Effect of gravitational exchange interaction in the cosmological photon gas

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Einstein's equations, written down for operators, are used to calculate the gravitational exchange interaction between photons in an isotropic model of the Universe. Expressions are obtained for the energy spectrum, distribution function, and energy density of the photons. A cosmological solution that takes into account the exchange process is found.

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INTRODUCTION

In a many-particle system, quantum gravitational effects become important at densities $\rho=\hbar/cl_g^4=10^{94}$ g/cm^3 and spacetime curvatures $R_i^k=l_g^{-2}=10^{66}$ cm^{-2} ($l_g=(\varkappa\hbar c)^{1/2}=10^{-33}$ cm). These effects are due either to the interaction of particles with the macroscopic gravitational field (such as pair creation; see, for example, [1–3]) or gravitational exchange interaction between the particles.[4,5] As was shown in[1], pair creation must be taken into account if the metric is not conformally Euclidean. The effect of gravitational exchange interaction does not depend on the type of the macroscopic metric, and a qualitative discussion of its consequences can be found in[4,5].

The analysis of quantum gravitational effects and their influence on the macroscopic geometry is of fundamental importance near singularities, especially in view of their inescapability in classical general relativity.[6]

In[7,8], a proposal was made for constructing a theory of gravitation associated with calculating the contribution of quantum processes to the macroscopic Lagrangian of the gravitational field. A somewhat different approach was discussed in[9]—a theory based on Einstein's equations written down for the Heisenberg operators, with the equations for observables being obtained by averaging these operator equations.

The present paper is based on the ideas put forward in[9]. In §1, the operator Einstein equations are transformed, after separation from the metric of the graviton operators (it is not assumed they are small), to a system of equations for the macrogeometry and the quantum field. In §2, gravitational exchange interaction between photons is calculated; in §3, the first term is calculated in the expansion of the energy density of the photons and a new cosmological solution differing from the Friedmann solution near $\rho\approx\rho_g$ and $R_i^k\approx l_g^{-2}$ is found.

§1. EQUATIONS OF THE THEORY

We introduce operators of the metric $\hat{g}_{ik}$ and $\hat{G}^{il}$ ($\hat{g}_{ik}\hat{G}^{il}=\hat{\delta}_k^l$), the connection $\hat{\Gamma}_{ik}^l$, and the curvature $\hat{R}_{ik}$, retaining for them the differential and algebraic relations of Riemannian geometry. Further, we represent the operators $\hat{g}_{ik}$ in the form

$$\hat{g}_{ik}=\hat{1}g_{ik}+h_{ik}, \tag{1.1}$$

where $g_{ik}=\langle\hat{g}_{ik}\rangle$ are the expectation values of the operator with respect to the density matrix and they are interpreted as the metric of macroscopic spacetime; $\hat{1}$ is the identity operator. The graviton operators h_{ik} are defined in such a way that under transformation of the macroscopic coordinates they behave like tensors in the g_{ik} space. This enables one to obtain an explicit expression for the inverse operator $\hat{G}^{ik}$ in terms of the macrotensor g^{ik} and invariants of the matrix h_i^k:

$$\eta^2\hat{G}^{ik}=g^{ik}(\hat{1}+I_1+I_2+I_3)-h^{ik}(\hat{1}+I_1+I_2)+h^{im}h_m^k(\hat{1}+I_1)-h^{im}h_m^sh_s^k, \tag{1.2}$$

where

$$\eta^2=\frac{\det\|\hat{g}_{ik}\|}{\det\|g_{ik}\|}=\hat{1}+I_1+I_2+I_3+I_4,$$
$$I_1=h_s^s=h,\quad I_2={}^1/_2(h^2-h_l^mh_m^l),$$
$$I_3={}^1/_6(h^3-3hh_m^lh_l^m+2h_m^lh_l^nh_n^m),\tag{1.3}$$
$$I_4={}^1/_{24}(h^4-6h^2h_m^lh_l^m+8hh_m^lh_l^nh_n^m+3h_l^mh_m^lh_l^sh_s^l-6h_l^mh_m^nh_n^sh_s^l).$$

