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Electrical properties of one-dimensional metals

A. A. Abrikosov and I. A. Ryzhkin

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences
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We construct a method for rigorously evaluating the properties of one-dimensional metals in the field of impurities taking both types of scattering into account, quasi-classical forward scattering of electrons and backward scattering (i.e., from the neighborhood of the momentum p_0 to the neighborhood of $-p_0$, where p_0 is the Fermi momentum). In contrast to the method proposed by Berezinskiĭ {Zh. Eksp. Teor. Fiz. 65, 1251 (1974) [Sov. Phys. JETP 38, 620 (1974)]} the present approach possesses a higher degree of automatism; it enables us to generalize to the case of a quasi-one-dimensional system and to take into account scattering by phonons. We give a detailed account of the method itself in the present paper and demonstrate how it can be applied by calculating as an example the conductivity and permittivity of a one-dimensional metal. We correct a result in Berezinskiĭ's paper.

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1. INTRODUCTION

Recently people have become interested in one-dimensional and quasi-one-dimensional problems. A distinctive feature of these problems is the fact that many approximate methods applicable to three-dimensional systems become unsuitable for one-dimensional ones. The exact solution of various problems for one-dimensional systems is connected with considerable difficulties and even when it is possible to find it the corresponding method makes it impossible to generalize it to the quasi-one-dimensional case (three-dimensional perturbation of a one-dimensional system).

One of those one-dimensional problems is the problem of the electrical resistivity of a one-dimensional metal in which the electrons are scattered by randomly distributed impurities. Berezinskiĭ[1] recently solved this problem. Unfortunately, the very ingenious method applied by him does not permit generalization to the quasi-one-dimensional case. At the same time such a generalization is of considerable interest as real systems are not purely one-dimensional. As an example we may mention the quasi-one-dimensional compounds which are the base of TCNQ, where the electrons have the possibility to make transitions between filaments. Another example is a semi-metal in a strong magnetic field where apart from the one-dimensional motion along the field there is a finite transverse motion described by an oscillator wavefunction.

We have been able to construct a new method for studying the properties of a one-dimensional system of electrons which interact with random impurities; this method enables us to generalize it to the quasi-one-dimensional case. In the present paper, the aim of which is an exposition of the method, we restrict ourselves to the problem of the electrical resistivity of a purely one-dimensional metal which was already solved by Berezinskiĭ.[1] In subsequent papers we shall consider quasi-one-dimensional systems.

§1. THE GREEN FUNCTION

We shall assume that the electrons have an energy spectrum

$$\varepsilon = p^2/2m. \tag{1}$$

We shall assume $T=0$ (if we neglect phonons, see[2], the temperature affects the results only when $T\sim\varepsilon_F$). In the equilibrium state the electrons are then degenerate and the Fermi momentum is connected with the electron density by the relation

$$n_e = 2p_0/2\pi. \tag{2}$$

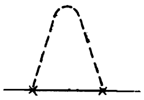

FIG. 1.

The electrons are scattered by randomly distributed impurities. The impurity concentration equals n_i. If it is not too large all physical effects will be determined by electrons with momenta in the immediate vicinity of $p=p_0$ and of $p=-p_0$. When an electron with momentum close to p_0 is scattered by an impurity it can either remain in that neighborhood or go over to the neighborhood of $-p_0$. The first process corresponds to a Born amplitude u_1 and the second one to u_2.

We consider the first approximation to the self-energy which is connected with the scattering of an electron by an impurity (Fig. 1). It has the form (see[2])

$$\Sigma(p)=n_i\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}G(p_1)|u(p-p_1)|^2=\Sigma_1+\Sigma_2,$$
$$\Sigma_1=n_i\int_{-\infty}^{\infty}|u_1|^2[\omega-vk+i\delta\,\mathrm{sign}\,\omega]^{-1}\frac{dk}{2\pi},\qquad(3)$$
$$\Sigma_2=n_i\int_{-\infty}^{\infty}|u_2|^2[\omega+vk+i\delta\,\mathrm{sign}\,\omega]^{-1}\frac{dk}{2\pi}.$$

The term Σ_1 corresponds to forward scattering when the electron remains in the vicinity of p_0. In that case $p_1=p_0+k$, $k\ll p_0$, $(p_1^2-p_0^2)/2m\approx vk$ $(v=p_0/m)$. The term Σ_2 corresponds to backward scattering when the electron goes over into the vicinity of $-p_0$. In that case $p_1=-p_0+k$, $k\ll p_0$, $(p_1^2-p_0^2)/2m\approx -vk$. Integrating we get

$$\Sigma_1=-in_i|u_1|^2\,\mathrm{sign}\,\omega/2v=-i\,\mathrm{sign}\,\omega/2\tau_1,$$
$$\Sigma_2=-in_i|u_2|^2\,\mathrm{sign}\,\omega/2v=-i\,\mathrm{sign}\,\omega/2\tau_2.\qquad(4)$$

We have introduced here the symbols τ_1 and τ_2.

It is well known (see, e.g., [1]) that in fact the diagrammatic method is inapplicable to the given problem. The reason is that when one evaluates the conductivity all diagrams are important and it is impossible to find some simple sequence of main diagrams and sum them. In view of this we formulate instead of our problem another one which is equivalent to it.

Since the important electron states are in the neighborhood of p_0 and $-p_0$ we introduce in our discussion states $\psi_1 e^{ikz}$ corresponding to momenta p_0+k, and $\psi_2 e^{ikz}$ corresponding to momenta $-p_0+k$, where $k\ll p_0$. The energy of a free particle reckoned from the chemical potential is $\xi=vk$ for $\psi_1 e^{ikz}$ and equal to $-vk$ for $\psi_2 e^{ikz}$. Hence, the energy operator of a free particle can be written in the form $vk\sigma_3$, where $\sigma_3=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ is a matrix with indices 1 and 2.

Instead of the interaction with the impurities we introduce random fields[1)] $\eta(z)$ and $\zeta(z)$. The interaction with the field $\eta(z)$ leaves an electron in the vicinity of p_0 if it was there, i.e., this interaction is diagonal in the indices 1 and 2. We can assume this interaction to be real and to have the following properties:

$$\langle\eta(z)\rangle=0,\quad\langle\eta(z)\eta(z')\rangle=\delta(z-z')v/\tau_1.\qquad(5)$$

If we change from a continuous variable z to a discrete set of points with spacing Δ: $\eta_i=\eta(z_i)\Delta$, Eqs. (5) take the form

$$\langle\eta_i\rangle=0,\quad\langle\eta_i\eta_k\rangle=v\Delta\delta_{ik}/\tau_1.\qquad(6)$$

We shall assume that the fields $\eta(z)$ and $\zeta(z)$ are Gaussian. Equations (5) and (6) correspond to an averaging over the functional

$$F[\eta]=\exp\left[-\int\eta^2(z)\,dz\frac{\tau_1}{v}\right]=\exp\left[-\sum_i\eta_i^2\frac{\tau_1}{v\Delta}\right].\qquad(7)$$

One checks easily that this corresponds to the Born approximation for the scattering. Indeed, expanding G in a series in η we have terms of the type

$$\int G_0(z-z_1)\eta(z_1)G_0(z-z_2)\eta(z_2)\ldots\eta(z_n)G_0(z_n-z')\,dz_1\ldots dz_n$$
$$=\sum_{i\ldots l}G_0(z-z_i)\eta_iG_0(z_i-z_k)\eta_k\ldots\eta_lG_0(z_l-z').$$

If η_i occurs only twice we get after averaging an expression with

$$\frac{v}{\tau_1}\Delta\sum_i\to\frac{v}{\tau_1}\int dz.$$

If, however, η_i recurs, e.g., four times, we get on averaging

$$\frac{v^2}{2\tau_1^2}\Delta^2\sum_i\to\Delta\frac{v^2}{2\tau_1^2}\int dz\to0\quad\text{as}\quad\Delta\to0.$$

The second field ζ transfers an electron from the vicinity of p_0 to the vicinity of $-p_0$. This field must be assumed to be complex as a "quantum" of such a field can be "emitted" only in the transition $1\to2$ and be "absorbed" only in the reverse transition. This representation corresponds to the fact that when we consider a real scattering the momentum transfer in the two crosses indicated in Fig. 1 must cancel. The interaction with the field thus has the form

$$\psi_1^+\psi_2\zeta+\psi_2^+\psi_1\zeta^*={}^1/_2\psi_\alpha^+[\sigma_{\alpha\beta}^{(+)}\zeta+\sigma_{\alpha\beta}^{(-)}\zeta^*]\psi_\beta,\qquad(8)$$

where $\sigma^{(\pm)}=\sigma_1\pm i\sigma_2$.

