terms of these last by means of the integral relation

$$\varphi_3 = \frac{t}{t}\varphi_3(u, v^*) - 2it\int_{v^*}^{v} \frac{[t^2(\bar{\sigma}\bar{\lambda}-\sigma\lambda)]}{t^2(u,y)}dy, \tag{A.8}$$

where v^* is a root of the equation $t(u, v^*) = t_0$. The integral operator in (A.8) is bounded, and therefore the function $\varphi_3 = 2t\varkappa$ is bounded. Thus, taking into account the boundedness of the functions $t\sigma$ and $t\lambda$ and assuming them continuous right up to $t=0$, we obtain the asymptotic estimates

$$\sigma \sim \alpha(x)t^{-1} + o(t^{-1}), \quad \lambda \sim \beta(x)t^{-1} + o(t^{-1}), \tag{A.9}$$

where $\alpha(x)$ and $\beta(x)$ are certain continuous functions that do not in general vanish identically. Between $\alpha(x)$ and $\beta(x)$ there is the relationship

$$\beta(x) = \bar{\alpha}(x)f(x) \tag{A.10}$$

($f(x)$ is given by Eq. (26)), which can be obtained by assuming for simplicity

$$\frac{\partial\sigma}{\partial t} = -\alpha(x)t^{-2} + o(t^{-2}) \tag{A.11}$$

and substituting (A.9) and (A.11) into Eq. (8) for σ.

One need not assume (A.11) but instead obtain (A.10) by using the integral relationship between λ and $\bar{\sigma}$:

$$\lambda(u,v) = \lambda_0(u)t^{-1/2}\exp\left(\int_{u_1}^{u}\mu_0(\xi)d\xi\right) + |\mu_0(u)|t^{-1/2}\exp\left(2\int_{u_1}^{u}\mu_0(\xi)d\xi\right)\int_{v_1}^{v}\bar{\sigma}(u,\xi)t^{-1/2}(u,\xi)d\xi.$$

Finally, if (A.9) is substituted with allowance for (A.10) into Eq. (A.8), we obtain for $\varkappa$ the asymptotic behavior

$$\varkappa \sim o(t^{-1}). \tag{A.12}$$

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Calculation of the Gell-Mann–Low function in scalar theory with strong nonlinearity

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The Gell-Mann–Low function in a scalar logarithmic theory with a polynomial interaction is found in all orders of perturbation theory in the limit of strong nonlinearity of the interaction. The existence of ultraviolet-stable points in the theory is proved.

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1. INTRODUCTION

It is well known that, in quantum electrodynamics and in most of the models of quantum field theory known until recently, the physical phenomenon of screening of the interaction as a result of quantum fluctuations of the vacuum occurs and has the result that the physical charges vanish if the bare charges are sufficiently small.[1] At the same time, in many currently popular nonabelian gauge models the opposite situation obtains, viz., for sufficiently small physical charges the bare charge vanishes.[2] These results were obtained by means of perturbation theory and therefore have a limited region of applicability. To investigate the consistency of quantum electrodynamics and the other traditional models of quantum field theory, and also to study the question of the infrared catastrophe in Yang–Mills theory, it is necessary to go beyond the framework of perturbation theory. The use of the renormalization-group apparatus proposed by Gell-Mann and Low (GML)[3] is of great benefit here. Thus, e.g., the question of the consistency of quantum electrodynamics reduces to the calculation, in all orders of perturbation theory, of the so-called GML function $\psi(\alpha)$, which, by definition is equal to

$$\psi(\alpha(k^2)) = \frac{d\alpha(k^2)}{d\ln(k^2/m^2)}, \quad \alpha(k^2) \equiv \alpha_c d_c(k^2/m^2), \tag{1}$$

where $\alpha_c = e_c^2/4\pi\hbar c$. The asymptotic form of the renormalized photon Green function is related to $d_c(k^2/m^2)$ by the formula

$$D_c^{\mu\nu}(k^2)|_{k^2\gg m^2} = -\frac{g_{\mu\nu}}{k^2}d_c(k^2).$$

As is well known, there are three possibilities, leading to different physical consequences: 1) the GML function (1) has a zero: $\psi(\alpha_0) = 0$. Then, when the bare charge is chosen equal to α_0, the renormalized charge does not vanish and the asymptotic form of the Green function $D_c^{\mu\nu}(k^2)$ is proportional to the free Green function:

$$D_c^{\mu\nu}(k^2)\,|_{k^2\to\infty} = -g_{\mu\nu}\alpha_0/k^2;$$

2) $\psi(\alpha)$ does not have a zero at any finite distance, but $\psi(\alpha\to\infty) < C\alpha\ln\alpha$. In this case the theory is consistent only when the bare charge is chosen equal to infinity. In this case the renormalized Green function $D_c^{\mu\nu}(k^2)$ falls off at large momenta more slowly than the free Green function; 3) $\psi(\alpha)$ does not have a zero at any finite distance and increases faster than $\alpha\ln\alpha$. Such properties of the GML function would signify an internal contradiction in quantum electrodynamics.[1]

An analogous situation obtains in scalar theories and in spinor models with scalar and pseudoscalar coupling.[4] Moreover, even for the Yang–Mills model, in the attempt to give the vector fields mass by means of the Higgs mechanism we again come up against the necessity of calculating the GML function in all orders of perturbation theory in order to validate the consistency of such a scheme.[5] Knowledge of the GML function would also be of great interest for the theory of second-order phase transitions.[6]

In the present paper we make an attempt to calculate the GML function in all orders of perturbation theory in the coupling constant, in the limit of strong nonlinearity of the interaction.

