Exact power-law solutions of the particle kinetic equations

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We find exact power-law stationary solutions of Boltzmann's kinetic equation which describe particle distributions with a flux from a source to a sink. We consider both direct and screened particle interactions and also a relativistic kinetic equation. The exponents in the distributions obtained are determined by the nature of the interaction and by the particle dispersion law. We study the locality of the obtained spectra. We show that in the case of Coulomb interactions a distribution with a constant flux is local. We analyze the dispersive properties of media with Coulomb interactions for the case of power-law distributions.

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1. INTRODUCTION

It is commonly assumed that interparticle collisions lead to equilibrium distributions. This conclusion is based upon the fact that the collision integral in the Boltzmann kinetic equation vanishes for equilibrium particle distributions and is a consequence of the detailed balancing principle (equality of the probabilities for direct and reverse transitions) and the conservation laws.

We have shown in brief communications[1] that collisions can also lead to non-equilibrium power-law particle distributions with a flux from a source to a sink region. In fact, the exact power-law solutions which we found make the Boltzmann collision integral vanish. It is necessary for the existence of such solutions that the dispersion law and the transition probability be homogeneous. These solutions are in a well defined sense analogous to the Kolmogorov spectrum in the inertial range[2] or, more precisely, to the weak-turbulence distributions in a system of waves[3–5] or of waves and particles.[6] However, that is a formal, methodological generality. The solutions found do not assume the existence of wave turbulence. They are established as the result of a direct interaction (collisions) of particles.

Earlier[1] we mainly paid attention to the applications of the power-law distributions to different physical problems, namely: the possibility to establish cosmic-ray spectra due to the direct interaction between particles (in contrast to the usually invoked mechanism of the interaction of the particles with waves in a turbulent plasma[7,8]), changes in such processes as Landau damping and the escape of particles, or the possibility of lowering the Lawson criterion. It is well known that one quite often encounters power-law and quasi-power-law distributions and that they may also be important in other cases (see in that connection[9–12]).

We must note that stationary solutions of the kinetic equation for particles that differ strongly from the Maxwell distribution have also been studied in the past (in that connection see, e.g., [13,14] and the literature cited). The basic difference between those works and[1] and the present paper is that in those papers we studied the relaxation of a small fraction of the particles on the background of an equilibrium distribution which corresponded to a different physical situation and allowed the linearization of the collision integral.

In the present paper we shall not touch upon applications but we obtain power-law distributions which have been used earlier[1] from the Boltzmann equation, and we discuss some general properties of such solutions. These exact solutions of the non-linear Boltzmann integral equation can be found relatively easily thanks to its symmetry properties which are used in an essential way. The first to suggest symmetry transformations of the collision integral, its inversion in particle space, was Zakharov (see[3]). We shall use more general vector transformations in **p**-space which are applied in weak turbulence theory (see[4] and the review by Kadomtsev and one of the authors[5]) as these transformations enable us to deal directly with matrix elements which are not averaged over angles which is very convenient, and also to obtain non-isotropic drift deviations from isotropic distributions. On the basis of such an approach we analyze in the present paper power-law solutions of the Boltzmann, Landau, Lenard–Balescu, Belyaev–Budker, and Klimontovich–Silin kinetic equations.

We have already noted above that the power-law solutions have a "Kolmogorov character" and their properties are determined solely by the internal symmetry of the non-linear collision integral while the particle (or energy) flux in velocity space is conserved. We find the distribution function for any interparticle interaction law, which possesses a well defined degree of homogeneity (the exponent of the power-law distribution depends on the degree of homogeneity of the interaction law). The above mentioned problems are dealt with in Secs. 2 to 4 of the present paper. We study in Sec. 5 locality problems, i.e., convergence problems of the collision integral for power-law distributions. We elucidate in Sec. 6 some characteristic peculiarities of the dispersive properties of a medium with particle power-law distributions for the case of Coulomb interactions between them.

2. POWER-LAW SOLUTIONS OF THE BOLTZMANN EQUATION

We write the Boltzmann equation in the form

$$\dot{n}_p = I\{n\} = \int d\tau\, w(\mathbf{pp}_1|\mathbf{p}_2\mathbf{p}_3) f(\mathbf{pp}_1|\mathbf{p}_2\mathbf{p}_3), \tag{2.1}$$

where n_p is the particle distribution function (for the sake of simplicity we consider for the present one kind of particles), $d\tau \equiv d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3$,

$$w_p = U(\mathbf{pp}_1|\mathbf{p}_2\mathbf{p}_3)\delta(\mathbf{p}+\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3)\delta(E+E_1-E_2-E_3) \tag{2.2}$$

is the scattering process probability,

$$f_p \equiv f(\mathbf{pp}_1|\mathbf{p}_2\mathbf{p}_3) = n_2 n_3 - n n_1, \quad n \equiv n_p, \quad n_1 \equiv n_{p_1} \text{ etc.}, \tag{2.3}$$

f_p is a quantity which is quadratic in the distribution functions and whose structure takes into account particle conservation during scattering, while momentum and energy conservation is taken into account by the δ-functions in the transition probability.

The transition probability w_p and the function f_p possess the obvious symmetry properties:

$$\begin{aligned} &w(\mathbf{pp}_1|\mathbf{p}_2\mathbf{p}_3) = w(\mathbf{p}_1\mathbf{p}|\mathbf{p}_2\mathbf{p}_3) = w(\mathbf{p}_2\mathbf{p}_3|\mathbf{pp}_1),\\ &f(\mathbf{pp}_1|\mathbf{p}_2\mathbf{p}_3) = f(\mathbf{p}_1\mathbf{p}|\mathbf{p}_2\mathbf{p}_3) = -f(\mathbf{p}_2\mathbf{p}_3|\mathbf{pp}_1). \end{aligned} \tag{2.4}$$

We shall assume the particle energy $E(\mathbf{p})$ and also the matrix element U_p and, thus the transition probability w_p to be homogeneous functions:

$$E(\lambda\mathbf{p}) = \lambda^\beta E(\mathbf{p}), \quad U(\lambda\mathbf{p}\lambda\mathbf{p}_1|\lambda\mathbf{p}_2\lambda\mathbf{p}_3) = \lambda^m U(\mathbf{pp}_1|\mathbf{p}_2\mathbf{p}_3), \tag{2.5}$$

and the system to be isotropic, as a result of which E and w are invariant under any rotation $\hat{g}$:

$$E(\hat{g}\mathbf{p}) = E(\mathbf{p}), \quad w_{\hat{g}p} = w_p. \tag{2.6}$$

We shall show below that the kinetic Eq. (2.1) in that case has not only the equilibrium but also two other stationary solutions of the form

$$n_p \sim p^{\beta s} \sim E^s, \quad s = \begin{cases} s_1 = -(m+3d)/2\beta \\ s_0 = s_1 + 1/2 \end{cases}. \tag{2.7}$$

Here d is the dimensionality of momentum space.