The averaging rules for the field ζ have the form

$$\langle\zeta(z)\zeta^*(z')\rangle=\delta(z-z')v/\tau_2,\quad\langle\zeta\rangle=0,$$
$$\langle\zeta(z)\zeta(z')\rangle=0.\qquad(9)$$

For an average of a product of a large number of ζ_i and ζ_i^* in the same point we stipulate the same requirements as for the η_i. This requirement can be realized by means of a Gaussian functional

$$\Omega[\zeta]=\exp\left[-\int|\zeta(z)|^2\,dz\,\tau_2/v\right].\qquad(10)$$

We find first of all the retarded Green function in the random field η. It satisfies the equation

$$\left[iv\sigma_3\frac{\partial}{\partial z}+\omega-\eta(z)\right]G_{\omega R}(z,z')=\delta(z-z').\qquad(11)$$

The solution of this equation which is the same when expanded in η as the perturbation theory result has the form

$$G_{\omega R}(z,z')=G^{(0)}_{\omega R}(z-z')\exp\left[-i\sigma_3\int_{z'}^{z}\eta(z_1)\frac{dz_1}{v}\right].\qquad(12)$$

Here $G^{(0)}$ is the free function, equal to

$$G^{(0)}_{\omega R}(z)=\int\frac{e^{ipz}\,dp/2\pi}{\omega-pv\sigma_3+i\delta}=-\frac{i}{v}e^{i\omega|z|/v}\,\theta(z\sigma_3).\qquad(13)$$

We can write Eq. (12) also in the form

$$G_{\omega R}(z,z')=-\frac{i}{2v}[1+\sigma_3\,\mathrm{sign}\,(z-z')]\exp\left[\frac{i\sigma_3}{v}\int_{z'}^{z}[\omega-\eta(z_1)]dz_1\right].\qquad(14)$$

We average the function $G_{\omega R}$. We write the index of the exponential in (12) as a sum:

$$-i\sigma_3\sum_{i\in(zz')}\frac{\eta_i}{v}.$$

We then get the product

$$\prod_{i\in(zz')}\exp\left(\frac{-i\sigma_3\eta_i}{v}\right)\approx\prod_{i\in(zz')}\left(1-\frac{i\sigma_3}{v}\eta_i-\frac{1}{2v^2}\eta_i^2\right).$$

We have expanded each exponent using what we have said earlier about high powers of η_i in a single point. We now average this expression. Since each factor occurring in the product gives the same on averaging there enters here the average of one of them to the power $|z-z'|/\Delta$. After this we let Δ tend to zero. As a result we have

$$\left\langle\prod_{i\in(z,z')}\exp\left(-\frac{i\sigma_3\eta_i}{v}\right)\right\rangle=\left(1-\frac{\Delta}{2v\tau_1}\right)^{|z-z'|/\Delta}\to\exp\left(\frac{-|z-z'|}{2v\tau_1}\right).\qquad(15)$$

This very obvious result can be obtained by different means. We gave the derivation here only to demonstrate our method.

We now consider the Green function when both fields η and ζ are present. We take some order in ζ and the exact expression in η. We now average over the field ζ. The result will correspond to some diagram of the type of Fig. 2 with dotted lines depicting the average of $\zeta_i\zeta_i^*$. The dotted lines connect points with the same z. Each point is connected with the Green functions (14), but while in the vicinity of a point with ζ_i we get from both Green functions a factor

$$\exp\left[2i\int^{z_i}\eta(z_1)\frac{dz_1}{v}\right],$$

near a point with ζ_i^* there arises a factor

$$\exp\left[-2i\int^{z_i}\eta(z_1)\frac{dz_1}{v}\right].$$

All such factors cancel and as a result the Green function will again be proportional to the same exponent which occurs in Eq. (14). One sees easily that the same

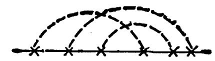

FIG. 2.

is valid for the average over ζ_i of a product of several Green functions. The result will depend solely on the end arguments. In that case, since Eq. (12) is valid for any Green function—retarded, advanced, or causal—we may deal with products of different kinds of Green functions. For the evaluation of the conductivity we need in what follows only closed loops of several Green functions. In such loops the factors (12) cancel completely.

The result of these discussions is thus: scattering described by the field η in which the electron remains in its original neighborhood does not affect the conductivity.

We further consider G_R in the field ζ. The appropriate equation has the form

$$\left[iv\sigma_3\frac{\partial}{\partial z}+\omega-\frac{1}{2}(\sigma^{(+)}\zeta+\sigma^{(-)}\zeta^*)\right]G_{\omega R}(z,z')=\delta(z-z').\qquad(16)$$

The difference from (11) consists in that field operators in different points do not commute with one another. In that case we can solve the equation symbolically. Let $z>z'$ and z_1 be some point between them, i.e., $z>z_1>z'$. We can then find from (16) a connection between $G_{\omega R}(z,z')$ and $G_{\omega R}(z_1,z')$. Since z' remains at one end we have from (16) when $z>z_1>z'$

$$G_{\omega R}(z,z')=S_\omega(z,z_1)G_{\omega R}(z_1,z'),\qquad(17)$$

where

$$S_\omega(z,z')=T_z\exp\left[i\int_{z'}^{z}\left(\sigma_3\omega-\frac{1}{2}\sigma^{(+)}\zeta(z_1)+\frac{1}{2}\sigma^{(-)}\zeta^*(z_1)\right)\frac{dz_1}{v}\right],\qquad(18)$$

The T_z-product means that if we changed from an integral to a sum, we would obtain S in the form of a product of factors. These factors, in contrast to the field η, do not commute and must be arranged from right to left in order of increasing z.

In Eq. (18) for S we can change to the interaction representation in ω, i.e., we can write

$$S_\omega(z,z')=\exp(i\omega\sigma_3z/v)\tilde{S}(z,z',[\zeta_\omega,\zeta_\omega^*])\cdot\exp(-i\omega\sigma_3z'/v),\qquad(18')$$

where

$$\frac{\sigma^{(+)}}{2}\zeta_\omega(z_i)=\exp\left(\frac{-i\omega\sigma_3z_i}{v}\right)\frac{\sigma^{(+)}}{2}\zeta(z_i)\exp\left(\frac{i\omega\sigma_3z_i}{v}\right)=\frac{\sigma^{(+)}}{2}\zeta(z_i)\exp\left(-\frac{2i\omega z_i}{v}\right).$$

The matrix $\tilde{S}$ is formally described by Eq. (18) but with $\omega=0$ and replacing ζ, ζ^* by ζ_ω, ζ_ω^*.

Let now the order of the points be $z'>z>z_1$. In that case it is clear that the connection between $G_{\omega R}(z,z')$ and $G_{\omega R}(z_1,z')$ remains as before (see (17), z' plays simply the role of a parameter). If we put the points in the order $z>z'>z_1$ we see that integrating from z_1 to z

we collect on the interval (z', z_1) a factor $S_\omega(z', z_1)$ relative to $G_{\omega R}(z_1, z')$. But after that the function has a jump by $-i\sigma_3/v$ and yet later the factor $S_\omega(z, z')$ is collected relative to the function with this contribution. We thus have for $z > z' > z_1$

$$G_{\omega R}(z, z') = S_\omega(z, z_1) G_{\omega R}(z_1, z') - iS_\omega(z, z')\sigma_3/v. \tag{19}$$

Equation (16) and Eqs. (17) and (19) which are simply the integral form of that first equation are valid for the retarded and for the advanced functions. We must now introduce boundary conditions in order to determine just the retarded function. We consider the structure of the perturbation theory series, e.g., for $G_{11}(z, z')$. The separate terms have the form

$$\int G^{(0)}_{\omega R11}(z-z_1)\zeta(z_1)G^{(0)}_{\omega R22}(z_1-z_2)\zeta^*(z_2) \times G^{(0)}_{\omega R11}(z_2-z_3)\ldots\zeta^*(z_n)G^{(0)}_{\omega R11}(z_n-z')\,dz_1\ldots dz_n. \tag{20}$$

It is important that each product under the integral sign starts with $G^{(0)}_{\omega R11}$ and ends with $G^{(0)}_{\omega R11}$. The integral over all z_i are from $-\infty$ to $+\infty$. However, according to (13),

$$G^{(0)}_{\omega R11}(z-z_1) = -\frac{i}{v}\theta(z-z_1)\exp\left[\frac{i\omega}{v}(z-z_1)\right],$$

so that the integral over z_1 goes from $-\infty$ to z, and the integral over z_n from z' to ∞. Hence, as $z \to -\infty$ or as $z' \to \infty$ any such term turns to zero. This refers also to the zeroth term. Thus

$$G_{\omega R11}(z, z') = 0 \quad \text{as} \quad z \to -\infty \quad \text{or} \quad z' \to \infty.$$

One can similarly show that

$$\begin{aligned} &G_{\omega R22}(z, z') = 0 \quad \text{as} \quad z \to \infty \quad \text{or} \quad z' \to -\infty,\\ &G_{\omega R12}(z, z') = 0 \quad \text{as} \quad z \to -\infty \quad \text{or} \quad z' \to -\infty,\\ &G_{\omega R21}(z, z') = 0 \quad \text{as} \quad z \to \infty \quad \text{or} \quad z' \to \infty. \end{aligned} \tag{21}$$

This just gives us the boundary conditions for the functions $G_{\omega R\alpha\beta}$.

Let in (19) $z_1 \to -\infty$; we then have because of the boundary conditions (21)

$$G_{\omega R\alpha\beta}(z, z') = S_{\omega\alpha 2}(z, -\infty)G_{\omega R2\beta}(-\infty, z') - i[S_\omega(zz')\sigma_3]_{\alpha\beta}\theta(z-z')/v.$$

We put $\alpha = 2$ and $z \to \infty$. The left-hand side then vanishes and we have

$$G_{\omega R2\beta}(-\infty, z') = i[S_\omega(\infty, z')\sigma_3]_{2\beta}/vS_{\omega 22}(\infty, -\infty).$$

Substituting this relation into the preceding equation we find

$$G_{\omega R\alpha\beta}(z, z') = \frac{i}{v}\left\{\frac{S_{\omega\alpha 2}(z, -\infty)[S_\omega(\infty, z')\sigma_3]_{2\beta}}{S_{\omega 22}(\infty, -\infty)} - [S_\omega(z, z')\sigma_3]_{\alpha\beta}\theta(z-z')\right. . \tag{22}$$

The matrix $S(z, z')$ is determined in (18) for $z > z'$. According to (22) this is sufficient since we only need such a matrix.