2. THE MODEL

The problem of determining the GML function in the theory with interaction Hamiltonian $H_{\rm int} = g:\varphi^4:/4!$ in four-dimensional space in all orders of perturbation theory encounters considerable calculational difficulties. At the present time only the first three terms of its series expansion are known.[7] One would like to generalize this model in such a way that a free parameter would appear in the theory but the theory would not lose the property of renormalizability. The simplest way of generalizing this model is to consider the theory with the interaction Hamiltonian

$$H_{int} = g:\varphi^n(x):/n!$$
$$\equiv g\exp\left(-\frac{1}{2}\int d^Dx_1\,d^Dx_2\,\Delta_0(x_1-x_2)\frac{\delta}{\delta\varphi(x_1)}\frac{\delta}{\delta\varphi(x_2)}\right)\varphi^n(x)/n! \qquad (2)$$

in a space-time with

$$D = 2n/(n-2) \qquad (3)$$

dimensions.

We shall consider the Euclidean variant of the theory, i.e., a Wick rotation will be assumed in all the Feynman diagrams.[8] In this case the Green function $\Delta_0(x_1 - x_2)$ has the following form:

$$\Delta_0(x_1-x_2) = \int\frac{d^Dp}{(2\pi)^D}\frac{\exp[ip(x_1-x_2)]}{m^2+p^2}. \qquad (4)$$

The choice of a space with the dimensionality (3) ensures that the coupling constant g is dimensionless and, as a consequence of this, that logarithmic divergences appear in the calculation of the physical charge as a function of the bare charge. Below, to ensure stability of the theory, we shall consider only even n:

$$n = 4, 6, 8, \ldots; \quad g > 0. \qquad (5)$$

The case $n\to 2$ does not correspond, generally speaking, to the free theory (since, according to formula (3), $D\to\infty$), but is not considered below. In the case $n = -2$ we would obtain $D = 1$, i.e., a soluble quantum-mechanical model.[9] The familiar scalar theories with $n = 4$, $D = 4$ and $n = 6$, $D = 3$ belong to the class (5). The latter theory is of interest in the problem of tricritical points.[10]

A cursory glance at the set of theories specified by the conditions (5) prompts the thought that the degree of nonlinearity of the interaction (n) might serve as the required parameter that could be made to tend to infinity, thereby obtaining a soluble model. Indeed, according to formula (3), as $n\to\infty$ the dimensionality of space tends to $D = 2$, and it is well known that in two-dimensional space certain field-theory models are soluble.[11] Such an approximation might be good even for the case of $g:\varphi^4:/4!$ theory, in which $n = 4$. In this paper we shall calculate the GML function in the limit

$$n\to\infty, \quad D\to 2 \qquad (6)$$

in all orders of perturbation theory.

In going over to the theory (6) there is a certain difficulty, associated with the necessity of introducing into the interaction Hamiltonian (2) extra counterterms in addition to those which appear in the $g:\varphi^4:/4!$ theory. Indeed, for large n there are power divergences in all Green functions with $r \le n-2$ legs, to cancel which we must add an arbitrary polynomial of degree $n-2$ in φ to the expression (2). Adding these counterterms does not lead, however, to a dependence of the GML function on the newly introduced interaction constants, since these constants are dimensional and, therefore, do not lead to logarithmically divergent diagrams in the calculation of the physical charge in terms of the bare charge. Below, therefore, we shall ignore these additions to the Hamiltonian (2).

3. SELECTING THE IMPORTANT DIAGRAMS

For the calculation of the GML function it is necessary to introduce the concept of the invariant charge. By analogy with the case of $g\varphi^4/4!$ theory[7] we introduce the following definition of the invariant charge:

$$g(p^2) \equiv g_\mu\Gamma_c(p^2/\mu^2)\,d_c^{n/2}(p^2/\mu^2), \qquad (7)$$

where g_μ is the charge at the normalization point μ:

$$\Gamma_c(1) = d_c(1) = 1, \qquad (8)$$

and $\Gamma_c(p^2/\mu^2, g_\mu)$ is the amputated n-point vertex function. The external momenta p_i are Euclidean and satisfy the relations[1)]

$$\sum p_i = 0, \quad p_i^2 = p^2, \quad p_ip_j = -p^2/(n-1), \quad i\ne j; \qquad (9)$$

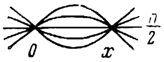

FIG. 1.

$\Delta_c = k^{-2} d_c(k^2/\mu^2)$ is the Green function of the scalar particle.

The parameter μ is chosen to be much greater than the particle mass m, so that in all the Feynman integrals only the asymptotic form of the Green function (4) at short distances will be important:

$$\Delta_0(x_1-x_2)\,|_{x_1\to x_2} = a(n)\,|x_1-x_2|^{-4/(n-2)},$$
$$a(n) = {}^1/_4 \pi^{-n/(n-2)} \Gamma(2/(n-2)), \tag{10}$$

where we have used the relation (3). For $n\to\infty$ the coefficient $a(n)$ is equal to

$$a(n)\,|_{n\to\infty} = \frac{n}{8\pi} - \frac{1}{4\pi}(1+\ln\pi + c_E) + O\left(\frac{1}{n}\right). \tag{11}$$

Here $c_E \approx 0.577$ is Euler's constant.[12]

The Gell-Mann–Low equations in the theory under consideration are written in the form

$$dg(p^2)/d\ln(p^2/\mu^2) = \psi(g(p^2)). \tag{12}$$

We shall consider the lowest orders of the GML function. For this, according to the definition (7), we must find the functions Γ and d. In the first order after the Born approximation, as in $g\varphi^4/4!$ theory too, only the correction to the vertex part (see Fig. 1) gives a contribution to the invariant charge. Taking into account the combinatoric factor, we obtain

$$g(p^2) = \sum_{k=1}^{\infty} g_\Lambda^k B_k\left(\frac{\Lambda}{p^2}\right), \quad B_1 = 1,$$
$$B_2 = -\frac{1}{2}\frac{n!}{\{(n/2)!\}^3} \int_{x^2>1/\Lambda} \frac{d^D x}{x^D} \{a(n)\}^{n/2} \exp\left[\frac{1}{2} ixn \frac{p}{(n-1)^{1/2}}\right]. \tag{13a}$$