The physical meaning of the power-law distributions consists in the fact that they describe distributions with a source (sink) at the origin, i.e., that they are solutions of the inhomogeneous equation $I\{n\} = -\varphi_1(p)\delta(\mathbf{p})$, where s_1 corresponds to a constant energy flux along the spectrum $\varphi_1 = J_1 E^{-1}(p)$, and s_0 to a constant particle flux $\varphi_0(p) = J_0 = \text{const}$. (Here J_0 and J_1 are the particle or energy power density of the source.) By using the structure of the collision integral and assuming that the distributions are local[1] and isotropic we can easily verify this, if we write the particle and energy conservation laws down:

$$\frac{\partial J_0}{\partial p} = -4\pi p^2 I\{n\}, \quad \frac{\partial J_1}{\partial p} = -4\pi p^2 E(p) I\{n\}. \tag{2.8}$$

Hence, taking into account that $I\{n\} \propto n^2$ we can see that

$$\begin{aligned} &n_p^{(1)} = a_1 |J_1|^{1/2} U_p^{-1/2}\, p^{-d/2},\\ &n_p^{(0)} = a_0 |J_0|^{1/2} U_p^{-1/2} E^{1/2}(p)\, p^{-d/2}, \quad a_{1,0} \lesssim 1. \end{aligned} \tag{2.9}$$

The fluxes J_1 and J_0 are equal to the source power. Comparing the exponents in (2.7) and (2.9) we see that s_1 corresponds to a constant energy flux and s_0 to a constant particle flux.

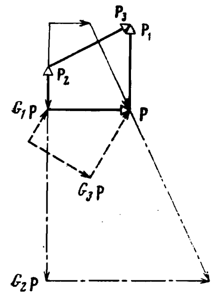

FIG. 1. Similarity transformation connected with the conservation laws for the scattering of identical particles. The transformations G_i transfer the vectors $\mathbf{p}_i$ into the fixed vector $\mathbf{p}$.

We change now to actually solving the integral equation (2.1) in the stationary case. To do this we consider a symmetry transformation. To be precise, we transfer in the quadrangle constructed on the vectors $\mathbf{p}$, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ the vector $\mathbf{p}_1$ to $\mathbf{p}$ by means of a rotation $\hat{g}_1$ and a dilatation by a factor $\lambda_1 = p/p_1$. The operation $G_1 = \lambda_1 \hat{g}_1$ changes a quadrangle into a similar one for a fixed side $\mathbf{p}$ corresponding to the external momentum. As $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are integration variables, we can consider this operation as a change of variables determined by the relations (see Fig. 1; the original variables are indicated by primes)

$$\mathbf{p}_1' = G_1^2\mathbf{p}_1, \quad \mathbf{p}_2' = G_1\mathbf{p}_3, \quad \mathbf{p}_3' = G_1\mathbf{p}_2 \quad (G_1\mathbf{p}_1 = \mathbf{p}). \tag{2.10}$$

The Jacobian of the transition is then equal to[2] λ_1^{4d} and the scattering probability transforms as $w_{G_1 p} = \lambda_1^{m-\beta-d} w_p$.

Indeed, both the matrix element and the δ-functions are invariant under the simultaneous rotation of all four momenta which is carried out under the transformation G_1, whereas the dilatation, because of the homogeneity properties, gives a factor λ_1^m from the matrix element, $\lambda_1^{-\beta}$ from the energy conservation law, and λ_1^{-d} from the momentum conservation law.

As a result of the change in variables the integral $I\{n\}$ transforms thus to

$$\begin{aligned} &I\{n\} = \int d\tau\, w_p f_{G_1 p} \lambda_1^r,\\ &f_{G_1 p} = f(\mathbf{p}G_1^2\mathbf{p}_1|G_1\mathbf{p}_3 G_1\mathbf{p}_2), \quad r = m+3d-\beta. \end{aligned} \tag{2.11}$$

We similarly introduce the transformations G_2 and G_3 which transfer $\mathbf{p}_2$ into $\mathbf{p}$ and $\mathbf{p}_3$ into $\mathbf{p}$, respectively. The corresponding changes in variables are then given by the relations

$$\begin{aligned} &\mathbf{p}_1' = G_2\mathbf{p}_3, \quad \mathbf{p}_2' = G_2^2\mathbf{p}_2, \quad \mathbf{p}_3' = G_2\mathbf{p}_1 \quad (G_2\mathbf{p}_2 = \mathbf{p}),\\ &\mathbf{p}_1' = G_3\mathbf{p}_2, \quad \mathbf{p}_2' = G_3\mathbf{p}_1, \quad \mathbf{p}_3' = G_3^2\mathbf{p}_3 \quad (G_3\mathbf{p}_3 = \mathbf{p}). \end{aligned} \tag{2.12}$$

Using all four transformations (the identical one and G_1, G_2, G_3) we can thus write the collision integral in the symmetrical form:

$$4I\{n\} = \int d\tau\, w_p [f_p + \lambda_1^r f_{G_1 p} + \lambda_2^r f_{G_2 p} + \lambda_3^r f_{G_3 p}]. \tag{2.13}$$

We shall look for isotropic solutions in the form $n = E^s$. For such functions $f_{Gp} = \pm \lambda^{2\beta s} f_p$ where the minus sign occurs for G_2 and G_3 which change the incoming to outgoing particles. For instance, after the transformation G_2 the function f_p becomes

$$f_{G_1p} \equiv f(\mathbf{p}G_2\mathbf{p}_3|G_2{}^2\mathbf{p}_2G_2\mathbf{p}_1) = f(G_2\mathbf{p}_2G_2\mathbf{p}_3|G_2\mathbf{p}G_2\mathbf{p}_1)$$
$$= \lambda_2^{2\beta s} f(\mathbf{p}_2\mathbf{p}_3|\mathbf{p}\mathbf{p}_1) = -(p/p_2)^{2\beta s} f(\mathbf{p}\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3) \equiv -\lambda_2^{2\beta s} f_p. \tag{2.14}$$

Proceeding similarly for G_1 and G_3 we reduce the collision integral to the form

$$I\{n\} = \frac{E^\nu}{4} \int d\tau w_p f_p [E^{-\nu} + E_1^{-\nu} - E_2^{-\nu} - E_3^{-\nu}], \quad \beta\nu = r + 2\beta s. \tag{2.15}$$

While the equilibrium solution corresponds to the vanishing of f_p, the vanishing of the square bracket in (2.15) leads to new solutions. To be precise, when $\nu(s) = 0$ ($s = s_0$) the bracket vanishes because of the particle number conservation law and when $\nu(s) = -1$ ($s = s_1$) because of the energy conservation law. Just as the equilibrium solutions are parametrized by the average energy, the new solutions are parametrized by the energy (or particle number) fluxes. We find below (Sec. 4) solutions in which small fluxes of several conserved quantities, such as the momentum when there is a small drift present, are present.