As is shown in[10,11], the representation (1.1) of the metric leads to the following relations for the connection and curvature:

$$\hat{\Gamma}_{ik}^l=\hat{1}\Gamma_{ik}^l+\mathscr{T}_{ik}^l,\quad \hat{R}_{ik}=\hat{1}R_{ik}+\mathscr{P}_{ik}, \tag{1.4}$$

$$\mathscr{T}_{ik}^l={}^1/_2\hat{G}^{lm}(h_{mi;k}+h_{mk;i}-h_{ik;m}), \tag{1.5}$$

$$\mathscr{P}_{ik}=\mathscr{T}_{ik;l}^{\ l}-\mathscr{T}_{il;k}^{\ l}+\mathscr{T}_{ik}^l\mathscr{T}_{lm}^m-\mathscr{T}_{il}^m\mathscr{T}_{km}^l. \tag{1.6}$$

In (1.4), Γ_{ik}^l and R_{ik} are expressed in terms of g_{ik} in the usual manner, and in (1.5) and (1.6) the semicolon denotes the covariant derivative in the macroscopic space.

Equations (1.1)–(1.6) define the quantities that occur on the left-hand side of the operator Einstein equations:

$$\hat{G}^{kl}\hat{R}_{il}-{}^1/_2\delta_i^k\hat{G}^{ml}\hat{R}_{ml}=\varkappa\hat{T}_i^k. \tag{1.7}$$

On the right-hand side of (1.7), the energy–momentum tensor must also be expressed in terms of field operators. For the electromagnetic field (to which we restrict ourselves in this paper)

$$\hat{T}_i^k=\frac{1}{4\pi}\left[-\hat{G}^{ks}\hat{G}^{lp}F_{il}F_{sp}+\frac{1}{4}\delta_i^k\hat{G}^{ms}\hat{G}^{lp}F_{ml}F_{sp}\right]. \tag{1.8}$$

Using a number of operator identities, one can obtain Maxwell's equations, which are contained in (1.7)–(1.8):

$$(\eta \hat{G}^{im}\hat{G}^{kl}F_{ml})_{;k}=0, \quad F_{[ik,l]}=0. \tag{1.9}$$

Finding the expectation value of (1.7) with respect to the density matrix, we arrive at equations for the macroscopic metric:

$$\langle\eta\hat{G}_l{}^k\rangle R_i{}^l-{}^1/_2\delta_i{}^k\langle\eta\hat{G}_l{}^m\rangle R_m{}^l+\langle\eta\hat{G}_l{}^k\mathscr{P}_i{}^l\rangle -{}^1/_2\delta_i{}^k\langle\eta\hat{G}_l{}^m\mathscr{P}_m{}^l\rangle=\varkappa\langle\eta\hat{T}_i{}^k\rangle. \tag{1.10}$$

The equations for the quantum gravitational field are obtained by subtracting (1.10) from (1.7). We write them in the form

$$L_i{}^k-{}^1/_2\delta_i{}^kL=\varkappa(\eta\hat{T}_i{}^k-\langle\eta\hat{T}_i{}^k\rangle), \quad L_i{}^k=\eta\hat{G}_l{}^k\mathscr{P}_i{}^l+\eta\hat{G}_l{}^kR_i{}^l-\langle\eta\hat{G}_l{}^k\mathscr{P}_i{}^l\rangle-\langle\eta\hat{G}_l{}^k\rangle R_i{}^l. \tag{1.11}$$

Equations (1.5)–(1.6) also enable us to obtain the following expression needed for concrete calculations:

$$\eta\hat{G}_l{}^k\mathscr{P}_i{}^l={}^1/_2[\eta\hat{G}_q{}^k\hat{G}_m{}^l(h^{qm}{}_{;i}+h_i^{m;q}-h_i^{q;m})]_{;l}-{}^1/_2(\eta\hat{G}_q{}^k\hat{G}_m{}^lh_l^{m;q})_{;i} +{}^1/_4\eta\hat{G}_q{}^k\hat{G}_l{}^p\hat{G}_m{}^s(h_p^{l;q}h_s^m{}_{;i}+2h_s^{q;l}h_p^m{}_{;i}-h_s^{l;q}h_p^m{}_{;i}-2h_{p;i}^qh_s^{m;l}). \tag{1.12}$$