Expression (22) is the solution we need which can be substituted in any product of the G_R and be averaged. We illustrate how this is done using as an example a single G_R function. We use Eq. (18′) for all S_ω and replace the integrals by sums. Each S will then be a product of factors containing $\zeta(z_i)$ in different points. We write

$$\zeta_i = -i\zeta(z_i)\frac{\Delta}{v}\exp\left(\frac{-2i\omega z_i}{v}\right).$$

Each factor in S connected with a fixed point can be expanded in ζ_i and ζ_i^* restricting ourselves to the second order:

$$\exp[{}^1/_2(\sigma^{(+)}\zeta_i + \sigma^{(-)}\zeta_i^*)] \approx 1 + {}^1/_2(\sigma^{(+)}\zeta_i + \sigma^{(-)}\zeta_i^*) + {}^1/_8|\zeta_i|^2.$$

We have neglected here also terms such as ζ_i^2 and ζ_i^{*2}. We consider the function $S_{22}(z, z')$. It is clear that it will be of the form

$$S_{22}(z, z') = \prod_{k\in(z,z')}\left(1 + \frac{1}{2}|\zeta_k|^2\right) + \sum_{l>i}\zeta_l^*\zeta_i \prod_{\substack{k\neq i,l\\ i,k,l\in(z,z')}}\left(1 + \frac{1}{2}|\zeta_k|^2\right) + \sum_{l>i>m>n}\zeta_l^*\zeta_i\zeta_m^*\zeta_n \prod_{\substack{k\neq i,m,n,l\\ i,l,k,m,n\in(z,z')}}\left(1 + \frac{1}{2}|\zeta_k|^2\right) + \ldots .$$

We split off

$$\prod_{k\in(z,z')}\left(1 + \frac{1}{2}|\zeta_k|^2\right)$$

and use again the rule about not retaining more than the bilinear terms such as $\zeta_i\zeta_i^*$ in one point. We have

$$S_{22}(z, z') \approx \prod_{k\in(z,z')}\left(1 + \frac{1}{2}|\zeta_k|^2\right)\left(1 + \sum_{l>i}\zeta_l^*\zeta_i + \sum_{l>i>m>n}\zeta_l^*\zeta_i\zeta_m^*\zeta_n\right).$$

It is convenient to use this formula. However, in Eq. (22) there occurs $[S_{22}(\infty, -\infty)]^{-1}$. It must be expanded in ζ_i. Up to terms of higher order we have

$$\left[\prod_{k\in(z,z')}\left(1 + \frac{1}{2}|\zeta_k|^2\right)\right]^{-1} \approx \prod_{k\in(z,z')}\left(1 - \frac{1}{2}|\zeta_k|^2\right). \tag{24}$$

However, the second factor in (23) is appreciably more difficult because in the expansion there arise products of separate terms in which partial cancellation may occur, i.e., indices of ζ and ζ^* may be the same, and also there may be an interchange of ζ^* and ζ. It is, however, important that in all terms the factor on the extreme left (i.e., with the largest z) is ζ^* and the one on the extreme right ζ.

In analogy with Eq. (23) we find

$$S_{12}(z, z') = \prod_{k\in(z,z')}\left(1 + \frac{1}{2}|\zeta_k|^2\right)\left(\sum_{i\in(z,z')}\zeta_i + \sum_{\substack{l>m>i\\ l,m,i\in(z,z')}}\zeta_l\zeta_m^*\zeta_i + \ldots\right), \tag{25}$$

$$S_{21}(z, z') = \prod_{k\in(z,z')}\left(1 + \frac{1}{2}|\zeta_k|^2\right)\left(\sum_{i\in(z,z')}\zeta_i^* + \sum_{\substack{l>m>i\\ l,m,i\in(z,z')}}\zeta_l^*\zeta_m\zeta_i^* + \ldots\right), \tag{26}$$

$$S_{11}(z, z') = \prod_{k\in(z,z')}\left(1 + \frac{1}{2}|\zeta_k|^2\right)\left(1 + \sum_{\substack{i>l\\ i,l\in(z,z')}}\zeta_i\zeta_l^* + \ldots\right). \tag{27}$$

We now take G_{22} according to Eq. (22) and average over ζ_i and ζ_i^*. First of all we note that in each term of the second factor of (23) all ζ_i are different. They can therefore not be averaged with one another. Since the last term in (22) is simply $S_{22}(z, z')$ we get, when it is averaged, only unity as the contribution from the second factor in (23). In the first term we might expect compensation of these factors. However, only $\langle|\zeta_i|^2\rangle$ is non-vanishing and according to (23) and (24) in the second factor in S_{22} and in $(S_{22})^{-1}$ all terms except unity start with ζ_i^*. There is then a ζ_i^* with the largest value of z_i for which there is no corresponding ζ_i. Hence it follows that it again gives unity as its contribution. After that it is now easy to see that when $z > z'$ the two terms in (22) cancel one another and when $z < z'$ we get (we took into account that $\langle|\zeta_i|^2\rangle = \Delta/v\tau_2$ and we introduced the "external" factors from Eq. (18'))

$$G_{\omega R22}(z,z')=\prod_{k\in(z,z')}\left(1-\frac{1}{2}\langle|\zeta_k|^2\rangle\right)G_{\omega R22}^{(0)}(z-z')$$
$$=\left(1-\frac{\Delta}{2v\tau_2}\right)^{|z-z'|/\Delta}G_{\omega R22}^{(0)}\to\exp\left(-\frac{|z-z'|}{2w\tau_2}\right)G_{\omega R22}^{(0)}. \tag{28}$$

Equation (22) is inconvenient to evaluate G_{R11}. In view of this we introduce the matrix $S_\omega^{-1}(z, z')$ which is defined such that $S_\omega^{-1}(z, z')S_\omega(z, z') = 1$. It is clear that

$$S_\omega^{-1}(z,z')=T_z^{-1}\exp\left[\frac{i}{v}\int_{z'}^{z}\left(-\omega\sigma_3+\frac{1}{2}\sigma^{(+)}\zeta(z_1)-\frac{1}{2}\sigma^{(-)}\zeta^*(z_1)\right)dz_1\right]. \tag{29}$$

Changing to the interaction representation in ω we can from this get

$$S_{\omega\alpha\beta}^{-1}(z,z',[\zeta,\zeta^*])=\exp[i\omega(z\sigma_{3\beta}-z'\sigma_{3\alpha})/v]\tilde{S}_{\alpha\beta}(z,z'[\zeta_\omega,\zeta_\omega^*]), \tag{29'}$$

where $\tilde{S}_{\alpha\beta}^{-1}(z, z'[\zeta_\omega, \zeta_\omega^*])$ corresponds to the matrix (29) with $\omega = 0$ and $\zeta(z)\to\zeta_\omega(z) = \zeta(z)\exp(-2i\omega z/v)$. When we use the discrete notation the factors in T_z^{-1} stand in the reverse order.

We multiply Eq. (19) from the left by $S_\omega^{-1}(z, z_1)$. When $z > z' > z_1$ we get

$$S_\omega^{-1}(z,z_1)G_{\omega R}(z,z')=G_{\omega R}(z,z')-iS_\omega^{-1}(z',z_1)\sigma_3/v.$$

Transposing terms and making the change $z\rightleftarrows z_1$ we have

$$G_{\omega R}(z,z')=S_\omega^{-1}(z_1,z)G_{\omega R}(z_1,z')+iS_\omega^{-1}(z',z)\sigma_3/v$$

when $z_1 > z' > z$. If, however, $z_1 > z > z'$ the last term on the right-hand side vanishes.