The solutions are very similar to those which occur in the weak turbulence theory for waves.[3-6] But, however deep the analogy is, in the case of the Boltzmann equation we are dealing with a physical effect of a principally different nature which occurs in a system of particles. In fact, the solutions discussed occur when there is no particle-wave interaction.

3. EXAMPLES OF POWER-LAW DISTRIBUTIONS

The simplest example of a system in which the solution found above can be realized is provided by a system of particles with a power-law interaction law, described by a potential $V(r) = V_0 r^{-\alpha}$. The scattering cross section $d\sigma/d\Omega$ for a power-law potential is (in the three-dimensional case) a homogeneous function of p of degree $-2\beta/\alpha$. (Indeed, the characteristic distances r defined by the condition $V(r) \sim E$ leads to the cross section[15] $d\sigma/d\Omega \propto r^2 \propto (V_0/E)^{2/\alpha}$.) Hence it follows for the transition probability that

$$U_p \sim vp^{-3}E(p)\frac{d\sigma}{d\Omega} \sim \frac{E^2}{p^4}\left(\frac{V_0}{E}\right)^{2/\alpha}.$$

The exponent of the degree of homogeneity of the square of the matrix element is thus equal to

$$m = -4 + 2\beta(1 - \alpha^{-1}). \tag{3.1}$$

The most important particular cases are the van der Waals interaction ($\alpha = 6$, $m = -\frac{2}{3}$) which leads to distributions with exponents $s_0^{\mathrm{vdW}} = -19/12$, $s_1^{\mathrm{vdW}} = -25/12$, and the Coulomb interaction ($\alpha = 1$, $m = -4$) thanks to which there appear distributions with $s_1^{\mathrm{Coul}} = -\frac{5}{4}$ and $s_0^{\mathrm{Coul}} = -\frac{3}{4}$. The latter distributions can also be obtained directly from the Landau equation (see Sec. 6).

It is convenient to use in the Coulomb case the Born approximation. The matrix element depends then only on the modulus of the transferred momentum $\hbar\mathbf{k} = \mathbf{p}_1 - \mathbf{p}_3$ (and must be symmetrized according to (24.)):

$$U(\mathbf{p}\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3) = \frac{2\pi}{\hbar}|V(\mathbf{k})|^2, \quad V(\mathbf{k}) = \int d\mathbf{r} V(\mathbf{r}) e^{i\mathbf{k}\mathbf{r}}, \tag{3.2}$$

whence it is clear that U_p is a homogeneous function of degree $m = 2(\alpha - d) = -4$. The Coulomb interaction makes it necessary to take into account the polarization of the medium which leads to screening.

The Lenard–Balescu equation which takes dynamical screening into account has the form (2.1) where the effective matrix element $V(\mathbf{k}, \omega)$ in contrast to (3.2) depends both on the momentum transferred and on the energy transferred. In the non-relativistic case we have

$$V_{ab}(\mathbf{k}, \omega) = e_a e_b / k_i k_j \varepsilon_{ij}(\omega, \mathbf{k}), \quad \hbar\omega = E_1 - E_3, \tag{3.3}$$

where ε_{ij} is the permittivity tensor of the medium, taking into account the temporal and spatial dispersion. The longitudinal permittivity

$$\varepsilon^l(\omega, \mathbf{k}) = 1 + \frac{4\pi e^2}{k^2} \int d\mathbf{p} \frac{\mathbf{k}\,\partial n/\partial \mathbf{p}}{\omega - \mathbf{k}\mathbf{v}}. \tag{3.4}$$

is thus important. In that case we have in the region of static screening ($kr_D \ll 1$, $k^2\overline{v^2} \gg \omega^2$) $\varepsilon^l \propto (kr_D)^{-2}$, where r_D is the effective Debye radius and $\overline{v^2}$ the mean squared velocity. The matrix element $V_{ab} \propto e_a e_b r_D^2$, which leads to $m = 0$ and $s_0^D = -\frac{7}{4}$, $s_1^D = -\frac{9}{4}$.

We now consider the relativistic case restricting ourselves only to scattering processes in which particles are conserved and neglecting pair creation and annihilation. The kinetic equation has in that case again the form (2.1) with a matrix element V which can be expressed in terms of the Fourier component of the Green function $\mathscr{G}$ of the Maxwell equations[16]:

$$V_{ab}(\mathbf{k}, \omega) = 4\pi e_a e_b v_i^a v_j^b \mathscr{G}_{ij}(\omega, \mathbf{k})/c^2. \tag{3.5}$$

Here $\mathbf{v}$ is the three-dimensional particle velocity, c the velocity of light, and $\omega, \mathbf{k}$ are as before connected with the transfer of energy and momentum in the collision. In the isotropic case $\mathscr{G}$ can be expressed in terms of the longitudinal ε^l and the transverse ε^{tr} parts of the permittivity ε_{ij}:

$$\mathscr{G}_{ij} = \frac{\delta_{ij} - k_i k_j / k^2}{(\omega/c)^2 \varepsilon^{tr}(\omega, \mathbf{k}) - k^2} + \frac{k_i k_j}{k^2} \frac{c^2}{\omega^2 \varepsilon^l(\omega, \mathbf{k})}.$$

If we neglect retardation $\varepsilon^l = \varepsilon^{tr} = 1$, whence follows (in three-dimensional notation) the Belyaev–Budker equation.[16] The derivation proposed in the preceding section refers thus completely also to the relativistic case in the region where the dispersion law and the transition probabilities are homogeneous. The latter means a restriction to the ultra-relativistic condition $\beta = 1$ ($E = cp$). The homogeneity exponent m for the case of a Coulomb interaction and when the Debye screening dominates remains the same as in the non-relativistic limit. We are thus led to the distributions

$$n^{(1)\mathrm{Coul}} \sim E^{-5/4}, \quad n^{(0)\mathrm{Coul}} \sim E^{-2},$$
$$n^{(1)D} \sim E^{-9/4}, \quad n^{(0)D} \sim E^{-4}. \tag{3.6}$$

These distributions turn out, however, to be non-local (see Sec. 5).