The meaning of the factor $\eta=(\hat{g}/g)^{1/2}$ introduced into (1.10)–(1.12) will be explained when we consider the conservation law of the particle number. The operator equation has the form

$$j^i{}_{;i}+\hat{\Gamma}_{ki}{}^ij^k=0, \tag{1.13}$$

where j^i is the operator of the four-current. Using the relation $\hat{\Gamma}_{ki}{}^i=(\ln\eta\sqrt{-g})_{,k}$ we can reduce (1.13) to the form

$$I^i{}_{;i}=0, \quad I^i=\eta j^i. \tag{1.14}$$

As follows from (1.14), the conserved quantity $\langle I^i\rangle=\langle\eta j^i\rangle$ is an observable. The observables in (1.10) are defined similarly.

In terms of its content, the theory considered here is in many respects similar to the theory of quantum gravitational processes in a "reference" classical space-time (see, for example, [12,13]). The nonlinear equations (1.9) and (1.11) describe the interactions and transformations of the particles—photons and gravitons. A new element is that the "reference" space is here not given—the quantum gravitational processes influence its geometry. The corresponding effects are described by the terms in Eq. (1.10) that are nonlinear in $h_i{}^k$; this equation must be solved simultaneously with (1.9) and (1.11). Equation (1.11) differs mathematically from the customary one only in that similar expectation values are subtracted from all the nonlinear field operator combinations in it. It will be seen that this leads to elimination of various divergences.[1)] We note finally that the averaging is performed everywhere with respect to the ensemble of real particles.

A solution of Eqs. (1.9)–(1.11) at particle energies $E<E_g=\hbar c/l_g=10^{27}$ eV can be obtained by perturbation theory. It follows from estimates of the gravitational operators in the zeroth ($h_{i(0)}^k\sim l_g/\lambda\approx(E/E_g)^{1/2}\sim\varkappa^{1/2}$) and first ($h_{i(1)}^k\sim l_g{}^2/\lambda^2\approx E/E_g\sim\varkappa$) approximations that Eqs. (1.9)–(1.11) can be expressed as series in the gravitational constant.

§2. GRAVITATIONAL EXCHANGE INTERACTION IN A SUPERDENSE PHOTON GAS

Let us consider the first nonvanishing quantum gravitational effects in a photon gas which fills an isotropic macroscopic universe with the metric

$$ds^2=a^2(\eta)(d\eta^2-dx^2-dy^2-dz^2). \tag{2.1}$$

The aim of the calculations is to obtain a quantum transport equation and the energy spectrum of photons with allowance for interaction, the gravitational contribution to their energy, and, finally, a cosmological solution $a=a(\eta)$.

To simplify the calculations, we restrict ourselves to a purely photon model of the Universe, i.e., we ignore the spontaneous creation of real gravitons in the macroscopic field and resulting from the collision of photons. In this model $h_{i(0)}^k=0$ and the operators $h_{i(1)}^k$ describe only virtual gravitons—fluctuations of the metric generated over distances of the order of the de Broglie wavelengths by the quantum uncertainties in the states of the photons. From (1.10) and (1.11) we readily obtain the estimate

$$h_i{}^k{}_{(1)}\sim\lambda^2R_i{}^m\sim E/E_g. \tag{2.2}$$

Equations (2.2) enable us to write down approximate equations that take into account the first nonvanishing quantum gravitational effects (the index (1) of $h_i{}^k$ is omitted):

$$-R={}^1/_4\langle h^{;k}h_{;k}+2h_m^{k;l}h_l{}^m{}_{;k}-h_m^{l;k}h_l{}^m{}_{;k}-2h_l{}^k{}_{;k}h^{;l}\rangle, \tag{2.3}$$

$${}^1/_2(h_i^{l;k}{}_{;l}+h_l^k{}_{;i}{}^{;l}-h_i^{k;l}{}_{;l}-h_{;i}{}^{;k})-{}^1/_2\delta_i{}^k(h_m^{l;m}{}_{;l}-h_{;m}{}^{;m}) =\frac{\varkappa}{4\pi}\left(-F_{il}F^{kl}+\frac{1}{4}\delta_i{}^kF_{ml}F^{ml}+\langle F_{il}F^{kl}\rangle-{}^1/_4\delta_i{}^k\langle F_{ml}F^{ml}\rangle\right), \tag{2.4}$$