Carrying out the renormalization in accordance with the rules (7) and (8) and substituting, for large n, the expression (11), we obtain for $\psi(g)$ in lowest order:

$$g(p^2) = g_\mu - g_\mu^2 \ln\frac{\mu^2}{p^2}\left(\frac{e}{\pi}\right)^{n/2} \frac{\sqrt{2}}{2\pi n} \exp\{-(1+c_E)\} + O(g_\mu^3),$$
$$\psi(g) \approx \left(\frac{e}{\pi}\right)^{n/2} \frac{\sqrt{2}}{2\pi n} \exp\{-(1+c_E)\} g^2. \tag{13b}$$

In the next order of perturbation theory it is necessary to take into account the renormalization of the Green functions in accordance with formula (7) (see Fig. 2):

$$\Delta(p^2) = p^{-2}[1-\Sigma(p^2)]^{-1},$$
$$\Sigma(p^2) = -\frac{g_\Lambda^2}{(n-1)!}[a(n)]^{n-1} \frac{1}{2D} \int_{p^{-2}\gg x^2\gg\Lambda^{-1}} \frac{d^D x}{x^D}\Bigg|_{\substack{n\to\infty\\ D\to 2}}$$
$$\approx 2\left(\frac{e}{8\pi}\right)^n \frac{\exp\{-2(1+c_E)\}}{(2\pi n)^{1/2}} g_\Lambda^2 \ln\frac{p^2}{\Lambda}. \tag{14a}$$

FIG. 2.

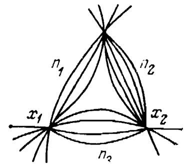

FIG. 3.

However, we shall compare the coefficient of the logarithm in (14a), multiplied by $n/2$, (this, according to formulas (7) and (12), is the lowest-order contribution of the Green function to $\psi(g)$) with the coefficient of the logarithm in the diagrams of Fig. 3 for the vertex parts:

$$\Gamma_2\left(\frac{p^2}{\Lambda}\right) = g_\Lambda^2 \sum_{n_1 n_2 n_3} \frac{1}{3!} \frac{n![a(n)]^n}{(n_1!)^2 (n_2!)^2 (n_3!)^2}$$
$$\times \int_{x_1^2, x_2^2 > \Lambda^{-1}} d^D x_1\, d^D x_2\, |x_1|^{-4_1/(n-2)} |x_2|^{-4n_2/(n-2)} |x_1-x_2|^{-4n_3/(n-2)}$$
$$\times \exp\left[ipx_1\left(\frac{n_2(n-n_2)}{n-1}\right)^{1/2}\right] \exp\left[ip'x_2\left(\frac{n_1(n-n_1)}{n-1}\right)^{1/2}\right], \tag{14b}$$
$$(p')^2 = p^2, \quad p'p = -p^2[n_1 n_2/(n_1+n_3)(n_2+n_3)]^{1/2},$$
$$\Gamma_2\left(\frac{p^2}{\Lambda}\right) \approx g_\Lambda^2 \frac{1}{3!}\left(\frac{9e}{8\pi}\right)^n \left(\frac{4\pi n}{3}\right)^{-3/2} \exp[-2(1+\ln\pi + c_E)]$$
$$\times 2^{1/2} \int_{\Lambda^{-1}\ll |x_i|^2 \ll p^{-2}} \frac{d^2 x_1\, d^2 x_2}{|x_1|^{4/3} |x_2|^{4/3} |x_1-x_2|^{4/3}},$$

where in the calculation of the asymptotic form for large n we have made use of the existence of the saddle point at $\tilde{n}_1 = \tilde{n}_2 = \tilde{n}_3 = n/3$ in the sum over the n_i, about which the important values are $n_i - \tilde{n}_i \sim n^{1/2}$, which enables us to replace the summation by an integration and also to take the integral over x_1 and x_2 at the saddle point $n_i = n/3$ outside the sum. Comparing (14a) and (14b) we see that the contribution of the renormalizations of the Green function to the invariant charge is exponentially small ($\sim 9^{-n}$) compared with the contribution of the renormalizations of the vertex function and, therefore, can be discarded in the limit $n\to\infty$. The situation remains analogous in higher orders of perturbation theory also. Thus, e.g., in the next order the main contribution is given by the sum of vertex diagrams of Fig. 4 with $n_{ij} \approx n/4$. In the calculation of the sum over the n_{ij} it is convenient to introduce a new, $(D+1)$-st coordinate l_i at each vertex, in order to write the condition $n = \sum_j n_{ij} + n_i$ in the form of the integral

$$\int \frac{dl_i}{2\pi i} \exp\left\{l_i\left(n - \sum_j n_{ij} - n_i\right)\right\}.$$

We shall do this in the next section.

The chief simplification achieved by taking the limit (6) is the following: in each order of perturbation theory

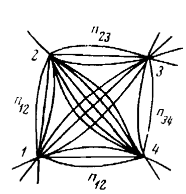

FIG. 4.

only one diagram turns out to be important, and this diagram contains only a singly-logarithmic divergence. Thus, in the sum (14b) the contribution of diagrams with $n_3 = n/2$, which contains a doubly-logarithmic divergence (the coefficient of which is, by virtue of Eq. (12), determined by the expression (13b)) is, in the calculation of the singly logarithmic terms, exponentially small compared with the contribution of diagrams with $n_i = n/3$. The second considerable simplification will be connected with the fact that the Feynman integrals obtained on taking the limit (6) are two-dimensional (cf. Sec. 5).

4. COMBINATORICS IN ARBITRARY ORDER

For the n-point Green function $G(x_1, x_2, \ldots, x_n)$ including the external two-point Green functions we have the following formal expression in k-th order of perturbation theory:

$$G_k = \frac{(-1)^k}{Jk!} \int \prod_x d\varphi(x) \prod_{i=1}^{n} \varphi(x_i) \int \prod_{r=1}^{k} d^D x^r \times H_{int}(\varphi(x^r)) \exp\left[-\frac{1}{2}\int d^D x (\partial_\mu \varphi)^2\right], \tag{15}$$

$$J = \int \prod_x d\varphi(x) \exp\left[-\frac{1}{2}\int d^D x (\partial_\mu \varphi(x))^2\right]. \tag{15a}$$

On calculation of the functional integral over $\varphi(x)$, the formula (15) contains both connected and disconnected graphs, and also graphs with vacuum loops. In the following we shall need only the connected diagrams. It turns out that it is precisely these diagrams which give the maximum contribution to (15) for $n \to \infty$. Therefore, in calculating the connected diagrams we can use this expression.