Above we considered collisions between only one kind of particles, e.g., between ions on a compensating electron background, assuming that the ion subsystem relaxes quasi-independently of the electron subsystem. Such a situation exists practically always for ions in a system with comparable ion and electron temperatures

($I_{ii} \gg I_{ie}$). For electrons, on the other hand, a relaxation which is independent of the ions is possible, as follows from an analysis of the relaxation times, [17] if $T_e \ll m_e T_i / m_i$. In the case of quasi-independence of the subsystems the second component is important in the sense that it plays the role of a background guaranteeing the quasi-neutrality of the medium and affecting its dispersive properties, the screening, and thereby the homogeneity exponent of the scattering probability.

In concluding the present section we estimate the characteristic times for establishing stationary distributions with a flux. We find τ_p^{-1} as the functional derivative $\delta I\{n\}/\delta n$ whence, assuming locality and using (2.7), we get

$$\tau_p^{-1} \sim \begin{cases} |J_1|^{1/3} U_p^{1/3} E^{-1}(p) p^{5/3} \\ |J_0|^{1/2} U_p^{1/2} E^{-1/2}(p) p^{3/2} \end{cases} \quad (3.7)$$

respectively, for the solution with an energy and a particle number flux. In particular, it follows from this for a local Coulomb distribution (with an energy flux) ($U_p^{\text{Coul}} = 2\pi(4\pi e^2/p^2)^2$

$$(\tau_p^{-1})^{\text{Coul}} \sim |J_1|^{1/3} m e^2 p^{-5/3}, \quad (3.8)$$

where m is the particle mass. If the electrons are relativistic we have

$$(\tau_e^{-1})^{\text{Coul}} \sim |J_1|^{1/3} m_e^{-3/2} e^2 c^{-5/3}. \quad (3.9)$$

With power-law particle distributions there is connected the possibility, mentioned earlier, [1] to explain the power-law spectra of cosmic rays[18] and correspondingly of the cosmic radio-emission of discrete sources without invoking for this ideas about "turbulent reactors."[7,8] The index γ of the differential flux density $I(E) = v n_p g(E) \propto E^{-\gamma}$ ($v = \partial E/\partial p$, $g(E) = d^3p/dE$) is then connected with the exponent of the distribution according to $\gamma^{\text{nonrel}} = -(1+s)$, $\gamma^{\text{rel}} = -(2+s)$.

The condition that over the characteristic dimensions of the system L Coulomb relaxation can take place $l = v\tau_p \ll L$ leads according to (3.8) to the condition $|J_1| L^2 \gg m^3 e^{-4} v^7$. Estimating the flux along the spectrum J_1 using the total power W of the source through $J_1 \sim W/L^3$ we are led to the inequality

$$\frac{W}{L} \gg \left(\frac{m}{m_e}\right)^3 \left(\frac{v}{c}\right)^7 10^{30}[\text{CGSE}],$$

which can be satisfied in powerful and compact cosmic objects.

4. MULTIPLE-FLUX DRIFT-TYPE DISTRIBUTIONS

The solutions describing the occurrence of small fluxes close to an equilibrium or a stationary distribution are, apparently, realized more often than the purely single-flux power-law solutions. Since the equilibrium Maxwell distribution is not self-similar, we restrict ourselves below to determining drift deviations from the above found single flux solutions, and also from a plateau which clearly possesses self-similarity properties. It is most important that these deviations can in the general case not be obtained by a Galileo change $E \to E - \mathbf{p}\cdot\delta\mathbf{u} - \delta\mu$ because the initial distributions are non-equilibrium ones. This fact has already been discussed earlier[4] for the weak turbulence case for waves.

We look for the solution of the Boltzmann equation (2.1) in the form

$$n_{\mathbf{p}} = n_E[1 + E^t\delta\mu + E^\delta(\mathbf{p}\delta\mathbf{u})], \quad n_E = E^s \quad (4.1)$$

and we shall restrict ourselves to terms linear in $\delta\mu$ and $\delta\mathbf{u}$. It is convenient to write the linearized collision integral and the function f_p in the form

$$I = I_0 + I_\mu\delta\mu + I_u\left(\frac{\mathbf{p}}{p}\delta\mathbf{u}\right), \quad f_p = f_p^0 + f_p^\mu\delta\mu + \mathbf{f}_p\delta\mathbf{u}, \quad (4.2)$$

where I_0 and f_0 have been considered earlier (see (2.15)) while the quantities f_p^μ and $\mathbf{f}_p$ are equal to

$$\begin{aligned} f_p^\mu &= n_{E_2} n_{E_3}(E_2^t + E_3^t) - n_E n_{E_1}(E^t + E_1^t), \\ \mathbf{f}_p &= n_{E_2} n_{E_3}(E_2^\delta \mathbf{p}_2 + E_3^\delta \mathbf{p}_3) - n_E n_{E_1}(E^\delta \mathbf{p} + E_1^\delta \mathbf{p}_1). \end{aligned} \quad (4.3)$$

We use the symmetry transformations introduced above. The integrals I_μ and I_u then factor. For the factoring of I_u the vector nature of the transformations used is essential. Finally, completely as in the case of waves, [4] we get

$$\begin{aligned} I_\mu &= \frac{E^{\nu_t}}{4}\int d\tau\, w_p f_p^\mu [E^{-\nu_t} + E_1^{-\nu_t} - E_2^{-\nu_t} - E_3^{-\nu_t}], \\ I_u &= \frac{E^{\nu_\delta - 1/\beta}}{4}\int d\tau\, w_p \mathbf{f}_p [E^{-\nu_\delta}\mathbf{p} + E_1^{-\nu_\delta}\mathbf{p}_1 - E_2^{-\nu_\delta}\mathbf{p}_2 - E_3^{-\nu_\delta}\mathbf{p}_3]. \end{aligned} \quad (4.4)$$

$$\nu_t = \nu(s) + t, \quad \nu_\delta = \nu(s) + \delta + 2/\beta.$$

We used essentially in the factoring the power-law character of the isotropic unperturbed solution. It is clear from the expression for I_0 that such solutions are flux ($s = s_0$, $s = s_1$) and plateau ($s = 0$) solutions. Linear deviations from these solutions describe the appearance of other small fluxes of conserved quantities. We consider them separately.