We now let $z_1\to\infty$. Using the boundary relation (21) we find

$$G_{\omega R\alpha\beta}(z,z')=S_{\omega\alpha1}^{-1}(\infty,z)G_{\omega R1\beta}(\infty,z')+i[S_\omega^{-1}(z',z)\sigma_3]_{\alpha\beta}\theta(z'-z)/v.$$

We take $\alpha = 1$, $z\to-\infty$. We then get, using (21)

$$G_{\omega R1\beta}(\infty,z')=-i[S_\omega^{-1}(z',-\infty)\sigma_3]_{1\beta}/vS_{\omega11}^{-1}(\infty,-\infty).$$

Substituting this expression into the previous equation we have

$$G_{\omega R\alpha\beta}(z,z')=-\frac{i}{v}\left\{\frac{S_{\omega\alpha1}^{-1}(\infty,z)[S_\omega^{-1}(z',-\infty)\sigma_3]_{1\beta}}{S_{\omega11}^{-1}(\infty,-\infty)}-[S_\omega^{-1}(z,z')\sigma_3]_{\alpha\beta}\theta(z'-z)\right\}. \tag{30}$$

In this form one can easily find $G_{\omega R11}$ and more difficultly $G_{\omega R22}$. For $G_{\omega R11}$ we get clearly the formula

$$G_{\omega R11}(z,z')=\exp(-|z-z'|/2v\tau_2)G_{\omega R11}^{(0)}(z-z'). \tag{31}$$

We give a few relations for unaveraged functions which turn out to be useful in what follows. From the definitions (18), (18') and (28), (29') we can up to terms in $|\zeta|^2$ get

$$S_{\omega11}^{-1}(z,z')=S_{\omega22}(z,z'),\quad S_{\omega12}^{-1}(z,z')=-S_{\omega12}(z,z'),$$
$$S_{\omega22}^{-1}(z,z')=S_{\omega11}(z,z'),\quad S_{\omega21}^{-1}(z,z')=-S_{\omega21}(z,z'). \tag{32}$$

In particular, it follows from these formulae that

$$S_{\omega11}(z,z')S_{\omega22}(z,z')-S_{\omega12}(z,z')S_{\omega21}(z,z')=1. \tag{33}$$

Using Eqs. (32) and the two definitions of $G_{\alpha\beta}$, and especially (22) and (30) we get the following relations:

$$G_{\omega R11}(z,z')=G_{\omega R22}(z',z),\quad G_{\omega R12}(z,z')=G_{\omega R12}(z',z),$$
$$G_{\omega R21}(z,z')=G_{\omega R21}(z',z). \tag{34}$$

In what follows we also need the functions G_A. By analogy with the preceding we easily get the formula

$$G_{\omega A\alpha\beta}(z,z')=\frac{i}{v}\left\{\frac{S_{\omega\alpha1}(z,-\infty)[S_\omega(\infty,z')\sigma_3]_{1\beta}}{S_{\omega11}(\infty,-\infty)}-[S_\omega(z,z')\sigma_3]_{\alpha\beta}\theta(z-z')\right\}. \tag{35}$$

As before, restricting ourselves to $|\zeta|^2$ terms, we get the relations

$$S_{\omega11}(z,z',[\zeta,\zeta^*])=S_{-\omega22}(z,z',[\zeta^*,\zeta]),$$
$$S_{\omega12}(z,z',[\zeta,\zeta^*])=S_{-\omega21}(z,z',[\zeta^*,\zeta]). \tag{36}$$

Using Eqs. (22) and (35), (36) we get from this

$$G_{\omega A11}(z,z',[\zeta,\zeta^*])=-G_{-\omega R22}(z,z',[\zeta^*,\zeta]),$$
$$G_{\omega A12}(z,z',[\zeta,\zeta^*])=-G_{-\omega R21}(z,z',[\zeta^*,\zeta]),$$
$$G_{\omega A22}(z,z',[\zeta,\zeta^*])=-G_{-\omega R11}(z,z',[\zeta^*,\zeta]),$$
$$G_{\omega A21}(z,z',[\zeta,\zeta^*])=-G_{-\omega R12}(z,z',[\zeta^*,\zeta]). \tag{37}$$

We can then always express G_A in terms of G_R, changing from ζ to ζ^* and vice versa. It is, however, somewhat better to use directly Eq. (35).

Finally we introduce the matrix S^+. It is clear that

$$S_\omega^+(z,z')=T_z^{-1}\exp\left[-i\int_{z'}^{z}\left(\sigma_3\omega+\frac{1}{2}\sigma^{(+)}\zeta(z_1)-\frac{1}{2}\sigma^{(-)}\zeta^*(z_1)\right)\frac{dz_1}{v}\right]. \tag{38}$$

One sees easily that

$$S_{\omega11}^+=S_{\omega22},\quad S_{\omega22}^+=S_{\omega11},\quad S_{\omega12}^+=S_{\omega12},\quad S_{\omega21}^+=S_{\omega21}. \tag{39}$$

We note that according to (32) the matrix S is non-unitary.

§2. ELECTRICAL PROPERTIES OF A ONE-DIMENSIONAL GAS

Let there be an electromagnetic field described by a vector potential $A(\omega_0)e^{-i\omega_0 t}$. By the standard method (see[3]) we get for the current

$$j(\omega_0, z) = \int Q_R(\omega_0, z, z_1) A(\omega_0, z_1) dz_1, \qquad (40)$$

where

$$Q_R = \begin{cases} Q, & \omega_0 > 0 \\ Q^*, & \omega_0 < 0 \end{cases},$$

$$Q = -\frac{e^2}{mc} n_e - i \frac{e^2}{4m^2c}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)\left(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_1'}\right) \times \int G_{\omega+\omega_0}(z, z_1') G_\omega(z_1, z') dz_1 \, d\omega/2\pi, \qquad (41)$$

where $z_1' \to z_1$, $z' \to z$.

We add to and subtract from the second term the analogous expression for the metal without impurities. The difference is then a convergent integral in which the vicinity of the Fermi surface and $\omega \approx 0$ are important. In view of this we can use in them the formulae for G obtained above. In that case

$$\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)\left(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_1'}\right) G_{\omega+\omega_0}(z, z_1') G_\omega(z_1, z') \to -4p_0^2 \, \mathrm{Sp}[\sigma_3 G_{\omega+\omega_0}(z, z_1) \sigma_3 G_\omega(z_1, z)]. \qquad (42)$$

We consider one of the terms in (42), e.g., $G_{\omega+\omega_0 11}(z, z_1) G_{\omega 11}(z_1, z)$. Since $G_\omega = G_{A\omega}$ when $\omega < 0$ and $G_\omega = G_{R\omega}$ when $\omega > 0$, the integral over the frequency of that product equals ($\omega_0 > 0$)

$$\int G_{\omega+\omega_0 11}(z, z_1) G_{\omega 11}(z_1, z) \frac{d\omega}{2\pi} = \int_{-\infty}^{-\omega_0} G_{A\omega+\omega_0 11}(z, z_1) G_{A\omega 11}(z_1, z) \frac{d\omega}{2\pi} + \int_{-\omega_0}^{0} G_{R\omega+\omega_0 11}(z, z_1) G_{A\omega 11}(z_1, z) \frac{d\omega}{2\pi} + \int_0^\infty G_{R\omega+\omega_0 11}(z, z_1) G_{R\omega 11}(z_1, z) \frac{d\omega}{2\pi}. \qquad (43)$$

We show that the first and third terms of that sum give zero when averaged over ζ. Indeed, let $z > z_1$. Using Eqs. (34) which are the same for G_R and G_A we can rewrite the first term so that it contains only $G_A(z_1, z)$. We have

$$\int_{-\infty}^{-\omega_0} G_{A\omega+\omega_0 22}(z_1, z) G_{A\omega 11}(z_1, z) \frac{d\omega}{2\pi}.$$

We substitute Eq. (35). We then get

$$\frac{1}{v^2} \int_{-\infty}^{-\omega_0} S_{\omega+\omega_0 21}(z_1, -\infty) S_{\omega+\omega_0 12}(\infty, z) S_{\omega 11}(z_1, -\infty) S_{\omega 11}(\infty, z) \times [S_{\omega+\omega_0 11}(\infty, -\infty) S_{\omega 11}(\infty, -\infty)]^{-1} \frac{d\omega}{2\pi}.$$

Since $z > z_1$ we see that the numerator contains at least one "unpaired" ζ in the interval (∞, z) and one ζ^* in the interval $(z_1, -\infty)$. This pair cannot be compensated when the denominator is taken into account as in it ζ always stands to the left and ζ^* on the right.

One can show that the same occurs when $z < z_1$. Similarly this can be shown for all other terms in Eq. (42) corresponding to the first and third terms in (43). There remains therefore solely the second term in Eq. (43) and similar terms with other indices in (42).

It is now relevant to remember that we subtracted and added the expression similar to the second term in (41) for a metal without impurities. As a whole we get together with the first terms in (41) from this addition to Q:

$$-\frac{e^2}{mc} n_e - \frac{ie^2 \cdot 4p_0^2}{4m^2c} \int_{-\omega_0}^{0} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} [G_{R11}^{(0)}(\omega+\omega_0, p_z) G_{A11}^{(0)}(\omega, p_z) + G_{R22}^{(0)}(\omega+\omega_0, p_z) G_{A22}^{(0)}(\omega, p_z)] + \frac{ie^2 p_0^2}{m^2 c} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} [G_{11}^{(0)}(\omega+\omega_0, p_z) \times G_{11}^{(0)}(\omega, p_z) + G_{22}^{(0)}(\omega+\omega_0, p_z) G_{22}^{(0)}(\omega, p_z)].$$

The last term gives zero when we take the integrals in the correct order (i.e., first over ω and then over p_z). The second term cancels the first term after integration, as $n_e = p_0/\pi$.