In order to calculate the functional integral in (15) explicitly it is convenient to introduce new variables l_r by the definition

$$H_{int}(\varphi(x^r)) = \oint \frac{dl_r}{2\pi i} \frac{\exp[l_r \varphi(x^r) - {}^1\!/_2 l_r^2 \Delta_0(0)]}{l_r^{n+1}} g. \tag{16}$$

Making a shift of variables in the Gaussian functional integral and then omitting the terms in $\Pi\varphi(x_i)$ that correspond to disconnected diagrams, we obtain

$$G_k(x_1, x_2 \ldots x_n) = \frac{(-1)^k}{k!} g^k \int \prod_{r=1}^{k} d^D x^r \frac{dl_r}{2\pi i l_r^{n+1}} \prod_{i=1}^{n} \sum_{r=1}^{k} l_r \Delta_0(x_i - x^r) \times \exp\left[\frac{1}{2}\sum_{r\neq r'} l_r l_{r'} \Delta_0(x^r - x^{r'})\right]. \tag{17}$$

The integral (17) is conveniently rewritten by introducing a new integration:

$$\int_{1/\Lambda}^{1/m^2} dy^2\, \delta\left(y^2 - \prod_r |x^r - x|^{2/k}\right) \int d^D x\, \delta^D\left(x - \frac{\Sigma x^r}{k}\right) = 1$$

and then making a change of variables: $x^r \to x^r y + x$, $l_i \to l_i y^{2/(n-2)}$. Using in addition the scale invariance of the Green function (10), we can transform (17) to the following form:

$$G_k(x_1 \ldots x_n) = \frac{(-1)^k}{k!} g^k \int_{1/\Lambda}^{1/m^2} \frac{dy^2}{y^2} \int d^D x \int \prod_{r=1}^{k} d^D x^r \times \frac{dl_r}{l_r^{n+1} 2\pi i} \prod_{i=1}^{n} \sum_{r=1}^{k} l_r \Delta_0(x_i - x + yx^r) \exp\left[\frac{1}{2}\sum_{r\neq r'} l_r l_{r'} \Delta_0(x^r - x^{r'})\right] \times \delta^D\left(\frac{\Sigma x^r}{k}\right) \delta\left(\frac{\Sigma \ln|x^r|^2}{k}\right). \tag{18}$$

If we are interested only in the ultraviolet divergence of the integral (18), i.e., $\Lambda^{-1/2} \ll y \ll |x_i - x_j|$, then the dependence on y in the arguments of the Green functions can be omitted. The integral

$$\int d^D x \prod_{i=1}^{n} \Delta_0(x_i - x)$$

corresponds in this limit to the external legs, i.e., when it is omitted we obtain the k-th coefficient of the expansion of the vertex function Γ. Differentiating the invariant charge with respect to $\ln\Lambda^{-1}$ (in this approach, $\Lambda^{1/2}$ plays the role of the normalization point μ), we obtain finally:

$$\psi(g) = \sum_{k=2}^{\infty} (-g)^k C_k(n), \tag{19}$$

$$C_k(n) = \frac{1}{k!} \int \prod_{r=1}^{k} d^D x^r \frac{dl_r}{l_r^{n+1} 2\pi i} \left(\sum_{r=1}^{k} l_r\right)^n \times \exp\left[\frac{1}{2}\sum_{r\neq r'} l_r l_{r'} \Delta_0(x^r - x^{r'})\right] \delta^D\left(\frac{\Sigma x^r}{k}\right) \delta\left(\frac{\Sigma \ln|x^r|^2}{k}\right). \tag{20}$$

We shall consider now the behavior of the integral (20) as $n \to \infty$ for fixed k. Taking into account that the Green function (10) in this limit can be written in the form

$$\Delta_0(x_1 - x_2) \approx \frac{n}{8\pi} - \frac{1}{4\pi}(1 + \ln\pi + c_E) - \frac{1}{2\pi}\ln|x_1 - x_2|, \tag{21}$$

we see that in the integrals over l_r in (20) there is a saddle point at

$$n\left[-\frac{1}{l_r} + 1 \Big/ \sum_{r=1}^{k} l_r + \sum_{r'\neq r} \frac{1}{8\pi} l_{r'}\right] = 0. \tag{22}$$

The system (22) has two solutions:

$$l_1^{-1} = l_2^{-1} = \ldots = l_k^{-1} = (k/8\pi)^{1/2}, \quad l_1^{-1} = l_2^{-1} = \ldots = l_k^{-1} = -(k/8\pi)^{1/2}. \tag{23}$$

Calculating the exponential at the saddle points (23) and the Gaussian integral over small deviations from these points, we can bring the expression (20) to the following form (cf. (13b), (14b)):

$$C_k(n)|_{n\to\infty} = \frac{1}{k!}\left(\frac{e}{8\pi} k^{(k+1)/(k-1)}\right)^{n(k-1)/2} \times \left(2\pi n \frac{k-1}{k}\right)^{-k/2} \exp[-(k-1)(1 + \ln\pi + c_E)] 2^{1/2} C_k, \tag{24}$$

$$C_k = \int \prod_{r=1}^{k} d^2 x_r \left(\prod_{r<r'} |x_r - x_{r'}|\right)^{-4/k} \delta^2\left(\frac{\Sigma x_r}{k}\right) \delta\left(\frac{\Sigma \ln|x_r|^2}{k}\right). \tag{25}$$

We note that by expanding the integrand in (20) in the neighborhood of the saddle point (23) it would be possible to calculate the $\sim 1/n$ corrections to the formula (24).