The deviation from the distribution with a constant energy flux has the form[3)]

$$n_{\mathbf{p}} = E^{s_1}[1 + E\delta\mu + E^{1-2/\beta}\mathbf{p}\delta\mathbf{u}]. \quad (4.5)$$

Here $\delta\mu$ has the meaning of a small particle number flux and $\delta\mathbf{u}$ that of a momentum flux along the spectrum. We check this somewhat later by using dimensionality considerations. In particular, for non-relativistic particles ($\beta = 2$)

$$n_{\mathbf{p}} = E^{s_1}[1 + E\delta\mu + \mathbf{p}\delta\mathbf{u}]. \quad (4.6)$$

and in the ultra-relativistic case

$$n_{\mathbf{p}} = E^{s_1}[1 + E\delta\mu + E^{-1}\mathbf{p}\delta\mathbf{u}] \quad (4.7)$$

(we remind ourselves that we have here dropped constants with dimensions). The impossibility of the substitution is clear: $(E - \delta\mu - \mathbf{p}\cdot\delta\mathbf{u})^{s_1}$ does not make the collision integral vanish.

The deviation from the solution with a constant particle flux $n_E = E^{s_0}$ has the form

$$n_{\mathbf{p}} = E^{s_0}[1 + E^{-1}\delta\mu + E^{-2/\beta}\mathbf{p}\delta\mathbf{u}], \quad (4.8)$$

where $\delta\mu$ has here the meaning of an energy flux. In

TABLE I.

Interactions	$E=p^{\beta}$: β	$U_{\lambda p}=\lambda^m U_p$: m	Distribution $n=E^s$. Particle flux in space $I(E)=E^{-\gamma}$. $s=s_1$: s_1	$s=s_1$: γ_1	$s=s_0$: s_0	$s=s_0$: γ_0	Deviations from the power-law distribution (4.1) $s=s_1$: t	$s=s_1$: δ	$s=s_0$: t	$s=s_0$: δ	Deviations from the plateau (4.11′): t_1	t_0	δ	Divergence of I_{coll} for a power-law distribution $I_{coll}\{E^s\}\sim\int dp^{\Delta}\ldots$ $s=s_1$: Δ′	$s=s_1$: Δ_1	$s=s_1$: Δ_2	$s=s_1$: Δ″	$s=s_0$: Δ′	$s=s_0$: Δ_1	$s=s_0$: Δ_2	$s=s_0$: Δ″
Non-relativistic systems																					
van der Waals interaction	2	−2/3	−25/12		−19/12		1	0	−1	−1											
Coulomb interaction	2	−4	−5/4	1/4	−3/4	−1/4	1	0	−1	−1	−5/2	−3/2	−5/2	1/2	1/2	−1/2	−5/2	3/2	3/2	1/2 *	−3/2
Debye screening	2	0	−9/4	5/4	−7/4	3/4	1	0	−1	−1	−9/2	−7/2	−9/2	−3/2 *	5/2	−5/2	−1/2	−1/2 *	7/2	−3/2	1/2 *
Relativistic systems																					
Coulomb interaction	1	−4	−5/2	1/2	−2	0	1	−1	−1	−2	−5	−4	−6	1/2	−1/2 *	1/2 *	−3/2	1	0	1 *	−1
Debye screening	1	0	−9/2	5/2	−4	2	1	−1	−1	−2	−9	−8	−10	−3/2 *	3/2	−3/2	1/2 *	−1 *	2	−1	1 *

*Non-local distributions.

the non-relativistic limit we get

$$n_p=E^{s_0}[1+E^{-1}(\delta\mu+\mathbf{p}\delta\mathbf{u})], \tag{4.9}$$

which in this case indeed corresponds to the first term of the expansion of $(E-\delta\mu-\mathbf{p}\cdot\delta\mathbf{u})^{s_0}$ in terms of the additional drift term. This is connected with the singularity of the quadratic dispersion law and the solution with a constant particle flux. In the ultra-relativistic case

$$n_p=E^{s_0}[1+E^{-1}\delta\mu+E^{-2}\mathbf{p}\delta\mathbf{u}]. \tag{4.10}$$

The deviation from the three-dimensional plateau $n_p=1$ (the solution corresponding to a zero flux) has the form

$$n_p=1+E\delta\tilde{\mu}+\mathbf{p}\delta\tilde{\mathbf{u}}+E^{-(m+3d)/\beta}[\delta\mu_1+E\delta\mu_0+E^{1-2/\beta}\mathbf{p}\delta\mathbf{u}]. \tag{4.11}$$

The first three terms in (4.11) make f_p vanish in the approximation which is linear in the drift parameter. The solution $n_p=1+E\delta\tilde{\mu}+\mathbf{p}\cdot\delta\tilde{\mathbf{u}}$ is not connected with the presence of self-similarity. The last terms in (4.11) make the factorizing factors in (4.4) vanish where $\delta\mu_0$ is proportional to the particle flux and $\delta\mu_1$ to the energy flux.

We give in Table I the values of the exponents for the distributions (4.1) and for the deviations from the plateau, taken in the form

$$n_p=1+E^{t_1}\delta\mu_1+E^{t_0}\delta\mu_0+E^{\delta}\mathbf{p}\delta\mathbf{u}. \tag{4.11′}$$

To elucidate the physical meaning of the additional terms we can use considerations similar to those given earlier.[4] We use then the differential form of the conservation laws (2.8). We restrict ourselves here to the simplest consideration, using the local nature and dimensionality arguments. Indeed, we can write the multiple-flux particle distribution in terms of a dimensionless function of the ratio of the fluxes F:

$$n_p=|J_1|^{1/2}E^{s_1}F_1\left(\frac{E\delta J_0}{J_1},\ \frac{E\mathbf{p}\delta\boldsymbol{\pi}}{J_1p^2}\right),\quad n_p=|J_0|^{1/2}E^{s_0}F_0\left(\frac{\delta J_1}{J_0E},\ \frac{\mathbf{p}\delta\boldsymbol{\pi}}{J_0p^2}\right),$$
$$n_p=n_0F(\delta J_1/x,\ E\delta J_0/x,\ E\mathbf{p}\delta\boldsymbol{\pi}/p^2x), \tag{4.12}$$

where $\delta\pi$ is the momentum flux, and x the magnitude of the dimensionality of the energy flux, expressed in terms of the "equilibrium" distribution $n_0=n/p_T^3$ (n the number of particles per unit volume, $p_T=(2\pi mT)^{1/2}$, T the "temperature") and the transition probability $x=p^{3d}U_pn_0^2$.

