in the integrand of expression (3.2); therefore, the value of the integral (3.1) at the focus is considerably larger than its value at any other point of space inside the paraboloid.

It is evident from Eq. (3.2) that for $\varepsilon/m\omega \gtrsim f$ the value of E_z is almost independent of the particle energy ε because in this case the value of the integral (3.2) is determined by the region $\eta\lesssim f$. However in the case $\varepsilon/m < \omega f$, the integral (3.2) is obviously determined by the region $\eta\lesssim\varepsilon/m\omega$ and for E_z one can obtain

$$E_z=-\frac{4ie}{\pi v}\left(\frac{\varepsilon}{m}\right)^2\frac{1}{f^2\omega}e^{2i\omega f}. \quad (3.3)$$

Now let the vacuum-conductor interface be half of a paraboloid that is cut along its symmetry plane $x=0$. As before the trajectory of the particle coincides with the paraboloid's axis of revolution. It is obvious that, just as before, the current density $j_\eta^{(1)}$ will be given by expression (2.3′). As to the current density $j_\eta^{(2)}$, it will be small in comparison with $j_\eta^{(1)}$ since there is no second scattering of the transition-radiation field. It follows from the symmetry of the problem that only two nonvanishing components of the field exist at the focus: E_z, which is equal to half of the corresponding expression (3.3) for the complete paraboloid, and E_x, which is given by

$$E_x(0,\omega)=i\omega f\int_0^\infty\int_{-\pi/2}^{\pi/2}\frac{(f-\eta)}{(f+\eta)^{5/2}}f^{1/2}j_\eta^{(1)}\exp\{i\omega f+i\omega\eta\}\cos\varphi\,d\eta\,d\varphi. \quad (3.4)$$

Confining ourselves to the most interesting case $\varepsilon/m < \omega f$, we obtain

$$E_x=\frac{ie}{2\pi vf}\frac{\varepsilon}{m}e^{2i\omega f}. \quad (3.5)$$

At $\omega f\gg 1$ one can neglect the influence of boundary effects on the value of the field near the paraboloid's focus in the first approximation.

Now let us consider the more general case of off-center entry of the particle into the paraboloid. Here we shall assume that the particle's trajectory does not coincide with the paraboloid's axis of revolution, but remains parallel to it and is displaced from it by a distance d. The particle field $\mathbf{H}_0$ will excite two nonvanishing components of the surface current:

$$j_\eta^{(1)}=\frac{1}{2\pi}H_0\cos(\psi-\varphi),\quad j_\varphi^{(1)}=\frac{1}{2\pi}H_0\sin(\psi-\varphi)\left(\frac{f}{\eta+f}\right)^{1/2}, \quad (3.6)$$

where the angle ψ is defined in the following way:

$$\cos\psi=\frac{\eta^{1/2}\cos\varphi-\delta^{1/2}}{[\delta+\eta-2(\eta\delta)^{1/2}\cos\varphi]^{1/2}}$$

and fulfillment of the condition $\psi=\varphi$ is required at $\delta=0$; here $d\equiv 2(f\delta)^{1/2}$. The corresponding expressions for the surface current $\mathbf{j}^{(2)}$ can be easily obtained from Eqs. (3.6) in analogy with the procedure of Sec. 2.

As a consequence of the symmetry of the problem the field at the focus is determined, to within terms of order $(\omega f)^{-1}$ by the component of the current $j_\eta^{(1)}$. To avoid cumbersome calculations, let us consider the simplest but nevertheless important case: $\varepsilon/m\ll\omega d$ and $\varepsilon/m\ll\omega f$. Here it is obvious that the surface current $j_\eta^{(1)}$ will be concentrated near the point (f,δ) on the surface of the paraboloid in a region of dimensions $\sim\varepsilon/m\omega$. This segment of the surface is seen from the focus at an angle θ determined by the relationship $\cos\vartheta=(f-\delta)/(f+\delta)$; therefore the waves arriving at the focus have wave vectors concentrated about the value

$$\mathbf{k}=\left\{\frac{\mathbf{x}}{x}\frac{2(f\delta)^{1/2}}{f+\delta};\ \frac{\mathbf{v}}{v}\frac{f-\delta}{f+\delta}\right\}\omega. \quad (3.7)$$

Calculations using Eqs. (3.6) give the following results for the components of the field $\mathbf{E}$ at the focus

$$E_z=-\frac{2ie}{vf^2T^5}(f\delta)^{1/2}\frac{\varepsilon}{m}e^{2i\omega f},$$
$$E_x=\frac{ie}{vf^2T^5}(f-\delta)\frac{\varepsilon}{m}e^{2i\omega f}, \quad (3.8)$$

where $T=(1+\delta/f)^{1/2}$. From Eqs. (3.7) and (3.8) it is evident that $\mathbf{E}\cdot\mathbf{k}=0$, that is, the field is transverse at the focus.