$$F_{im}{}^{;m}=(h_m{}^lF_{il}+h_i{}^lF_{lm}-{}^1/_2hF_{im})^{;m}. \tag{2.5}$$

The terms ignored in (2.3) have order $\varkappa^3$ and in (2.4) and (2.3) the order $\varkappa^2$. It is convenient to obtain the solution of Eqs. (2.4) and (2.5) on the isotropic background (2.1) in the second quantization representation, expanding the operators with respect to three-dimensional scalar, vector, and tensor plane waves (the Lifshits expansion[14]). However, if one chooses the Coulomb gauge for the potentials of the electromagnetic field, $A_0=0$, then the right-hand side of (2.4) contains a bilinear combination of only vector functions. By virtue of the parity conservation law, particular solutions of Eq. (2.4) corresponding to virtual gravitons $h_{i(1)}^k$ do not contain terms proportional to vector functions. Eliminating three-vector gravitons at once from the treatment, we write down the expansions for the operators in the form

$$A_\alpha={}^{(3)}A_\alpha=(2\pi\hbar c)^{1/2}\sum_{\mathbf{k},\sigma}\frac{S_\alpha(\mathbf{k},\sigma)}{\omega_k{}^{1/2}}a_{\mathbf{k}}(\sigma)e^{i\mathbf{kx}}+\text{H.c.}, \quad A^\alpha=-\frac{1}{a^2}{}^{(3)}A^\alpha, \tag{2.6}$$

$$h_\alpha^\beta = {}^{(3)}h_\alpha^\beta = \frac{1}{a^2}\sum_{\mathbf{p},\tau}\left[\frac{\delta_\alpha^\beta}{3}(\mu_\mathbf{p}+\lambda_\mathbf{p}) - \frac{p_\alpha p^\beta}{p^2}\lambda_\mathbf{p} + Q_\alpha^\beta(\mathbf{p},\tau) f_\mathbf{p}(\tau)\right] e^{i\mathbf{p}\mathbf{x}} + \text{H.c.}, \tag{2.7}$$

$$h_\alpha^0 = {}^{(3)}h_\alpha = \frac{1}{a}\sum_\mathbf{p} \frac{p_\alpha}{p}\sigma_\mathbf{p} e^{i\mathbf{p}\mathbf{x}} + \text{H.c.}, \quad h_0^\alpha = -\frac{1}{a^2}\,{}^{(3)}h^\alpha, \tag{2.8}$$

$$h_0^0 = {}^{(3)}\chi = \frac{1}{a^2}\sum_\mathbf{p} \chi_\mathbf{p} e^{i\mathbf{p}\mathbf{x}} + \text{H.c.}, \tag{2.9}$$

In the expansions (2.6)–(2.9) the index (3) means that we have introduced three-dimensional quantities. All operations with spatial indices are performed here and in what follows by the unit tensor $\gamma_{\alpha\beta} = \text{diag}\,(1, 1, 1)$. The transverse vector $S_\alpha(\mathbf{k}, \sigma)$ and transverse tensor $Q_\alpha{}^\beta(\mathbf{p}, \tau)$ have appropriate polarization indices σ and τ. The operators $a_\mathbf{k}^+(\sigma)$, $a_\mathbf{k}(\sigma)$ in the zeroth approximation (for free photons) satisfy the usual commutation relations. Below, we shall omit the indices σ and τ, and double the result of averaging over the polarizations.

In this approximation, after substitution of (2.6)–(2.9) into (2.4) we can perform a Fourier transformation with respect to the time. Contraction of Eq. (2.4) gives the connection between the operators:

$$\nu^2 b_{\mathbf{p}\nu} - {}^2/{}_3 p^2 c_{\mathbf{p}\nu} = 0, \tag{2.10}$$

where

$$b_{\mathbf{p}\nu} = \mu_{\mathbf{p}\nu} - \frac{2p}{\nu}\sigma_{\mathbf{p}\nu} - \frac{p^2}{\nu^2}\chi_{\mathbf{p}\nu}, \quad c_{\mathbf{p}\nu} = \mu_{\mathbf{p}\nu} + \lambda_{\mathbf{p}\nu}. \tag{2.11}$$

In all the other equations obtained from (2.4) by Fourier transformation with respect to scalar functions, and also in Eqs. (2.3) and (2.5), the expansion coefficients again occur only in the form of the combinations (2.11), which are invariant under gauge transformations. This enables us to solve the problem without recourse to additional conditions on the field $h_i{}^k$.