In the approximation which is linear in the small fluxes we get from (4.12)

$$n_p=|J_1|^{1/2}E^{s_1}\left[1+\frac{E\delta J_0}{J_1}+\frac{E\mathbf{p}\delta\boldsymbol{\pi}}{J_1p^2}\right]=|J_1|^{1/2}E^{s_1}\left[1+E\delta\mu_0+\frac{E\mathbf{p}\delta\mathbf{u}}{p^2}\right],$$
$$n_p=|J_0|^{1/2}E^{s_0}\left[1+\frac{\delta J_1}{J_0E}+\frac{\mathbf{p}\delta\boldsymbol{\pi}}{J_0p^2}\right]=|J_0|^{1/2}E^{s_0}\left[1+E^{-1}\delta\mu_1+\frac{\mathbf{p}\delta\mathbf{u}}{p^2}\right], \tag{4.13}$$
$$n_p=n_0[1+\delta J_1/x+E\delta J_0/x+E\mathbf{p}\delta\boldsymbol{\pi}/p^2x].$$

Hence we can easily establish the coefficients which have dimensions in the exact solutions of the integral equations obtained above.

5. STUDY OF THE CONVERGENCE OF THE COLLISION INTEGRAL

We study the convergence of the collision integral for the power-law distributions (4.1) which we have found, restricting ourselves to the case, which is important for applications, where the matrix element of the scattering probability depends on the transferred momentum and energy. The conservation laws in (2.2) allow one or simultaneously two of the momentum variables (e.g., p_1 and p_3) to vanish and correspondingly three (p_1, p_2, p_3) or two (p_1, p_3) from them to become infinite. Because of the symmetry of the integrand it is sufficient to study the convergence in the regions

$$\text{I) } p_1\to0,\quad \text{II) } p_1,p_3\to0,\quad \text{III) } p_1,p_3\to\infty,\quad \text{IV) } p_1,p_2,p_3\to\infty.$$

We need then the asymptotic behavior of the transition probability U_p in the indicated regions while their symmetry properties which, when taken into account, lead to an improvement of the convergence (cf.[4]), are important.

In the case of interest to us the matrix element $V(\mathbf{k},\omega)$ in (3.2) depends solely on the momentum (and energy) transfer. This enables us to find the general form of the asymptotic behavior without turning to the explicit expression for $V(\mathbf{k}, \omega)$. Let $p_1 \ll p, p_2, p_3$. We can then drop $\mathbf{p}_1$ both in the conservation laws and in the momentum transfer and, hence,

$$U(\mathbf{pp}_1|\mathbf{p}_2\mathbf{p}_3)=\frac{2\pi}{\hbar}\frac{1}{4}[|V(p_2)|^2+|V(-p_2)|^2+2\rightleftarrows 3]\left[1+O\left(\frac{p_1}{p}\right)\right],$$

$$V(p)=V\left(\frac{\mathbf{p}}{\hbar}, \frac{E_\mathbf{p}}{\hbar}\right) \tag{5.1}$$

The asymptotic behavior of U_p in the region of two small (large) momenta depends on the sign of the degree of homogeneity m of that function as we can for $m>0$ drop in U_p terms with a small momentum transfer, while for $m<0$ we can drop those with a large transfer. Using this we easily obtain the asymptotic behavior of U_p which we write in a form which is convenient for what follows

$$U(\mathbf{pp}_1|\mathbf{p}_2\mathbf{p}_3)=(|\mathbf{p}_1||\mathbf{p}_3|)^{m_1/2}(|\mathbf{p}||\mathbf{p}_2|)^{m_2/2}u(\mathbf{p}+\mathbf{p}_2|\mathbf{p}_1,\mathbf{p}_3),$$
$$m_1=(m-|m|)/2,\quad m_2=(m+|m|)/2.\quad p_1,p_3\ll p,p_2. \tag{5.2}$$

The function $u(\mathbf{p}|\mathbf{p}_1,\mathbf{p}_3)$ is of order unity and possesses the following symmetry properties:

$$u(\mathbf{p}|\mathbf{p}_1,\mathbf{p}_3)=u(\mathbf{p}|\mathbf{p}_3,\mathbf{p}_1)=u(\mathbf{p}|-\mathbf{p}_1,-\mathbf{p}_3).$$

Its explicit form can be established by comparing (5.2), e.g., for $m>0$:

$$(|\mathbf{p}||\mathbf{p}_2|)^{m/2}u(\mathbf{p}+\mathbf{p}_2|\mathbf{p}_1,\mathbf{p}_3)=\frac{1}{4}\frac{2\pi}{\hbar}[|V(\mathbf{p})|^2+|V(-\mathbf{p})|^2+\mathbf{p}\rightleftarrows\mathbf{p}_2]\times\left[1+O\left(\frac{p_{1,3}}{p}\right)\right].$$

We consider the convergence of the collision integral in region I. If $n_\mathbf{p}\to\infty$ as $p\to 0$, the dangerous term in $I\{n\}$ is proportional to nn_1. (As $n_\mathbf{p}\to 0$ the collision integral certainly converges in region I.) Taking initially only the main term of the asymptotic Eq. (5.1) into account we find that the collision integral converges as $p_1\to 0$ simultaneously with integrals such as

$$\int d\mathbf{p}_1 n_{\mathbf{p}_1},\quad p_1\ll p.$$

Hence we get the condition for the convergence for isotropic power-law distributions $n_p=E^s$:

$$\Delta'\equiv d+\beta s>0. \tag{5.3}$$

We get by the substitution $s\to s+t$ from (5.3) the condition for the convergence to zero for a small isotropic addition to the distribution, proportional to an additional flux $\delta\mu$:

$$\Delta'+\beta t>0.$$

For distributions with small momentum fluxes $\propto \delta\mathbf{u}$

$$n_p=E^s(1+E^\delta\mathbf{p}\delta\mathbf{u})$$

the main term of the asymptotic behavior of w_p does not contribute to the integral of the anisotropic part of $n_\mathbf{p}$. Taking the next term of the expansion of w_p into account we are led to the condition for the integral of the anisotropic addition to converge to zero:

$$\Delta'+\beta\delta+2>0. \tag{5.4}$$

Using the same asymptotic behavior (5.1) with the substitution $\mathbf{p}_1\to\mathbf{p}$ and dropping p in the conservation laws we get similarly to the preceding the conditions for the convergence of the collision integral in the region IV ($p_1, p_2, p_3 \gg p$):

$$\Delta''<0,\quad \Delta''+\beta t<0,\quad \Delta''+\beta\delta<0,\quad \Delta''\equiv m+2d-\beta+\beta s. \tag{5.5}$$

The three inequalities (5.5) correspond, respectively, to the convergence of each of the three terms of the distribution (4.1).