In the language of perturbation-theory diagrams, the expression (25) corresponds in k-th order of perturbation theory to one diagram with the same number of lines ($n_{ij}=n/k$) joining each pair of interaction points. In the next section we find the coefficients C_k in arbitrary order of perturbation theory. The possibility of calculating the integrals (25) analytically is connected with the fact that they are two-dimensional.

5. CALCULATION OF THE COEFFICIENTS C_k

The expression (25) can be rewritten in the following equivalent form after making a shift of variables $x_r \to x_r + x_k$ ($r=1, \ldots, k-1$) and eliminating the integral over x_k on account of the δ-function:

$$C_k = \lim_{\Lambda\to\infty} \frac{1}{2\ln\Lambda} \int \prod_{r=1}^{k-1} \frac{d^2x_r}{x_r^{4/k}} \prod_{r<r'} |x_r - x_{r'}|^{-4/k}, \tag{26}$$

where the integration is carried out over the region

$$\frac{1}{\Lambda} \ll \Big[\prod_{r=1}^{k-1} (x_r+\bar{x})^2 \bar{x}^2\Big]^{1/k} \ll \Lambda, \quad \bar{x} = \Big(-\frac{1}{k}\Big) \sum_{r=1}^{k-1} x_r. \tag{27}$$

When one of the variables in the integral (26) is fixed, e.g., under the condition $\Lambda^{-1} \ll x_1^2 \ll \Lambda$, in a substantial region of the integration over the remaining variables $x_2, x_3, \ldots, x_{k-1}$ we have $|x_r| \sim |x_1|$. This means that the condition (27) is fulfilled automatically and can be omitted. At the same time, the integration over the regions $x_1^2 \gtrsim \Lambda$ and $x_1^2 \lesssim \Lambda^{-1}$ does not, by virtue of the restriction (27), lead to a divergence of the integral over x_1. Thus, the expression (25) can be represented in the following equivalent form:

$$C_k = \lim_{\Lambda\to\infty} \frac{1}{2\ln\Lambda} \int \prod_{r=1}^{k-1} \frac{d^2x_r}{x_r^{4/k}} \prod_{r<r'} |x_r - x_{r'}|^{-4/k}, \tag{28}$$

$$1/\Lambda \ll x_1^2 \ll \Lambda. \tag{28a}$$

In the integral (28) we go over to integration over a pseudo-Euclidean space by rotating the contour of integration over the second component of each of the vectors x_r in a counterclockwise direction in the complex plane ($x_r^1 \equiv x_r$, $x_r^2 = it_r$):

$$C_k = \lim_{\Lambda\to\infty} \frac{i^{k-1}}{2\ln\Lambda} \int \prod \frac{dx_r\, dt_r}{[-t_r^2 + x_r^2 + i\varepsilon]^{2/k}} \prod_{r<r'} [-(t_r - t_{r'})^2 + (x_r - x_{r'})^2 + i\varepsilon]^{-2/k}, \tag{29}$$

where the $i\varepsilon$ in the denominators has been added in order that we obtain the Euclidean integral (28) by the reverse rotation. Next, in (29) we introduce variables on the light cone:

$$\alpha = t - x, \quad \beta = t + x. \tag{30}$$

We then obtain

$$C_k = \lim_{\Lambda\to\infty} \frac{1}{2\ln\Lambda} \Big(\frac{i}{2}\Big)^{k-1} \int \prod_{r=1}^{k-1} \frac{d\alpha_r\, d\beta_r}{[-\alpha_r\beta_r + i\varepsilon]^{2/k}} \times \prod_{r<r'} [-(\alpha_r - \alpha_{r'})(\beta_r - \beta_{r'}) + i\varepsilon]^{-2/k}. \tag{31}$$

In accordance with the fact that we must take the restriction (28a) into account in the integration in the Euclidean space, in the expression (31) we fix the integration variables α_1 and β_1. We change the scale of the remaining variables $\alpha_r \to \alpha_r\alpha_1$, $\beta_r \to \beta_r\beta_1$ ($r=2, 3, \ldots, k-1$) and take $(\alpha_1\beta_1)^{2/k}$ outside each denominator in (31). Then the integration over α_1 and β_1 becomes logarithmic:

$$\frac{i}{2} \int \frac{d\alpha_1\, d\beta_1}{-\alpha_1\beta_1 + i\varepsilon}$$

and, after we change to the Euclidean space, gives, in the region (28a), $\pi 2\ln\Lambda$. Thus, we obtain

$$C_k = \Big(\frac{i}{2}\Big)^{k-2} \pi \int \prod_{r=2}^{k-1} \frac{d\alpha_r\, d\beta_r}{[\alpha_r\beta_r + i\varepsilon]^{2/k}} \prod_{r<r'} [(\alpha_r - \alpha_{r'})(\beta_r - \beta_{r'}) + i\varepsilon]^{-2/k},$$
$$r, r' = 1, 2, \ldots, k-1, \tag{32}$$

where, by definition, $\alpha_1 = \beta_1 = 1$.