The total expression for Q thus has the form

$$Q = \frac{ie^2 v^2}{c} \int_{-\omega_0}^{0} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} dz_1 \, \mathrm{Sp}(\sigma_3 G_{R\omega+\omega_0}(z, z_1) \sigma_3 G_{A\omega}(z_1, z)), \qquad (44)$$

where we must substitute for G the Green functions with impurities found above. We split the integration over z_1 into the regions $z_1 > z$ and $z_1 < z$ and using Eqs. (34) we express everything in terms of G where the smaller of z and z_1 stands as the first argument. We then get

$$Q = \frac{ie^2 v^2}{c} \int_{-\omega_0}^{0} \frac{d\omega}{2\pi} \Big\{ \int_{-\infty}^{z} [G_{R\omega+\omega_0 22}(z_1, z) G_{A\omega 11}(z_1, z) + G_{R\omega+\omega_0 11}(z_1, z) G_{A\omega 22}(z_1, z) - G_{R\omega+\omega_0 12}(z_1, z) G_{A\omega 21}(z_1, z) - G_{R\omega+\omega_0 21}(z_1, z) G_{A\omega 12}(z_1, z)] dz_1 + \int_z^\infty [G_{R\omega+\omega_0 11}(z, z_1) G_{A\omega 22}(z, z_1) + G_{R\omega+\omega_0 22}(z, z_1) G_{A\omega 11}(z, z_1) - G_{R\omega+\omega_0 12}(z, z_1) G_{A\omega 21}(z, z_1) - G_{R\omega+\omega_0 21}(z, z_1) G_{A\omega 12}(z, z_1)] dz_1 \Big\}. \qquad (45)$$

We use Eqs. (22) and (35) for G. We also use the fact that when we substitute S in the form (18′) and average over ζ we can assume the integrand to be a function of $\omega + \omega_0 - \omega = \omega_0$ only:

$$Q = \frac{ie^2 \omega_0}{2\pi c} \Big\{ \int_{-\infty}^{z} [S_{\omega_0 22}(z_1, -\infty) S_{11}(z_1, -\infty) - S_{\omega_0 12}(z_1, -\infty) S_{21}(z_1, -\infty)] \times [S_{\omega_0 22}(\infty, z) S_{11}(\infty, z) - S_{\omega_0 21}(\infty, z) S_{12}(\infty, z)] dz_1 + \int_z^\infty [S_{\omega_0 22}(z, -\infty) S_{11}(z, -\infty) - S_{\omega_0 12}(z, -\infty) S_{21}(z, -\infty)] \times [S_{\omega_0 22}(\infty, z_1) S_{11}(\infty, z_1) - S_{\omega_0 21}(\infty, z_1) S_{12}(\infty, z_1)] dz_1 \Big\} \times [S_{\omega_0 22}(\infty, -\infty) S_{11}(\infty, -\infty)]^{-1}. \qquad (46)$$

The formula obtained contains integrals of averages such as

$$A_{\omega_0}(z_1, z_2, z_3, z_4) = [S_{22\omega_0}(z_1, z_2) S_{11}(z_1, z_2) - S_{21\omega_0}(z_1, z_2) S_{12}(z_1, z_2)][S_{22\omega_0}(z_3, z_4) S_{11} \cdot (z_3, z_4) - S_{12\omega_0}(z_3, z_4) S_{21}(z_3, z_4)] / [S_{22\omega_0}(z_1, z_4) S_{11}(z_1, z_4)], \qquad (47)$$

where $z_1 > z_2 > z_3 > z_4$, and $A_{\omega_0}(z_1, z_2, z_3, z_4) \to A_{\omega_0}(z_2, z_3, z_4)$ as $z_1 \to z_2$:

$$A_{\omega_0}(z_2, z_3, z_4) = [S_{22\omega_0}(z_3, z_4) S_{11}(z_3, z_4) - S_{12\omega_0}(z_3, z_4) S_{21}(z_3, z_4)] / [S_{22\omega_0}(z_2, z_4) \cdot S_{11}(z_2, z_4)]. \qquad (48)$$

$$A(z_2, z_3, z_4) \to A(z_3, z_4) \quad \text{as} \quad z_2 \to z_3:$$
$$A_{\omega_0}(z_3, z_4) = [S_{22\omega_0}(z_3, z_4) S_{11}(z_3, z_4) - S_{12\omega_0}(z_3, z_4) S_{21}(z_3, z_4)] / [S_{22\omega_0}(z_3, z_4) \cdot S_{11}(z_3, z_4)]. \tag{49}$$

Finally, $A(z_3, z_4) \to 1$ as $z_3 \to z_4$.

We use these properties as follows. We get a differential equation to determine $A(z_3, z_4)$ as function of $z_3 - z_4$ and, solving it we find $A(z_3, z_4)$. After that we get a differential equation for $A(z_2, z_3, z_4)$ in the variable z_2 and we use the value of $A(z_3, z_4)$ which we have found as a boundary condition. Finally, we find $A(z_1, z_2, z_3, z_4)$ in a similar way.

This is the general idea. However, it is not possible to carry it out directly because the method which we apply works only when the external variable (in (47) this is z_1 and z_4) occurs only in one kind of $S_{\alpha\beta}(z_i, z_k)$, i.e., say, $S_{\alpha\beta}(z_1, z_2)$. These variables occur in Eq. (47) at once in two kinds of $S_{\alpha\beta}$, say $S(z_1, z_2)$ and $S(z_1, z_4)$. To avoid this we write $[S_{22\omega_0}(z_1, z_4)S_{11}(z_1, z_4)]^{-1}$ in the following form:

$$[S_{22\omega_0}(z_1, z_4) S_{11}(z_1, z_4)]^{-1} = [S_{22\omega_0}(z_1, z_2) S_{22\omega_0}(z_2, z_4) + S_{21\omega_0}(z_1, z_2) \times S_{12\omega_0}(z_2, z_4)]^{-1} [S_{11}(z_1, z_2) S_{11}(z_2, z_4) + S_{12}(z_1, z_2) S_{21}(z_2, z_4)]^{-1}$$
$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} (-1)^{n_1+n_2} \frac{[S_{21\omega_0}(z_1, z_2) S_{12\omega_0}(z_2, z_4)]^{n_1}}{[S_{22\omega_0}(z_1, z_2) S_{22\omega_0}(z_2, z_4)]^{n_1+1}} \frac{[S_{12}(z_1, z_2) S_{21}(z_2, z_4)]^{n_2}}{[S_{11}(z_1, z_2) S_{11}(z_2, z_4)]^{n_2+1}}. \tag{50}$$

We further draw attention to the fact that in the first factor under the summation sign there occur necessarily n_1 unpaired ζ^* in the interval (z_1, z_2). This circumstance remains also in the case where we take into account the first factor in (47) referring to the same interval. Hence these unpaired ζ^* can on averaging be compensated only if we take into account the ζ occurring in the second term in the sum in (50), i.e., after averaging there remain only terms with $n_1 = n_2$. Thus

$$[S_{22\omega_0}(z_1, z_4) S_{11}(z_1, z_4)]^{-1} \to \sum_{n=0}^{\infty} \frac{[S_{21\omega_0}(z_1, z_2) S_{12}(z_1, z_2)]^n}{[S_{22\omega_0}(z_1, z_2) S_{11}(z_1, z_2)]^{n+1}} \times \frac{[S_{12\omega_0}(z_2, z_4) S_{21}(z_2, z_4)]^n}{[S_{22\omega_0}(z_2, z_4) S_{11}(z_2, z_4)]^{n+1}}.$$

Equation (47) becomes

$$A(z_1, z_2, z_3, z_4) = \sum_{n=0}^{\infty} [B_n(z_1, z_2) - B_{n+1}(z_1, z_2)] C_n(z_2, z_3, z_4), \tag{51}$$

where

$$B_n(z_1, z_2) = \frac{[S_{21\omega_0}(z_1, z_2) S_{12}(z_1, z_2)]^n}{[S_{22\omega_0}(z_1, z_2) S_{11}(z_1, z_2)]^n}, \tag{52}$$

$$C_n(z_2, z_3, z_4) = \frac{[S_{12\omega_0}(z_2, z_4) S_{21}(z_2, z_4)]^n}{[S_{22\omega_0}(z_2, z_4) S_{11}(z_2, z_4)]^{n+1}} \times [S_{22\omega_0}(z_3, z_4) S_{11}(z_3, z_4) - S_{12\omega_0}(z_3, z_4) S_{21}(z_3, z_4)]. \tag{53}$$

We find below the equation which describes the z_2 dependence of C_n. The boundary condition for this equation will be the value as $z_2 \to z_3$, i.e., $B_n'(z_3, z_4) - B_{n+1}'(z_3, z_4)$, where

$$B_n'(z_3, z_4) = \left[\frac{S_{12\omega_0}(z_3, z_4) S_{21}(z_3, z_4)}{S_{22\omega_0}(z_3, z_4) S_{11}(z_3, z_4)}\right]^n. \tag{54}$$

When evaluating B_n' we can use the fact that the average is invariant under the transformation $\zeta \rightleftarrows \zeta^*$. Using Eqs. (36) and the fact that in expressions such as (54) we can add to each $S_{\alpha\beta}$ the same frequency we get

$$B_n'(z_3, z_4) = B_n(z_3, z_4). \tag{55}$$

We consider $B_n(0, z)$ $(z < 0)$ and substitute $S_{\alpha\beta}$ in the interaction representation (18′). We then get

$$B_n(0, z) = \left[\frac{\tilde{S}_{21\omega_0}(0, z) \tilde{S}_{12}(0, z)}{\tilde{S}_{22\omega_0}(0, z) \tilde{S}_{11}(0, z)}\right]^n \exp\left(-\frac{2i\omega_0 nz}{v}\right).$$