The study of the convergence of the collision integral in the region II is somewhat more complicated. This is caused by the mutual cancelling of the contributions from two dangerous terms in f_p. As before assuming that $n_\mathbf{p}\to\infty$ as $p\to 0$ we expand f_p in terms of the small momentum transfer $\mathbf{q}=\mathbf{p}_1-\mathbf{p}_3$, $\mathbf{p}_2=\mathbf{p}+\mathbf{q}$:

$$f_p=n_2n_3-nn_1=n(n_3-n_1)+\mathbf{q}\frac{\partial n}{\partial\mathbf{p}}n_3+O(n_3q^2). \tag{5.6}$$

Restricting ourselves to the main term in the asymptotic expression for w_p and integrating over $\mathbf{p}_2$, using momentum conservation, we are led to an integral of the form

$$\int_{p_{1,3}\ll p} d\mathbf{p}_1\,d\mathbf{p}_3(p_1p_3)^{m_1/2}p^{m_2}u(\mathbf{p}|\mathbf{p}_1,\mathbf{p}_3)\,\delta\left(E_1-E_3-\mathbf{q}\frac{\partial E}{\partial\mathbf{p}}\right),\quad \mathbf{q}=\mathbf{p}_1-\mathbf{p}_3. \tag{5.7}$$

By virtue of the symmetry of $u(\mathbf{p}|\mathbf{p}_1,\mathbf{p}_3)$ under the substitution $1\rightleftarrows 3$ the first term of the expansion of f_p in (5.6) does not contribute to the integral which we have written down. The integral of the second term also vanishes as it is odd in $\mathbf{q}$ and the remaining part of the integrand does not change under the substitution $\mathbf{p}_1\to-\mathbf{p}_1$, $\mathbf{p}_3\to-\mathbf{p}_3(\mathbf{q}\to-\mathbf{q})$. It is thus necessary to retain the next term in the transition probability, in particular, to expand the argument of the δ-function of the energy conservation law up to terms which are quadratic in $\mathbf{q}$:

$$E+E_1-E_2-E_3\approx E_1-E_3-\mathbf{q}\frac{\partial E}{\partial\mathbf{p}}-\frac{1}{2}q_iq_j\frac{\partial^2E}{\partial p_i\partial p_j}.$$

The first terms in f_p then give a non-vanishing contribution to the collision integral and the corresponding integrals converge more slowly than the integral of $O(n_3q^2)$ in (5.6). The convergence conditions can be established by calculating the powers, and this leads to the following inequalities for distributions such as (4.1):

$$\Delta_1>0,\quad \Delta_1+\beta t>0,\quad \Delta_1+\beta\delta+2-\beta>0,\quad \Delta_1\equiv m_1+2d+\beta-1+\beta s,$$
$$m_1=(m-|m|)/2. \tag{5.8}$$

In region III ($p_1, p_3 \gg p, p_2$) there occurs no cancellation in f_p and the symmetry of w_p under a change in sign of all arguments improves the convergence of the integral of the anisotropic part of the distribution by one power of p. The convergence conditions are given by the inequalities

$$\Delta_2<0, \quad \Delta_2+\beta t<0, \quad \Delta_2+\beta\delta<0, \quad \Delta_2\equiv m_2+d+1-\beta+\beta s, \quad m_2=(m+|m|)/2. \tag{5.9}$$

We note that we have not discussed the region of small momentum transfers in which the Coulomb matrix element is singular. The presence of this singularity is not connected with the form of the distribution function and can be removed, e.g., by changing to the Landau collision integral (6.1).

We apply the convergence criteria which we have found to the stationary particle distributions obtained above. One usually calls a distribution local which is such that the convergence of the collision integral is guaranteed in all regions I to IV. The main contribution to the integral comes for a local distribution from integration over the region $p_1, p_2, p_3 \sim p$. According to (5.3), (5.4), (5.5), (5.8), and (5.9) the distribution with a constant energy flux in a non-relativistic Coulomb plasma ($m=-4$, $s=-\frac{5}{4}$) is local in that sense.[1] The addition to it which is produced by a small momentum flux is also local, and thereby also the distribution $n_p = E^{-5/4}(1+\mathbf{p}\cdot\delta\mathbf{u})$. The distribution with a constant particle flux in the non-relativistic case for $m=-2$ also has the property of being local (together with anisotropic terms). The other distributions are non-local, of course, within the framework of the Born approximation used here. The degrees of divergence are equal to the appropriate values of Δ (see the table).

Non-local distributions, being formal solutions of the kinetic equation, nevertheless require an additional study because to discover them we must operate with divergent expressions and the problem of their existence remains open. The difference between local and non-local solutions is very clear from the example of the Landau kinetic equation.

6. LANDAU COLLISION INTEGRAL. DISPERSIVE PROPERTIES OF A SYSTEM OF PARTICLES WITH POWER-LAW ENERGY DISTRIBUTIONS

Systems of charged particles with Coulomb interactions possess singularities which are connected with the divergence of the scattering cross section for small momentum transfers. This leads to the fact that one can restrict oneself to the diffusion approximation and write the collision integral in the Landau form[16]:

$$I\{n\}=-\operatorname{div}\mathbf{j}, \quad j_i=\pi e^4\lambda\int d\mathbf{p}'\left(\frac{\delta_{ik}}{u}-\frac{u_iu_k}{u^3}\right)\left(n_p\frac{\partial n_{p'}}{\partial p_k'}-n_{p'}\frac{\partial n_p}{\partial p_k}\right),$$
$$\lambda=\ln\frac{1}{n}\left(\frac{\bar{E}}{e^2}\right)^3, \tag{6.1}$$

where $\mathbf{u}=\mathbf{v}-\mathbf{v}'$, λ is the Coulomb logarithm, and $\mathbf{j}$ the particle flux density in momentum space. For a power-law distribution function $n_p = Ap^{2s}$ we find easily that

$$I\{n\}=16\pi^2e^4m\lambda A^2p^{4s}\left\{\frac{(4s+3)(4s+5)}{(2s+2)(2s+3)(2s+5)}\right.$$
$$\left.+\frac{s}{3}\left[\frac{2s}{2s+3}\left(\frac{p_1}{p}\right)^{2s+3}+\frac{2s+1}{2s+2}\left(\frac{p_2}{p}\right)^{2s+2}\right]\right\}, \tag{6.2}$$

where p_2 and p_1 are the maximum and minimum momenta of the power-law distribution. It is clear from Eq. (6.1) that for the case of Coulomb interactions the distribution function with $s=-\frac{5}{4}$ is local (i.e., the collision integral remains finite as $p_1\to 0$, $p_2\to\infty$) as was shown in Sec. 5.