If the relations (2.10) are used, all the scalar equations in (2.4) in this approximation are equal to each other and give

$$\frac{p^2}{3} c_{\mathbf{p}\nu} = \frac{\varkappa\hbar c}{4}\sum_{\mathbf{k},\omega,\mathbf{q},\Omega} \frac{(\omega\Omega + \mathbf{k}\mathbf{q})\delta_\mu{}^\gamma - k^\gamma q_\mu}{(\omega_k\omega_q)^{1/2}}\Phi_\gamma{}^\mu. \tag{2.12}$$

The equation for the tensor gravitons is obtained by projecting the (α, β) components of (2.4) onto $Q_\beta{}^{\alpha *} e^{-i\mathbf{p}\cdot\mathbf{x} + i\nu\eta}$:

$$(\nu^2 - p^2) f_{\mathbf{p}\nu} = \varkappa\hbar c \sum_{\mathbf{k},\omega,\mathbf{q},\Omega} \frac{(\omega\Omega + \mathbf{k}\mathbf{q}) Q_\beta{}^{\alpha *}(\mathbf{p}) - k_\gamma q^\mu Q_\mu{}^{\gamma *}(\mathbf{p})\delta_\beta{}^\alpha - k_\gamma q^\alpha Q_\beta{}^{\gamma *}(\mathbf{p}) - k_\beta q^\gamma Q_\gamma{}^\alpha(\mathbf{p})}{(\omega_k\omega_q)^{1/2}}\Phi_\alpha{}^\beta. \tag{2.13}$$

In (2.12)–(2.13) we have introduced the notation

$$\begin{aligned}\Phi_\alpha{}^\beta = &-S_\alpha(\mathbf{k}) S^\beta(\mathbf{q}) a_{\mathbf{k}\omega} a_{\mathbf{q}\Omega}\delta(\mathbf{k}+\mathbf{q}-\mathbf{p})\delta(\omega+\Omega-\nu)\\ &+ S_\alpha^*(\mathbf{k}) S^\beta(\mathbf{q}) (a_{\mathbf{k}\omega}^+ a_{\mathbf{q}\Omega} - \langle a_{\mathbf{k}\omega}^+ a_{\mathbf{q}\Omega}\rangle)\delta(\mathbf{q}-\mathbf{k}-\mathbf{p})\delta(\Omega-\omega-\nu)\\ &+ S_\alpha(\mathbf{k}) S^{\beta *}(\mathbf{q}) (a_{\mathbf{k}\omega} a_{\mathbf{q}\Omega}^+ - \langle a_{\mathbf{k}\omega} a_{\mathbf{q}\Omega}^+\rangle)\delta(\mathbf{k}-\mathbf{q}-\mathbf{p})\delta(\omega-\Omega-\nu).\end{aligned} \tag{2.14}$$

Real photons surround themselves with a cloud of virtual gravitons, which are described by Eqs. (2.12)–(2.13). The energy spectrum of the "dressed" photons and their density matrix can be calculated directly from the Maxwell equations (2.5). After Fourier transformations,

$$\begin{aligned}(k^2-\omega^2) a_{\mathbf{k}\omega} = \frac{1}{2a^2}\sum_{\mathbf{p},\nu,\mathbf{q}\Omega} k[&F(\mathbf{p},\mathbf{k},\mathbf{q}) a_{\mathbf{q}\Omega}\delta(\mathbf{p}+\mathbf{q}-\mathbf{k})\delta(\nu+\Omega-\omega)\\ &+F^+(\mathbf{p},\mathbf{k},\mathbf{q}) a_{\mathbf{q}\Omega}\delta(\mathbf{p}+\mathbf{k}-\mathbf{q})\delta(\nu+\omega-\Omega) + F(\mathbf{p},\mathbf{k},\mathbf{q}) a_{\mathbf{q}\Omega}^+\delta(\mathbf{p}-\mathbf{q}-\mathbf{k})\delta(\nu-\Omega-\omega)],\end{aligned} \tag{2.15}$$