4. SPECTRAL AND ANGULAR DISTRIBUTION OF THE TRANSITION RADIATION

At large distances from the paraboloid and for central entry of the particle, the distribution of the radiation can be easily obtained from Eq. (2.7) with allowance for the bounded nature of the region of existence of the current $j_\eta^{(2)}$. At large distances the vector potential is given by the expression

$$A_\eta=\frac{e^{i\omega R}}{R}\int j_\eta^{(2}(\eta',\varphi')\exp\{-i\omega\cos\vartheta(f-\eta')$$
$$-i\omega\sin\vartheta\cdot 2(f\eta')^{1/2}\cos(\varphi-\varphi')\}\cdot 2f\,d\eta'\,d\varphi', \quad (4.1)$$
$$\mathbf{R}/R=\{\cos\varphi\sin\vartheta;\sin\varphi\sin\vartheta;\cos\vartheta\}.$$

The rapidly oscillating integrand in (4.1) has a stationary-phase point at $\varphi'=\varphi$ and $\eta'=f\tan^2(\vartheta/2)$; hence for A_η we have

$$A_\eta=\frac{e}{2\pi vR\omega}\left(\frac{2m\omega f}{\varepsilon v}\operatorname{ctg}\frac{\vartheta}{2}\right)K_1\left(\frac{2m\omega f}{\varepsilon v}\operatorname{ctg}\frac{\vartheta}{2}\right)\frac{\exp\{i\omega R+2i\omega f\}}{\sin(\vartheta/2)}. \quad (4.2)$$

Consequently, we obtain the following result for the energy emitted per element $d\Omega$ of solid angle in the frequency interval $d\omega$:

$$\frac{d\mathscr{E}}{d\Omega\,d\omega}=\frac{e^2}{4\pi^2v^2}\left(\frac{2m\omega f}{\varepsilon v}\operatorname{ctg}\frac{\vartheta}{2}\right)^2K_1^2\left(\frac{2m\omega f}{\varepsilon v}\operatorname{ctg}\frac{\vartheta}{2}\right)\left(\sin\frac{\vartheta}{2}\right)^{-2}. \tag{4.3}$$

Here it is obvious that $\pi/2\leqslant\vartheta\leqslant\pi$ because the radiation originates in the left half space. In the case of small radius of the field, $\varepsilon/m\omega\ll f$, approximate integration of (4.3) over the angles gives

$$\frac{d\mathscr{E}}{d\omega}\sim\frac{e^2}{\pi}\left(\frac{\varepsilon}{m\omega f}\right)^2, \tag{4.4}$$

which is found to be in agreement with Eq. (3.3).

Direct conversion from the derived formulas to the case of a planar boundary cannot be achieved simply by the limiting transition $f\to\infty$. The difference between a paraboloid and a plane is that the beam is reflected twice from the surface of a paraboloid, but only once from a plane. As $f\to\infty$ the second reflection point moves to infinity. For the correct conversion to the planar case, it is necessary to use the first-order approximation (2.3) for the current density and find the field created by it. The result agrees with the results obtained in[1-4].

5. DISCUSSION OF THE RESULTS

The strong dependence of the transition radiation at the focus of a paraboloid on the particle's energy allows one to utilize this effect for the detection of high-energy particles. An important circumstance here is that this dependence arises at all frequencies ranging from the radio to the optical band. The optimal size of the paraboloid obviously corresponds to the case when the transverse radius of the self-field is comparable with the focal distance, $\varepsilon/m\omega\sim f$. In this connection it is obvious that it is sufficient to use half of a truncated paraboloid for measurement of the field at the focus (the paraboloid truncated in the focal plane is next cut along a plane passing through the symmetry axis). In the case $\varepsilon/m\omega\lesssim f$ such a surface will act as half of an infinite paraboloid, and formula (3.5) will be valid for the field at the focus. An amount of energy

$$d\mathscr{E}\sim\frac{e^2}{f^2}\left(\frac{\varepsilon}{m}\right)^2\frac{d\omega}{\omega^2} \tag{5.1}$$

passes through a detector of area ω^{-2} near the focus in the frequency interval $d\omega$. Let the band of frequencies measured by the detector lie between ω_1 and ω_2. Then the total energy registered by the detector is

$$\mathscr{E}\sim e^2\left(\frac{\varepsilon}{m}\right)^2\frac{1}{f^2}\left(\frac{1}{\omega_1}-\frac{1}{\omega_2}\right). \tag{5.2}$$

For radiation from a beam of particles or relativistic nuclei with large Z, the radiation intensity will be sufficient to be detected even in the optical region. It is interesting to note that interference between the transition radiation at optical frequency, due to passage of an electron beam through two planar surfaces, has already recently been observed.[10] However, to detect the transition radiation from a single proton it is more convenient to use radio frequencies. Here, however, it is necessary to take into account the increase in the geometrical dimensions of the paraboloid with increasing ε/m. Thus, for $\varepsilon/m\sim10^3$ and centimeter waves, the dimensions of the paraboloid should reach tens of meters.

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