We shall fix the variables β_r in the expression (32). Then the integrals over α_r will be nonzero only under the condition

$$0 < \beta_r < 1. \tag{33}$$

Otherwise, all the singularities in one of the integrals over α_r will be either above or below the contour of integration and this integral will vanish. Using the symmetry of the expression (32) under the replacements $\beta_r \rightleftarrows \beta_{r'}$ and restricting the integration to the region indicated in (35), which is narrower than (33), we can rewrite (32) in the following form:

$$C_k = \pi (k-2)!\, B_k A_k, \tag{34}$$

where the quantities β_k and A_k are given by the formulas

$$B_k = \int \prod_{r=2}^{k-1} \frac{d\beta_r}{\beta_r^{2/k}} \prod_{r<r'} (\beta_r - \beta_{r'})^{-2/k}, \quad 0 < \beta_{k-1} < \beta_{k-2} < \ldots < \beta_2 < \beta_1 = 1; \tag{35}$$

$$A_k = \Big(\frac{i}{2}\Big)^{k-2} \int \prod_{r=2}^{k-1} d\alpha_r \prod_{r<r'} [\alpha_r - \alpha_{r'} + i\varepsilon]^{-2/k}, \tag{36}$$
$$r, r' = 1, 2, \ldots, k, \quad \alpha_k = 0, \quad \alpha_1 = 1.$$

Thus, we have simplified considerably the problem of calculating the integral (35) by writing it in the factorized form (34). We now transform the contour integral for A_k to a real form. For this we divide the region of integration in (36) into a sequence of subregions depending on the relative magnitudes of the α_r. In the region $0 < \alpha_{k-1} < \alpha_{k-2} < \ldots < \alpha_2 < \alpha_1 = 1$ we have

$$A_k = \Big(\frac{i}{2}\Big)^{k-2} \int \prod_{r=2}^{k-1} \frac{d\alpha_r}{\alpha_r^{2/k}} \prod_{r<r'} (\alpha_r - \alpha_{r'})^{-2/k} = \Big(\frac{i}{2}\Big)^{k-2} B_k, \tag{37}$$
$$0 < \alpha_{k-1} < \ldots < \alpha_2 < 1.$$

In an arbitrary region $-\infty < \alpha_{i_k} < \alpha_{i_{k-1}} < \cdots < \alpha_k = 0 < \cdots < \alpha_1 = 1 < \cdots < \alpha_{i_1} < \infty$ ($i_r = 1, 2, \ldots, k$), we have, taking into account the phases that arise as a *result of going* round the branch points of the integrand in (36),

$$A_k = \Big(\frac{i}{2}\Big)^{k-2} \exp\Big(-\frac{2\pi i}{k}\eta_p\Big) \int \prod_{r=2}^{k-1} \frac{d\alpha_r}{|\alpha_r|^{2/k}} \prod_{r<r'} |\alpha_r - \alpha_{r'}|^{-2/k}, \tag{38}$$
$$-\infty < \alpha_{i_k} < \ldots < \alpha_k < \ldots < \alpha_1 < \ldots < \alpha_{i_1} < \infty,$$

where η_p is the number of single interchanges of neighboring elements that must be made in order to get from the normal order $(k, k-1, \ldots, 2, 1)$ to the given order $(i_k, i_{k-1}, \ldots, i_1)$ (obviously, the order of the numbers k and 1 cannot be changed). We shall prove first of all that for any ordering of the α_r the integral over the α_r in formula (38) coincides with B_k. In fact, by relabeling the variables we can reduce it to the form

$$B_k' = \int \prod_{r=1}^{k} d\beta_r \prod_{r<r'} (\beta_r - \beta_{r'})^{-2/k} \delta(\beta_i)\delta(\beta_j - 1), \qquad (39)$$
$$-\infty < \beta_k < \beta_{k-1} < \ldots < \beta_1 < \infty, \quad j < i.$$

Replacing β_j in the argument of the δ-function by $\beta_j - \beta_i$ and making a shift of variables, we can write (39) in the form

$$B_k' = \int \prod_{r=1}^{k} d\beta_r \prod_{r<r'} (\beta_r - \beta_{r'})^{-2/k} \delta(\beta_j - \beta_i - 1)\, \delta\left(\frac{\Sigma \beta_r}{k}\right). \qquad (40)$$

Introducing next

$$\int_0^\infty d\beta\, \delta(\beta_k - \beta_1 - \beta) = 1$$

and making the replacement $\beta_r \to \beta_r \beta$ in the integral (40), we can transform (40) to the form

$$B_{k'} = \int \frac{\prod d\beta_r}{\prod (\beta_r - \beta_{r'})^{2/k}} \int_0^\infty \frac{d\beta}{\beta} \delta(\beta_k - \beta_1 - 1)\, \delta[\beta(\beta_j - \beta_i) - 1]\, \delta\left(\frac{\Sigma \beta_r}{k}\right). \qquad (41)$$

After integrating over β we obtain $B_k' = B_k$ (cf. (35)).

Thus, the expression (34) for C_k can be written in the form

$$C_k = \pi (k-2)!\, B_k^2 D_k, \qquad (42)$$

$$D_k = \left(\frac{i}{2}\right)^{k-2} \sum_{\{p\}} \exp\left(-i\eta_p \frac{2}{k}\pi\right). \qquad (43)$$

In formula (43) the summation runs over all permutations $\{p\}$ in which the ordering of the numbers k and 1 is not changed, and η_p is the number of elementary interchanges (of two neighboring numbers) in the product of interchanges by means of which the identity permutation

$$\{1\} = \begin{Bmatrix} k, k-1, \ldots, 1 \\ k, k-1, \ldots, 1 \end{Bmatrix}$$

can be brought to the given permutation.

We introduce the quantity p_n^k—the number of permutations with the same value $\eta_p = n$. The number of permutations with a fixed order of appearance of the elements k and 1 is easily expressed in terms of the total number Q_m^{k-2} of possible permutations, with a given $\eta_p = m$, of the $k-2$ elements $k-1, k-2, \ldots, 2$, by the formula

$$p_n^k = \sum_{m=\max(0, n-k+2)}^{n} (n - m + 1) Q_m^{k-2}. \qquad (44)$$

In fact, any permutation with a fixed order of appearance of the elements k and 1 can be carried out in two stages: one first permutes the elements $k-1, \ldots, 2$ by m elementary interchanges, and then interchanges the numbers k and 1 with these elements. For a fixed total number n of elementary interchanges and a fixed number $m \leq n$ of elementary interchanges of the elements $k-1, \ldots, 2$, the interchange of the elements k and 1 with the other elements can be performed in $n - m + 1$ ways. The lower limit in the sum (44) follows from the fact that, for $m < n - k + 2$, to obtain a permutation with the given n we should have to interchange the elements k and 1, which, from the definition of $\{p\}$, is not possible. The total number Q_m^{k-2} of possible permutations of $k-2$ elements with a fixed number m of elementary interchanges is easily expressed, by analogy with the derivation of formula (44), in terms of the number of permutations of $k-3$ elements:

$$Q_m^{k-2} = \sum_{m'=\max(0, m-k+3)}^{m} Q_{m'}^{k-3}, \quad Q_m^1 = \delta_{m0}, \quad Q_m^2 = \delta_{m0} + \delta_{m1}. \qquad (45)$$