We substitute for $\tilde{S}_{\alpha\beta}$ Eqs. (23), (25)–(27). The factors $\Pi_i(1 + \frac{1}{2}|\zeta_i|^2)$ in the numerator and in the denominator then cancel. We split off the dependence on the ζ on the extreme right, which we denote by ζ_1. We then get

$$B_n(0, z) = [\tilde{S}_{21\omega_0}(0, z+\Delta) + \zeta_1^* e^{2i\omega_0 z/v} \tilde{S}_{22\omega_0}(0, z+\Delta)]^n [\tilde{S}_{12}(0, z+\Delta) + \tilde{S}_{11}(0, z+\Delta)\zeta_1]^n \cdot [\tilde{S}_{22\omega_0}(0, z+\Delta) + \zeta_1 e^{-2i\omega_0 z/v} \tilde{S}_{21\omega_0}(0, z+\Delta)]^{-n} [S_{11}(0, z+\Delta) + S_{12}(0, z+\Delta)\zeta_1^*]^{-n} \cdot e^{-2i\omega_0 nz/v}. \tag{56}$$

We expand up to terms of first order in ζ_1 and ζ_1^* and retain terms which do not contain ζ_1 or are proportional to $|\zeta_1|^2$. In the term which does not contain ζ_1 we write the factor $e^{-2i\omega_0 nz/v}$ in the form $e^{-2i\omega_0 n(z+\Delta)/v}(1 + 2i\omega_0 n\Delta/v)$. After that we substitute $\langle |\zeta_1|^2 \rangle = \Delta/v\tau_2$. Equation (56) becomes

$$B_n(0, z) = B_n(0, z+\Delta)(1 + 2i\omega_0 n\Delta/v) + n^2(\Delta/v\tau_2)[B_{n-1}(0, z+\Delta) + B_{n+1}(0, z+\Delta) - 2B_n(0, z+\Delta)].$$

This recurrence relation becomes as $\Delta \to 0$ a differential equation. Introducing the variable $t = -z/v\tau_2 > 0$, we get

$$\partial B_n/\partial t = n^2(B_{n-1} + B_{n+1} - 2B_n) + i\beta n B_n, \tag{57}$$

where $\beta = 2\omega_0\tau_2$.

We now introduce the generating function

$$B(x, t) = \sum_{n=1}^{\infty} B_n(t) x^{n-1}. \tag{58}$$

For this function we get the equation (using the fact that $B_0(t) = 1$ by definition)

$$\frac{\partial B}{\partial t} = i\beta \frac{\partial}{\partial x}(xB) + 1 + \frac{\partial}{\partial x}\left[x \frac{\partial}{\partial x}(B(1-x)^2)\right]. \tag{59}$$

The boundary condition for Eq. (59) is the value at $t = 0$. According to (52) we find that $B_n(0) = \delta_{n,0}$. Hence, from (58), $B(x, 0) = 0$. Moreover, the solution must be regular as $x \to 0$.

We perform the following transformation of variables. We write:

$$u = x/(1-x), \quad R = B/(u+1)^2. \tag{60}$$

Equation (59) now becomes

$$\frac{\partial R}{\partial t} = \frac{1}{(u+1)^2} + \frac{\partial}{\partial u} u(u+1) \frac{\partial R}{\partial u} + i\beta \frac{\partial}{\partial u} u(u+1) R. \tag{61}$$

We note that $x = 0$ corresponds to $u = 0$, while $x \to 1 - 0$

corresponds to $u\to\infty$. We get rid of the inhomogeneous term. To do that we put

$$R(t,\ u)=R_1(t,\ u)+r(u) \tag{62}$$

and we choose $r(u)$ such that the inhomogeneous term is cancelled. The equation for $r(u)$ has a solution that is regular as $u\to 0$ and does not grow as $u\to\infty$, in the form

$$r(u)=\frac{1}{u+1}+i\beta\int_0^\infty\frac{e^{i\beta u_1}du_1}{u_1+u+1}=\frac{1}{u+1}-i\beta e^{-i\beta(1+u)}\{\mathrm{ci}[\beta(1+u)]+i\,\mathrm{si}[\beta(1+u)]\}. \tag{63}$$

The boundary condition for $R(t, u)$ is $R(0, u)=0$. Hence $R_1(t, u)$ satisfies the homogeneous equation corresponding to (61), namely,

$$\frac{\partial R_1}{\partial t}=\frac{\partial}{\partial u}u(u+1)\frac{\partial R_1}{\partial u}+i\beta\frac{\partial}{\partial u}u(u+1)R_1 \tag{64}$$

with the boundary condition

$$R_1(0,\ u)=-r(u). \tag{65}$$

We Laplace transform (64) with respect to t:

$$R_1(u,t)=\int_0^\infty e^{-st}R_1(u,s)\,ds,$$

$$\frac{\partial}{\partial u}u(u+1)\frac{\partial R_1(u,s)}{\partial u}+i\beta\frac{\partial}{\partial u}u(u+1)R_1(u,s)+sR_1(u,s)=0. \tag{66}$$

It is difficult to solve this equation in the general case. We consider first of all the case $\beta=0$. R_1 then satisfies the hypergeometric equation and its solution is

$$R_\lambda=F(1/2+i\lambda,\ 1/2-i\lambda,\ 1,\ -u),\quad s=1/4+\lambda^2. \tag{67}$$

These functions have orthogonality properties, namely

$$\int_0^\infty R_\lambda(u)R_{\lambda'}(u)\,du=\delta(\lambda-\lambda')/2\lambda\,\mathrm{th}\,\pi\lambda. \tag{68}$$

This makes it possible to satisfy the boundary condition easily. According to (63) (with $\beta=0$) and (65)

$$\int_0^\infty R_1(u,s)\,ds=-1/(1+u).$$

By virtue of (68) we get

$$R_1(u,\lambda)=-2\lambda\,\mathrm{th}\,\pi\lambda R_\lambda(u)\int_0^\infty\frac{R_\lambda(u_1)\,du_1}{1+u_1}.$$

Evaluating the integral we find

$$R_1(u,t)=-\int_0^\infty d\lambda\frac{2\pi\lambda\,\mathrm{sh}\,\pi\lambda}{\mathrm{ch}^2\pi\lambda}e^{-(1/4+\lambda^2)t}F(1/2+i\lambda,1/2-i\lambda,1,-u). \tag{69}$$

The function $R(u, t)$ determines the static conductivity. Indeed, in view of the fact that $j=\sigma E=(\sigma i\omega_0/c)A$ we have

$$\sigma_0=\lim_{\omega_0\to 0}\frac{Q(\omega_0)c}{i\omega_0}. \tag{70}$$

As $\omega_0\to 0$ the expressions in the brackets in the integrals in (46) become, according to (33), equal to unity. For a conductor of length L we get therefore

$$\sigma_0=\frac{e^2L}{2\pi}\langle[S_{22}(L,0)S_{11}(L,0)]^{-1}\rangle. \tag{71}$$

However, according to (33)

$$[S_{22}(L,0)S_{11}(L,0)]^{-1}=\frac{S_{22}(L,0)S_{11}(L,0)-S_{12}(L,0)S_{21}(L,0)}{S_{22}(L,0)S_{11}(L,0)}=1-B_1(L,0).$$

From (61) it follows that

$$B(x,t)=\frac{1}{(1-x)^2}R\left(\frac{x}{1-x},t\right).$$

In view of that

$$B_1(t)=R(0,\ t)=r(0)+R_1(0,\ t)=1+R_1(0,\ t).$$

Collecting this all together we get

$$\sigma=-\frac{e^2L}{2\pi}R_1(0,t)=e^2L\int_0^\infty\lambda\,d\lambda\frac{\mathrm{sh}\,\pi\lambda}{\mathrm{ch}^2\pi\lambda}e^{-(1/4+\lambda^2)t},$$

where $t=L/v\tau_2$. When $t\gg 1$ small values of λ are important. Evaluating the integral we have

$$\sigma_0=(\pi^{3/2}/4)e^2(v\tau_2)^{3/2}L^{-1/2}\exp(-L/4v\tau_2). \tag{72}$$

The static conductivity is thus exponentially small for a large sample.

Mott[4] had already predicted that the static conductivity would vanish; it raises the problem of the evaluation of $\sigma(\omega_0)$. It follows from Eq. (69) that $R_1(u, t)$ decreases exponentially with t for large t. The same remains valid also when $\beta\neq 0$. Bearing this in mind, we consider Eq. (51). The values of the end arguments in which we are interested are $z_1=L$, $z_4=0$. The points z_2 and z_3 correspond to z and z_1 in (46). Below we shall see that the functions $C_n(z_2, z_3, z_4)$ decrease exponentially when $z_2-z_3\gg v\tau_2$. The distance between z and z_1 in (46) is then small, i.e., the current in the point z is determined by the field in a vicinity of the order of $v\tau_2$. It is clear that the conductivity can change with the coordinate only near the ends of the sample and as that effect is of no interest we must take z (and then also z_1) at a large distance from the points 0 and L. But it is then clear that the exponentially decreasing parts of the functions $B_n(z_1, z_2)$ and $B_n(z_3, z_4)$ do not contribute at all and can be limited only by parts independent of t. This relieves us of the necessity to evaluate R_1 for $\beta\neq 0$ and we can restrict ourselves to $R(u, t)=r(u)$, i.e., to Eq. (63).