where

$$F(\mathbf{p},\mathbf{k},\mathbf{q}) = \frac{p^2-\nu^2}{2p^2} W_1(\mathbf{k},\mathbf{q}) c_{\mathbf{p}\nu} + W_2(\mathbf{p},\mathbf{k},\mathbf{q}) f_{\mathbf{p}\nu}, \tag{2.16}$$

$$W_1(\mathbf{k},\mathbf{q}) = \frac{(\omega_k\omega_q + \mathbf{k}\mathbf{q})\delta_\alpha{}^\beta - k^\beta q_\alpha}{(\omega_k\omega_q)^{1/2}} S^{\alpha *}(\mathbf{k}) S_\beta(\mathbf{q}), \tag{2.17}$$

$$\begin{aligned}W_2(\mathbf{p},\mathbf{k},\mathbf{q}) = \{[&k^\gamma q_\mu Q_\gamma{}^\mu(\mathbf{p})\delta_\alpha{}^\beta - (\omega_k\omega_q - \mathbf{k}\mathbf{q}) Q_\alpha{}^\beta(\mathbf{p})\\ &- k^\gamma q_\alpha Q_\gamma{}^\beta(\mathbf{p}) - k^\beta q_\gamma Q_\alpha{}^\gamma(\mathbf{p})] S^{\alpha *}(\mathbf{k}) S_\beta(\mathbf{q})\}/(\omega_k\omega_q)^{1/2}.\end{aligned} \tag{2.18}$$

The adopted approximations have enabled us in the calculation of the right-hand side of (2.15) to perform a Fourier transformation with respect to only the high-frequency time dependence. In the reduction of the kernels to the form (2.16)–(2.18), we have used the connection (2.10) between the operators, and we have also assumed that the photons have a dispersion relation. To find the observables, we must multiply Eq. (2.18) from the left by $a_{\mathbf{k}'\omega'}^+$ and average. The expectation values on the right-hand side are calculated by means of (2.12) and (2.13). To within terms $\sim \varkappa^2$

$$\begin{aligned}(k^2-\omega^2)\langle a_{\mathbf{k}'\omega'}^+ a_{\mathbf{k}\omega}\rangle = \frac{\varkappa\hbar c}{a^2}\sum_{\substack{\mathbf{p},\nu;\mathbf{q},\Omega;\\ \mathbf{l},\varphi;\mathbf{m},\psi}} &k\delta(\mathbf{k}+\mathbf{l}-\mathbf{m}-\mathbf{q})\delta(\omega+\varphi-\psi-\Omega)\\ \times\{&V(\mathbf{p},\nu;\mathbf{k},\mathbf{q};\mathbf{l},\mathbf{m})\delta[\mathbf{p}+(\mathbf{q}-\mathbf{k})\,\text{sign}\,(\Omega-\omega)]\delta[\nu+(\Omega-\omega)\,\text{sign}\,(\Omega-\omega)]\\ &\times[\langle a_{\mathbf{k}'\omega'}^+ a_{\mathbf{l}\varphi}^+ a_{\mathbf{m}\psi} a_{\mathbf{q}\Omega}\rangle - \langle a_{\mathbf{k}'\omega'}^+ a_{\mathbf{q}\Omega}\rangle\langle a_{\mathbf{l}\varphi}^+ a_{\mathbf{m}\psi}\rangle + \langle a_{\mathbf{k}'\omega'}^+ a_{\mathbf{m}\psi} a_{\mathbf{l}\varphi}^+ a_{\mathbf{q}\Omega}\rangle\\ &-\langle a_{\mathbf{k}'\omega'}^+ a_{\mathbf{q}\Omega}\rangle\langle a_{\mathbf{m}\psi} a_{\mathbf{l}\varphi}^+\rangle] + V(\mathbf{p},\nu;\mathbf{k},\mathbf{l};\mathbf{q},\mathbf{m})\delta(\mathbf{p}-\mathbf{l}-\mathbf{k})\delta(\nu-\varphi-\omega)\\ &\times\langle a_{\mathbf{k}'\omega'}^+ a_{\mathbf{l}\varphi} a_{\mathbf{m}\psi} a_{\mathbf{q}\Omega}^+\rangle\},\end{aligned} \tag{2.19}$$