If we introduce the generating function

$$\varphi^{k-2}(z) = \sum_{m=0}^{\infty} Q_m^{k-2} z^m,$$

we can rewrite Eq. (45) in the form

$$\varphi^{k-2}(z) = \frac{1 - z^{k-2}}{1 - z} \varphi^{k-3}(z) = \prod_{r=1}^{k-2} \frac{1 - z^r}{1 - z}. \qquad (45a)$$

Using the relation (44) and formula (45a), we can represent the expression (43) in the following form:

$$D_k = \left(\frac{i}{2}\right)^{k-2} \varphi^{k-2}(z) \frac{d}{dz} \frac{1 - z^k}{1 - z}\Big|_{z = e^{-i2\pi/k}}$$
$$= \left(\frac{i}{2}\right)^{k-2} \frac{k}{(1 - z)^k} \prod_{r=1}^{k-1} (1 - z^r)\Big|_{z = e^{-i2\pi/k}}. \qquad (46)$$

After substituting $z = e^{-i2\pi/k}$ and using the well-known formula

$$\prod_{m=1}^{k-1} \sin \frac{\pi m}{k} = 2^{-(k-1)} k, \qquad (47)$$

we finally obtain the following expression for D_k:

$$D_k = 2^{-2(k-1)} k^2 (\sin(\pi/k))^{-k}. \qquad (48)$$

We turn our attention now to the calculation of the integral for B_k (cf. (35)). Using the symmetry of the integrand and the equality, proved above, of all the integrals (39), we rewrite it in the following form:

$$B_k = \lim_{\varepsilon \to +0} \frac{2}{k!} \int_{-\infty}^{+\infty} \prod_{i=1}^{k} d\beta_i \prod_{i>j} \frac{1}{|\beta_i - \beta_j|^{2/k}} \delta(\beta_k)\, \delta(\beta_1 - \varepsilon)\, \varepsilon, \qquad (49)$$

where the factor in front of the integral takes account of the fact that, unlike in (35), the integration in (49) is performed over regions with all possible orderings of the β_i (with the condition that $\beta_1 > \beta_k$); the number of these regions is $k!/2$ and each of them gives a contribution B_k' equal to B_k.

It is obvious that when $\varepsilon \to 0$ the result of the integration for points β_i lying on a straight line coincides with the result of the integration when these points are positioned on the unit circle:

$$B_k=\frac{2}{k!}\lim_{\varepsilon\to+0}\frac{1}{2\pi}\int\prod_{i=1}^{k}d\varphi_i\prod_{i>j}\frac{1}{|\varphi_i-\varphi_j|^{2/k}}\delta(\varphi_1-\varphi_k-\varepsilon)\varepsilon,\quad 0<\varphi_i<2\pi, \tag{50}$$

where we have omitted the factor $\delta(\varphi_k)$, averaging the result of the φ_k integration (introducing the factor $1/2\pi$).

To calculate the expression (50) we shall make use of a conjecture of Dyson,[13] proved by Gunson[14] and Wilson,[15] which is that, for any k, the equality

$$\Psi_k(\gamma)\equiv(2\pi)^{-k}\int\prod_{i=1}^{k}d\varphi_i\prod_{i>j}|e^{i\varphi_i}-e^{i\varphi_j}|^{\gamma}=\frac{\Gamma(1+\gamma k/2)}{[\Gamma(1+\gamma/2)]^k}. \tag{51}$$

is valid.

We shall consider $\Psi_k(\gamma)$ for $\gamma\to-2/k$:

$$\Psi_k(\gamma)|_{\gamma\to-2/k}=\left(1+\frac{1}{2}\gamma k\right)^{-1}\left[\Gamma\left(1-\frac{1}{k}\right)\right]^{-k}. \tag{52}$$

On the other hand, when the variable $\varphi_1-\varphi_k$ is fixed the integration over the other variables in (51) gives a finite answer for $\gamma\to-2/k$, and a divergence arises in the last integration as $\varphi_1-\varphi_k\equiv\varepsilon\to 0$:

$$\Psi_k(\gamma)|_{\gamma\to-2/k}=(2\pi)^{-k}\int\limits_{|\varepsilon|\ll1}d\varepsilon\int\prod d\varphi_i\prod_{i>j}|\varphi_i-\varphi_j|^{\gamma}\delta(\varphi_1-\varphi_k-\varepsilon)|_{\gamma\to-2/k}. \tag{53}$$

Making the change of variables $\varphi_i\to|\varepsilon|\varphi_i$ in the integral (53) and comparing it with the expression (50) with an analogous change of variables, we obtain

$$\Psi_k(\gamma)|_{\gamma\to-2/k}=\frac{B_k}{(2\pi)^k}\frac{k!}{2}\int\limits_{|\varepsilon|\ll1}d\varepsilon|\varepsilon|^{k-2+k(k-1)\gamma/2}=B_k\frac{k!(1+\gamma k/2)^{-1}}{(2\pi)^{k-1}(k-1)}. \tag{54}$$

Thus, from formulas (52) and (54) we find

$$B_k=\frac{(2\pi)^{k-1}}{(k-2)!k}\left\{\Gamma\left(1-\frac{1}{k}\right)\right\}^{-k}. \tag{55}$$

Substituting the expressions (48) and (55) into formula (42), after elementary transformations we have, finally,

$$C_k=\frac{\pi^{k-1}}{\Gamma(k-1)}\left\{\Gamma\left(\frac{1}{k}\right)\Big/\Gamma\left(1-\frac{1}{k}\right)\right\}^k. \tag{56}$$

In the next section we shall investigate the properties of the GML function in the theory under consideration.