We thus consider $C_n(z_2, z_3, z_4)$ as function of z_2. The equation for C_n can be obtained in a similar manner as we obtained the equation for B_n. However, we must take into account that the powers of S in the numerator and in the denominator are different and that in that connection there appears in the numerator a spare factor $(1-|\zeta_1|^2)$ and the total frequency factor is $e^{i\omega_0(2n+1)z/v}$. As a result the equation takes the form[2)]

$$dC_n/dt=n^2C_{n-1}+(n+1)^2C_{n+1}-2n(n+1)C_n -C_n+(n+{}^1/_2)i\beta C_n. \quad (73)$$

Introducing

$$C(x,t)=\sum_{n=0}^{\infty}C_n(t)x^n, \quad (74)$$

we get an equation for $C(x, t)$

$$\frac{\partial C}{\partial t}=x(x-1)^2\frac{\partial^2 C}{\partial x^2}+(x-1)(3x-1)\frac{\partial C}{\partial x}+(x-1)C+\frac{i\beta}{2}C-i\beta x\frac{\partial C}{\partial x}. \quad (75)$$

Making a change of variables

$$u=x/(1-x), \quad \Phi=C/(1+u), \quad (76)$$

we get an equation for $\Phi(u, t)$:

$$\frac{\partial\Phi}{\partial t}=\frac{\partial}{\partial u}u(u+1)\frac{\partial\Phi}{\partial u}+i\beta[u(u+1)]^{1/2}\frac{\partial}{\partial u}[u(u+1)]^{1/2}\Phi. \quad (77)$$

We note that this equation is homogeneous and therefore Φ in the final result decreases exponentially for large t.

The boundary condition for C_n for $z_2=z_3$ is $B_n(z_3, z_4) - B_{n+1}(z_3, z_4)$. Hence it follows that

$$C(t_2, t_3, t_4, x)=1-(1-x)B(t_3, t_4, x)$$

or, from (76)

$$\Phi(t_2, t_3, t_4, x)=1/(1+u)-R(t_3-t_4, x).$$

However, it follows from what has been said above that we must retain for R only the part which is independent of t_3-t_4 which is given by Eq. (63). Hence it follows that $\Phi(t_2, t_3, t_4, x)$ depends only on $t=t_2-t_3$ and satisfies for $t=0$ the condition

$$\Phi(0,u)=-i\beta\int_0^\infty\frac{\exp(i\beta u_1)}{u_1+u+1}du_1\equiv-r_1(u). \quad (78)$$

We now turn to Eq. (51). In order to perform the summation we use the following method:

$$A=\sum_n(B_n-B_{n+1})C_n=\frac{1}{2\pi}\int_0^{2\pi}d\varphi\sum_{n_1}(B_{n_1}-B_{n_1+1})e^{in_1\varphi} \times\sum_{n_2}C_{n_2}e^{-in_2\varphi}=\frac{1}{2\pi}\int_0^{2\pi}[1-(1-x_1)B(x_1)]C(x_2)d\varphi,$$

where $x_1=e^{i\varphi}$, $x_2=e^{-i\varphi}$. Substituting here the expressions for B and C in terms of R and Φ and changing to the variable $u=x_2/(1-x_2)=(e^{i\varphi}-1)^{-1}$ we get

$$A=\frac{i}{2\pi}\int_C du\, r_1[-(u+1)]\Phi(u), \quad (79)$$

where the contour C is a straight line parallel to the imaginary axis and intersecting the real axis at $u=-\frac{1}{2}$. The functions $C(x)$ and $\Phi(x)$ are analytical in a circle with radius $|x|=1$. The transformation $u=x/(1-x)$ maps the unit circle into the right-hand half-plane from the straight line $\mathrm{Re}\,u=-\frac{1}{2}$. The contour C can thus be displaced to the right. The residue in the integral (78) then gives a contribution and as a result we get

$$A=-i\beta\int_0^\infty e^{i\beta u}\Phi(u)du. \quad (80)$$

There occur two terms in Eq. (46). One sees easily that they are both equal to one another and are integrals of the expression A over t. Substituting (80) into Q we have

$$Q=\frac{e^2v}{2\pi c}\beta^2\int_0^\infty du\, e^{i\beta u}\int_0^\infty dt\,\Phi(u,t). \quad (81)$$

The solution of Eq. (77) is complicated in the general case. We therefore consider only the limiting cases: $\beta\ll1$ and $\beta\gg1$. We start with the case $\beta\ll1$. We perform the Laplace transform which is the inverse of the one we performed in (66); namely, we multiply (77) by e^{-st} and integrate from 0 to ∞. As a result we get

$$-\Phi(t=0)+s\Phi(s)=\frac{\partial}{\partial u}u(u+1)\frac{\partial\Phi(s)}{\partial u}+i\beta[u(u+1)]^{1/2}\frac{\partial}{\partial u}[u(u+1)]^{1/2}\Phi(s).$$

According to (81) we need know only $\Phi(s=0)$. Denoting this quantity by f and using the boundary condition for Φ we find

$$\frac{\partial}{\partial u}u(u+1)\frac{df}{du}+i\beta[u(u+1)]^{1/2}\frac{d}{du}[u(u+1)]^{1/2}f=r_1(u)=i\beta\int_0^\infty\frac{\exp(i\beta u_1)}{1+u+u_1}du_1. \quad (82)$$

We can integrate by parts in Eq. (81) in order that in it only the derivative df/du occurs. Using the fact that $f(\infty)=0$, we get

$$Q=-\frac{e^2vi\beta}{2\pi c}\int_0^\infty(1-e^{i\beta u})\frac{df}{du}du. \quad (83)$$

We now must find df/du. Let $\beta\ll1$. We introduce two regions: $u<U$ and $u>U$, where $1\ll U\ll1/\beta$. In the first of these regions we can neglect the second term on the left-hand side of Eq. (82). The remaining equation can easily be solved for df/du with the condition of regularity at $u=0$. We find

$$\frac{df}{du}=-\frac{\ln(1+u)}{u(1+u)}+\frac{1}{u+1}\int_0^\infty\frac{e^{i\beta u_1}du_1}{(1+u_1)(1+u+u_1)}.$$

We shall assume that $1\ll u\ll1/\beta$ and we expand this expression first in β, $\beta(1+u)$ restricting ourselves to second-order terms, and then in u^{-1}, restricting ourselves to terms of order β/u^2 or β^2/u. We then get

$$\frac{df}{du}\approx\frac{i\beta}{u}\left(\ln\frac{1}{\gamma\beta u}+\frac{i\pi}{2}+1\right)-\frac{i\beta}{u^2}\left(\ln\frac{1}{\gamma\beta}+\frac{i\pi}{2}+2\right) +\beta^2\left(\frac{3}{4}+\frac{i\pi}{4}+\ln\frac{1}{\gamma\beta u}\right)+\frac{\beta^2}{u}\left(-\frac{1}{4}+\frac{i\pi}{4}+\ln\frac{1}{\gamma\beta u}\right). \quad (84)$$

We now turn to Eq. (82) and introduce a new variable $\eta=\beta u$. We get

$$\eta(\beta+\eta)f''+(\beta+2\eta)f'+i\eta(\beta+\eta)f'+i(\eta+\beta/2)f=i\beta\int_0^\infty\frac{e^{i\zeta}}{\beta+\eta+\zeta}d\zeta.$$

In the region $u>U$ we have $\eta\gg\beta$. We shall therefore look for f in the form $f=\beta f_1+\beta^2f_2+\ldots$. We restrict ourselves to the first two terms in that expansion. We then get

$$\eta^2 f_1'' + 2\eta f_1' + i\eta^2 f_1' + i\eta f_1 = i\int_0^\infty \frac{e^{i\zeta}}{\zeta+\eta} d\zeta, \tag{85}$$

$$\eta^2 f_2'' + 2\eta f_2' + i\eta^2 f_2' + i\eta f_2 = -\eta f_1'' - f_1' - i\eta f_1 - \frac{i}{2} f_1 - i\int_0^\infty \frac{e^{i\zeta}}{(\eta+\zeta)^2} d\zeta. \tag{86}$$

As in Eq. (83) only df/du occurs we must convert that equation to one for $F_1 = df_1/du$, $F_2 = df_2/du$. To do this we divide Eq. (85) by η and differentiate with respect to η. We have

$$\eta F_1'' + (3+i\eta) F_1' + 2iF_1 = \frac{\partial}{\partial\eta} \frac{i}{\eta} \int_0^\infty \frac{e^{i\zeta}}{\eta+\zeta} d\zeta. \tag{87}$$

We proceed as follows with Eq. (86). We use Eq. (85) to get an expression for f_1 in it. We then get

$$\eta^2 f_2'' + 2\eta f_2' + i\eta^2 f_2' + i\eta f_2 = -\frac{i}{2\eta}\int_0^\infty \frac{e^{i\zeta}}{\eta+\zeta} d\zeta - i\int_0^\infty \frac{e^{i\zeta}}{(\eta+\zeta)^2} d\zeta - \frac{\eta}{2}(F_1' + iF_1). \tag{88}$$

After that, dividing by η and differentiating we can get an equation for F_2. The boundary conditions for these equations are $F_1, F_2 \to 0$ as $\eta \to \infty$ and the joining up with the solution of (77) when $u = U$. To do that we change in Eq. (84) to the variable $\eta = \beta u$ and use the fact that in the joining-up region $\eta \ll 1$. As a result we get

$$F_1 \to \frac{i}{\eta}\left(\ln\frac{1}{\gamma\eta} + \frac{i\pi}{2} + 1\right), \tag{89}$$

$$F_2 \to -\frac{i}{\eta^2}\left(\ln\frac{1}{\gamma\beta} + \frac{i\pi}{2} + 2\right). \tag{90}$$