where

$$V(\mathbf{p},\nu;\mathbf{k},\mathbf{q};\mathbf{l},\mathbf{m}) = \frac{3(p^2-\nu^2) W_1(\mathbf{k},\mathbf{q}) W_1(\mathbf{l},\mathbf{m})}{16 p^4(\omega_k\omega_q\omega_l\omega_m)^{1/2}} + \frac{W_2(\mathbf{p},\mathbf{k},\mathbf{q}) W_2(\mathbf{p},\mathbf{l},\mathbf{m})}{2(p^2-\nu^2)(\omega_k\omega_q\omega_l\omega_m)^{1/2}}. \tag{2.20}$$

It is readily seen that Eq. (2.19) describes the interaction of two photons, $(\mathbf{k}, \omega_k)$, $(\mathbf{l}, \omega_l) \rightleftarrows (\mathbf{q}, \omega_q)$, $(\mathbf{m}, \omega_m)$, through the virtual graviton $(\mathbf{p}, \nu)$.

The energy spectrum of the photons is obtained by separating the real part from (2.19). To calculate the spectrum, it is necessary to assume that the operators in (2.19) are normally ordered—this enables one to eliminate the gravitational effect of a photon on itself. After decomposition of the expectation values of four operators into products of expectation values of two operators, averaging over the polarizations, and the transition from summation to integration, we have

$$\omega_k{}^2 = k^2 - \frac{\varkappa\hbar c}{a^2}\int \omega_k W(\mathbf{k},\mathbf{q}) N_\mathbf{q}\frac{d^3q}{(2\pi)^3}. \tag{2.21}$$

In (2.21),

$$W(\mathbf{k},\mathbf{q}) = \frac{(\omega_k\omega_q + \mathbf{k}\mathbf{q})^2(\omega_k\omega_q - \mathbf{k}\mathbf{q})}{\omega_k\omega_q}\left(\frac{1}{(\mathbf{k}-\mathbf{q})^4} - \frac{1}{(\mathbf{k}+\mathbf{q})^4}\right) + \frac{\omega_k{}^2\omega_q{}^2 - (\mathbf{k}\mathbf{q})^2}{\omega_k\omega_q}\left(\frac{1}{(\mathbf{k}+\mathbf{q})^2} + \frac{1}{(\mathbf{k}-\mathbf{q})^2}\right), \tag{2.22}$$

$N_{\mathbf{q}}$ is the photon distribution function defined by

$$N_{\mathbf{q}}\delta(\mathbf{q}-\mathbf{q}')\delta(\Omega-\omega_q)=\int\langle a^{+}_{\mathbf{q}'\Omega'}a_{\mathbf{q}\Omega}\rangle e^{i(\Omega'-\Omega)\eta}d(\Omega'-\Omega).$$

A transport equation for the distribution function is derived as follows: From (2.19) separate the imaginary part, express the expectation values of four operators in terms of those of six, decompose the latter into all possible products of expectation values of two operators. Then, average again over the polarizations and go over to integration, obtaining[2)]

$$\frac{\partial N_{\mathbf{k}}}{\partial\eta}=\frac{(\varkappa\hbar c)^2}{a^4}\int U(\mathbf{k},\mathbf{q},\mathbf{l},\mathbf{m})\,(N_{\mathbf{k}}N_{\mathbf{l}}N_{\mathbf{m}}+N_{\mathbf{k}}N_{\mathbf{q}}N_{\mathbf{l}}$$
$$-N_{\mathbf{m}}N_{\mathbf{q}}N_{\mathbf{l}}-N_{\mathbf{k}}N_{\mathbf{m}}N_{\mathbf{q}}+N_{\mathbf{k}}N_{\mathbf{l}}-N_{\mathbf{m}}N_{\mathbf{q}})\,\delta(\mathbf{k}+\mathbf{l}-\mathbf{q}-\mathbf{m})\cdot$$
$$\times\delta(\omega_k+\omega_l-\omega_q-\omega_m)\frac{d^3q\,d^3l\,d^3m}{(2\pi)^9},\qquad(2.23)$$