6. THE GELL-MANN-LOW FUNCTION IN THE SCALAR THEORY WITH HAMILTONIAN $H_{\text{int}}=g\varphi^n/n!$

As can be seen from formulas (19), (24) and (56), the series determining the GML function is asymptotic. It diverges like $(k!)^{n/2-1}$ at large k. The appearance of this divergence is not connected with our method of calculation (we find $C_k(n)$ for fixed values of k, and $n\to\infty$). It can be shown (this will be done in a subsequent paper) that, when the limits are taken in the opposite order, the coefficients

$$\tilde{C}_k(n)=\lim_{n\to\infty}\lim_{k\to\infty}C_k(n)$$

coincide with those found above.

Thus, the question of determining the origin of the series (19) arises, i.e., of finding the function for which this series is the asymptotic expansion in the vicinity of the point $g=0$. It is obvious that, given such a strong divergence of the series, we can find many such functions. A correct prescription should be obtained directly from the functional integral determining the Green function. Since methods of calculating the functional integral without the use of perturbation-theory methods do not exist at the present time, we shall make the following assumption: the correct analytic continuation of the perturbation-theory series is the Watson–Sommerfeld transform with respect to the variable $\ln g$. In particular, for the GML function the following representation should be valid:

$$\psi(g)=(-2i)^{-1}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{dk}{\sin\pi k}g^kC_k(n),\quad 1<\sigma<2. \tag{57}$$

It is not difficult to see that, in the case when $C_k(n)$ is analytic to the right of the integration contour, for $g\to0$ the integral (57) converges and the function $C_k(n)$ is the analytic continuation of the coefficients $C_k(n)$ (24); the asymptotic series for the function (57) coincides with the series (19). Next we choose the simplest analytic continuation with integer points $k=2, 3, \ldots$ that satisfies the conditions indicated above (cf. (24), (56)):

$$C_k(n)=\frac{2^{1/2}}{\Gamma(k+1)\Gamma(k-1)}\left(\frac{e}{8\pi}k^{(k+1)/(k-1)}\right)^{n(k-1)/2}$$
$$\times\left(2\pi n\frac{k-1}{k}\right)^{-k/2}e^{-(k-1)(1+c_E)}\left\{\Gamma\left(\frac{1}{k}\right)\Big/\Gamma\left(1-\frac{1}{k}\right)\right\}^k. \tag{58}$$

Support for this continuation is provided by the fact that the integral (57) coincides with the perturbation-theory series (19) in the region $n\leqslant2$, where the series is convergent. With another continuation they would not coincide.

We shall consider now the properties of the GML function (57).

The asymptotic form of the integral (57) for $g\to0$ is determined by the poles at the points $k=2, 3, \ldots$:

$$\psi(g)|_{g\to0}=2^{-1/2}(e/\pi)^{n/2}(\pi n)^{-1}e^{-(1+c_E)}g^2-O(g^3). \tag{59}$$

In the opposite limit $g\to\infty$ the asymptotic form of the integral is determined by the singularity at $k=1$:

$$C_k(n)|_{k\to1}=(\pi n)^{-1/2}(k-1)^{3/2}, \tag{60}$$

so that

$$\psi(g)|_{g\to\infty}=-(2\pi)^{-1}n^{-1/2}g(\ln g)^{-3/2}. \tag{61}$$

Comparing formulas (59) and (61), we see that the function changes sign as we go from the region of small g to the region of large g, i.e., there *is at least* one ultraviolet-stable point: $g(p^2\to\infty)=g_0$, where $\psi(g_0)=0$. In fact, for large n the GML function has a number of zeros, and this number increases like $n^{1/4}$ with increase of n. This is connected with the fact that in the integral (57) there are two saddle points, satisfying the equation

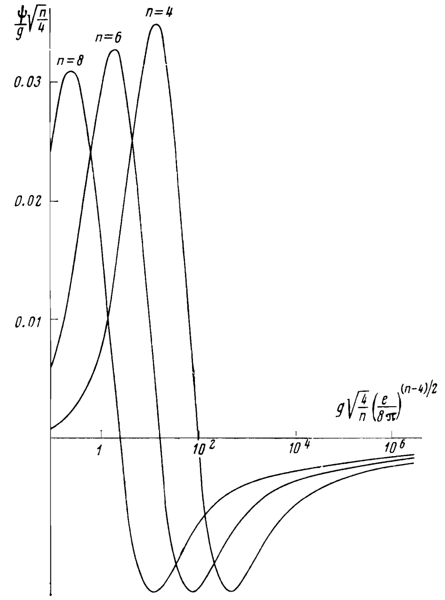

FIG. 5.

$$\ln\left(\frac{g}{g_0(n)}\right)+\frac{n}{2}\left(\ln k_0+\frac{1}{k_0}-1\right)=0,\quad g_0(n)=\left(\frac{8\pi}{e^3}\right)^{n/2}. \tag{62}$$

These saddle points move toward each other with increase of g and collide when $k=1$; they then move away into the complex plane, and this, in turn, leads to oscillations of $\psi(g)$ in a certain range of variation of g.

We have plotted the GML function (57) for the physically interesting cases $n=4$ and $n=6$ (Fig. 5). In both cases it has only one ultraviolet point ($g_0\approx 103$ for $n=4$ and $g_0\approx 187$ for $n=6$). It is possible to believe that the exact graph for the GML functions in the $g\varphi^4/4!$ and $g\varphi^6/6!$ theories will differ from the graph of Fig. 5 by small corrections $\sim 1/n$, so that the qualitative result obtained in this work concerning the existence of the ultraviolet-stable points will remain valid.

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[1)]In the limit as $n\to\infty$, as will be shown below, the principal role in the calculation of the GML function is played by maximally connected diagrams, the contribution of which does not depend on the ratios of the invariants, i.e., in place of (9) we can use any other set of invariants.

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