We solve Eq. (87). We note first of all that the corresponding homogeneous equation has the solution $1/\eta^2$. We therefore put $F_1 = C(\eta)/\eta^2$. We have then

$$C'' + \left(i - \frac{1}{\eta}\right) C' = \eta \frac{d}{d\eta} \frac{i}{\eta} \int_0^\infty \frac{e^{i\zeta}}{\eta+\zeta} d\zeta.$$

Solving this equation we find

$$F_1 = \frac{1}{\eta^2}\int_\infty^\eta \eta_1 e^{-i\eta_1} \int_\infty^{\eta_1} e^{i\eta_2} \frac{d}{d\eta_2} \frac{1}{\eta_2} \int_0^\infty \frac{e^{i\zeta}}{\eta_2+\zeta} d\zeta + \frac{C_1}{\eta^2},$$

where the constant C_1 is selected from the joining-up condition at $\eta \ll 1$. Transforming the integral we get finally

$$F_1 = -\frac{i}{\eta}\int_0^\infty \frac{e^{i\zeta\eta}}{\zeta+1} \ln(\zeta+1) d\zeta + \frac{1}{\eta^2}\int_0^\infty \frac{e^{i\zeta\eta}-1}{\zeta(\zeta+1)} \ln(\zeta+1) d\zeta. \tag{91}$$

One checks easily that this formula satisfies condition (89) for small η and, on the other hand, satisfies Eq. (87).

We now can substitute this expression into Eq. (88). We get for F_2 an equation like (87) which differs only in the right-hand side, which turns out to equal

$$-\frac{d}{d\eta}\frac{i}{\eta^2}\left[\int_0^\infty \frac{e^{i\zeta\eta}}{1+\zeta} d\zeta + \int_0^\infty \frac{e^{i\zeta\eta}}{(1+\zeta)^2} d\zeta + \int_0^\infty \frac{e^{i\zeta\eta}}{1+\zeta} \ln(1+\zeta) d\zeta + \left(\frac{1}{2} + \frac{i}{\eta}\right)\int_0^\infty \frac{e^{i\zeta\eta}-1}{\zeta(1+\zeta)} \ln(1+\zeta) d\zeta\right].$$

We shall not continue with this rather complicated calculation and note merely that the function F_1 makes it possible to determine the contribution of order β^2 and F_2 the contribution of order β^3 in Q. The main term among those terms is the one proportional to $\beta^2 \ln^2\beta$. To determine it, it turns out that one needs Eq. (90).

We rewrite Eq. (83) in the form

$$Q = -\frac{e^2 v}{2\pi c} i\beta \int_0^\infty (1 - e^{i\eta})(\beta F_1 + \beta^2 F_2) d\eta.$$

When $\eta \ll 1$ we have $F_1 \propto 1/\eta$, whence it follows that small η do not play a separate role in determining the term of order β^2. Substituting Eq. (91) and integrating first over η and afterwards over ζ we find

$$Q_1 = e^2 v \beta^2 \zeta(3)/\pi c.$$

Substituting here $\beta = 2\omega_0\tau_2$ and using the fact that the real part of Q determines the dielectric permittivity: $\mathrm{Re}\,Q = \varepsilon\omega_0^2/4\pi c$, we find

$$\varepsilon_0 = 16\zeta(3) e^2 v \tau_2^2. \tag{92}$$

On the other hand, if we take F_2 from Eq. (90) we get a logarithmic integral over η with the limits from β to 1. Hence, the region $\eta \gtrsim 1$ turns out to be unimportant for the logarithmic accuracy. To that accuracy we thus get

$$Q_2 = \frac{e^2 v}{2\pi c} i\beta^3 \ln^2\frac{1}{\beta}.$$

Using the fact that $\mathrm{Im}\,Q = i\omega_0\sigma/c$ we get from this

$$\sigma \approx (4/\pi) e^2 v \omega_0^2 \tau_2^3 \ln^2(1/\omega_0\tau_2). \tag{93}$$

The value of the dielectric permittivity obtained by Berezinskiĭ[1] has the same dependence on the variables but differs from (92) by a numerical factor which is, apparently, the result of a numerical error. The correct value of the constant in Q_1 (or ε_0) can be found in the paper by Gogolin, Mel'nikov, and Rashba[2] (the difference of a factor 2 is explained by the fact that in[2] two spin projections are taken into account). The value of σ found by Berezinskiĭ[1] is the same as (93) after bringing the notations in agreement.

We now go to the case $\beta \gg 1$. We expand first of all the boundary condition (78) in $1/\beta$:

$$\Phi(0,u) = \frac{1}{1+u} + \frac{1}{i\beta(1+u)^2} - \frac{2}{\beta^2(1+u)^3} + \dots \tag{94}$$

We turn to Eq. (77) and in the first approximation we drop the term on the right-hand side in which there is no β. We obtain a first-order equation which can easily be solved by the characteristics method. The solution has the form

$$\Phi = [u(u+1)]^{-1/2}\varphi\left(\frac{u+1}{u} e^{-i\beta t}\right),$$

where $\varphi(x)$ is an arbitrary function. For $t = 0$ we have, when $\beta \gg 1$, $\Phi \to 1/(u+1)$ (see (94)). In view of that $\varphi(x) = x^{-1/2}$ and we finally get

$$\Phi_0 = e^{i\beta t/2}/(u+1).$$

Moreover, assuming the term in (77) without β to be small and using the method of successive approximations we can also find the next-order term which has the form

$$\Phi_1 = e^{i\beta t/2}\left(\frac{1}{i\beta(1+u)^2} - \frac{t}{1+u}\right).$$

As a whole up to terms of order $1/\beta$ or t we can write the result in the form

$$\Phi(t,u) \approx e^{-(1-i\beta/2)t}\left[\frac{1}{1+u} + \frac{1}{i\beta(1+u)^2}\right] \approx e^{-(1-i\beta/2)t}\Phi(0,u). \quad (95)$$

We emphasize that this result is already invalid for terms of order β^{-2} or t^2. The transfer of t to the argument of the exponential is justified by the fact that it is clear from what preceded that Φ must be exponentially damped at very large t. This means that when we take integrals over t we must put $i\beta \to i\beta - \gamma$ with $\gamma \ll \beta$. Equation (95) automatically guarantees that.

Further we have[3)]

$$\int_0^\infty \Phi(t,u)\,dt \approx \frac{\Phi(0,u)}{1-i\beta/2} \approx -\frac{2}{i\beta(1+u)} - \frac{4}{(i\beta)^2(1+u)} - \frac{2}{(i\beta)^2(1+u)^2}.$$

Substituting this expression into (81) we find

$$Q = -e^2v/\pi c + 4ie^2v/\pi c\beta. \quad (96)$$

Hence, connecting this with what preceded, we find the asymptotic behavior of the dielectric permittivity and of the conductivity:

$$\varepsilon = -4e^2v/\omega_0^2 = -4\pi n_e e^2/m\omega_0^2,$$
$$\sigma = 2e^2v/\pi\omega_0^2\tau_2 = 2n_e e^2\tau_2/m(\omega_0\tau_2)^2,$$

where $n_e = p_0/\pi = mv/\pi$ is the number of electrons. As should be the case, when $\omega\tau_2 \gg 1$ the permittivity is given by the formula for free electrons.

1)The introduction of the mutually uncorrelated fields η and ζ as well as Eq. (11) for G which is of first order in $\partial/\partial z$ is valid provided $1/\tau \ll \varepsilon_F$.

2)One should note that equations very close to (57) and (73) were obtained also by Berezinskiĭ.[1] However, in view of the fact that the numerical value of $Q(\omega_0)$ found in[1] is incorrect and that our method for solving the equations is somewhat different we thought it useful to give here the complete calculation right to the end.

3)We note that the same result is obtained if we take the expression for Φ without transferring t to the exponent and put $i\beta \to i\beta - \gamma$, where $\gamma \to +0$.

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Roton spectrum in superfluid He^3–He^4 solutions

V. V. Lebedev

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences

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The roton spectrum in superfluid He^3–He^4 solutions is considered by taking into account interactions between impurity excitations and rotons. An equation for the self-energy function of the rotons is obtained within the framework of a model in which this interaction is assumed to be a point interaction. The equation is solved by numerical integration with a computer. The solutions are used to determine the thermodynamic characteristics of the rotons and the energy dependence of the cross sections of various scattering processes in which rotons take part.

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INTRODUCTION

It is known that in superfluid He^3–He^4 solutions there are two excitation branches—Fermi (impurity) and Bose. We are interested in temperatures at which the role of the phonons is negligible, i.e., the only Bose excitations considered are rotons. Information on the spectrum of these excitations in He^3–He^4 solutions can be obtained from measurements of the density of the normal component by the method of the oscillating stack of disks,[1–3] the velocity of fourth sound,[4] or mobility of the positive ions.[5] In the interpretation of the experimental data, the authors of the cited papers have concluded that the roton gap decreases strongly with increasing impurity concentration.

However, the results of experiments on the scattering of photons[6,7] and neutrons[8] by superfluid He^3–He^4