Solving Eq. (2.5) and using Eq. (6.2) for the collision integral we get the energy flux in momentum space J_1:

$$J_1=-\frac{32\pi^3e^4\lambda}{(2s+2)(2s+3)}A^2p^{4s+5}$$
$$\times\left\{\frac{4s+3}{2s+5}+\frac{s}{3}\left[2s\left(\frac{p_1}{p}\right)^{2s+3}+(2s+1)\left(\frac{p_2}{p}\right)^{2s+2}\right]\right\}, \tag{6.3}$$

whence it is clear that the energy flux is in the direction of small momenta, while the constant A in the local distribution ($s=-\frac{5}{4}$) is given in terms of the flux as follows (as $p_1\to 0$, $p_2\to\infty$):

$$A=\alpha_1|J_1|^{1/2}, \quad \alpha_1=\left[\frac{5}{(8\pi)^3}\right]^{1/2}\frac{1}{e^2\lambda^{1/2}}\approx\frac{0.018}{e^2\lambda^{1/2}}. \tag{6.4}$$

The normalization factor in (2.9) is correspondingly equal to $a_1\approx 0.57/\lambda^{1/2}$ while the appearance of the Coulomb logarithm in the estimate for a_1 is connected with the Coulomb divergence.

Power-law particle distributions show up first of all in those properties of the medium which are sensitive to the presence of particles in the "tail." For instance, the dielectric properties of a system of charged particles depends strongly on the particle velocity distribution function. As an example we consider the longitudinal permittivity (3.4) for the case of an isotropic power-law particle distribution corresponding to a non-vanishing flux in momentum space. We have studied earlier[1] the damping of Langmuir waves under such conditions when the number of particles in the region of the power-law distribution is small compared to the total number of particles but they determine the imaginary part of the permittivity. The expression for the imaginary part of the frequency then has the following form[1]:

$$\operatorname{Im}\omega=-\frac{\pi^2\alpha|J|^{1/2}}{n}m^{2s+3}u_\Phi^{2s+3}\omega_0 \qquad \operatorname{Im}\omega\ll\omega_0, \; v_1<u_\Phi=\frac{\omega}{k}<v_2. \tag{6.5}$$

The particle number density n and the frequency ω_0 are determined by the equilibrium part of the distribution. We note that for $s>-\frac{3}{2}$ (which includes both Coulomb power-laws) the damping increases with increasing phase velocity (in contrast to the usual Landau damping in an equilibrium system). The problem of the effect of this fact on the stability and also on the collisionless non-linear relaxation of the Langmuir waves requires special attention.

Here we wanted to draw attention to the possibility to evaluate the dielectric permittivity for particle power-law distributions without making the assumption that the number of particles in the power-law region is small. This is caused by convergence of the appropriate integrals (see (3.4)) for the power-law functions n_p. Because of the absence of a characteristic velocity scale parameter in the power-law distribution the imaginary and real parts of the permittivity will be of the same order of magnitude in that frequency and wavelength region where the dispersive properties of the medium and the damping of the oscillations are determined by the interaction with the particles which obey the power-law distribution.

As the power-law distribution can be realized only in a limited velocity range $v_1 < v < v_2$ while the dielectric permittivity (3.4) contains an integration over the whole of **p**-space, it is for the evaluation of $\varepsilon(\omega, \mathbf{k})$ in general necessary to know the complete distribution, including also the source ($v \gtrsim v_2$) and sink ($v \lesssim v_1$) regions. However, if the integral in (3.4) converges for the power-law distribution $n_p = Ap^{2s}$ both as $p \to 0$ and as $p \to \infty$ the contribution to $\varepsilon(\omega, \mathbf{k})$ can be found without a detailed knowledge of the complete distribution function. Below we restrict ourselves to considering just such a situation which, as one can easily verify, corresponds to $-\frac{3}{2} < s < -\frac{1}{2}$ and, in particular, includes the case of the Coulomb distribution with an energy flux, $s = -\frac{5}{4}$.

For an isotropic distribution n_p we are after integrating over the transverse momentum led to the following form for $\delta\varepsilon \equiv \varepsilon - 1$ $(n(\infty) = 0)$

$$\delta\varepsilon'(\omega, k) = -\frac{16\pi^2 e^2}{k^2} \int_0^\infty dp \frac{pvn(p)}{u^2 - v^2},$$
$$\delta\varepsilon''(\omega, k) = 8\pi^3 e^2 m^2 k^{-2} u n(mu), \quad u = \omega/k, \tag{6.6}$$

whence we get, in particular the above given convergence conditions for $n \propto p^{2s}$.

We consider the phase velocity region $v_1 \ll u \ll v_2$. To find the contribution from the power-law part of the distribution to $\delta\varepsilon$ we can in this case replace the integration over the region $p_1 < p < p_2$ by an integration over the semi-axis because the integral converges for $-\frac{3}{2} < s < -\frac{1}{2}$. In this case the real part of $\delta\varepsilon$ will be of the same order of magnitude as the imaginary part. In particular, for $n = Ap^{2s}$, $s = -\frac{3}{4}$, $-\frac{5}{4}$ the principal value integral can easily be evaluated and we get for $\delta\varepsilon$

$$\delta\varepsilon(\omega, k) = \frac{8\pi^3 e^2}{m\omega^2} A \left(m\frac{\omega}{k}\right)^{1 \pm 1/2} \left(\mp\frac{1}{2} + i\right), \tag{6.7}$$

where the upper sign refers to a particle flux ($s = -\frac{3}{4}$) and the lower one to an energy flux ($s = -\frac{5}{4}$). For Coulomb distributions the dispersion equation $\varepsilon(\omega, k)$ thus leads to $\mathrm{Im}\,\omega \sim \mathrm{Re}\,\omega$ and there are no branches such as Langmuir waves.

In conclusion we must note that as in nature, and recently also in experiments, power-law distributions or at least power-law tails of distributions are very often encountered, one may think that they are formed by a unique and very common mechanism. In the present paper we have shown that when there is a source present such a mechanism may be provided by the direct collisional interactions between particles taking their screening by a self-consistent field into account. The exact power-law solutions of the Boltzmann kinetic equation with a source may thus serve as a basis to explain power-law distributions in different systems from unique and rather general positions.

1)Indeed (see Sec. 5), it turns out that only the distribution corresponding to an energy flux for the Coulomb interaction of non-relativistic particles, including also the case when there is a small momentum flux, superimposed upon the basic solution (see section 4), is local. Nonetheless all formal solutions correspond to the structure (2.9).

2)We can establish that the Jacobian is independent of the angles by direct calculations. Indeed, this property becomes clear after going over to integration over the internal angles of the polygon which do not change under the similarity transformations G_i.

3)In the corresponding formula of our earlier paper[1] there is an error.

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