where

$$U(\mathbf{k},\mathbf{q},\mathbf{l},\mathbf{m})=\frac{1}{\pi^3}\,[V(\mathbf{q}-\mathbf{k},\omega_q-\omega_k;\mathbf{k},\mathbf{q};\mathbf{l},\mathbf{m})$$
$$+V(\mathbf{m}-\mathbf{k},\omega_m-\omega_k;\mathbf{k},\mathbf{m};\mathbf{q},\mathbf{l})+V(\mathbf{l}+\mathbf{k},\omega_l+\omega_k;\mathbf{k},\mathbf{l};\mathbf{q},\mathbf{m})]^2.\qquad(2.24)$$

The reciprocal lifetime of the single-photon state with energy ω_k is determined by the expression

$$\gamma(\mathbf{k})=\frac{(\varkappa\hbar c)^2}{a^4}\int U(\mathbf{k},\mathbf{q},\mathbf{l},\mathbf{m})N_{\mathbf{l}}(N_{\mathbf{m}}+1)(N_{\mathbf{q}}+1)\delta(\mathbf{k}+\mathbf{l}-\mathbf{m}-\mathbf{q})$$
$$\times\delta(\omega_k+\omega_l-\omega_m-\omega_q)\frac{d^3q\,d^3l\,d^3m}{(2\pi)^9}.\qquad(2.25)$$

The system (2.21)–(2.25) describes gravitational effects in the photon gas in the two-photon interaction approximation. A solution of these equations can be obtained by successive approximations. We note first of all that the equilibrium solution of (2.23) is the Planck distribution:

$$N_{\mathbf{k}}=(\exp(\omega_k/k_0)-1)^{-1},\qquad(2.26)$$

where k_0 is a parameter related to the temperature by $T=\hbar c k_0/a$. The relaxation time has the order $1/\gamma(k_0)$, and in ordinary units

$$\tau\sim a^5/ck_0{}^5l_g{}^4=\hbar^3c^2/\varkappa^2T^5.$$

To find the energy spectrum of equilibrium photons on the right-hand side of (2.21) it is sufficient to use only the dispersion relation $\omega_k=k$ for "bare" particles. As a result of calculating the integral in (2.21), we obtain

$$\omega_k{}^2=k^2-\frac{\varkappa\hbar c}{2a^2}\left\{\sum_{n=1}^{\infty}\frac{1}{\pi^2}\left[\left(\frac{k_0k^3}{n}+\frac{3k_0{}^3k}{n^3}+\frac{3k_0{}^5}{n^5k}\right)\left(e^{-nk/k_0}\mathrm{Ei}\left(\frac{nk}{k_0}\right)\right.\right.\right.$$
$$\left.\left.-e^{nk/k_0}\mathrm{Ei}\left(-\frac{nk}{k_0}\right)\right)+\left(\frac{3k_0{}^4}{n^4}+\frac{2k_0{}^2k^2}{n^2}\right)\left(e^{-nk/k_0}\mathrm{Ei}\left(\frac{nk}{k_0}\right)+e^{nk/k_0}\mathrm{Ei}\left(-\frac{nk}{k_0}\right)\right)\right]$$
$$\left.+\frac{\pi^2k_0{}^2}{15}+\frac{k_0{}^2k^2}{12}\right\}.\qquad(2.27)$$

Note that the expressions for the energy spectrum in the region of large and small k differ weakly from one another:

$$\omega_k{}^2=k^2\left(1-\frac{17}{72}\frac{k_0{}^2\varkappa\hbar c}{a^2}\right),\quad k\ll k_0,$$
$$\omega_k{}^2=k^2\left(1-\frac{1}{8}\frac{k_0{}^2\varkappa\hbar c}{a^2}\right),\quad k\gg k_0.\qquad(2.28)$$

The distribution function of the photons with allowance for their gravitational interaction is obtained by substituting (2.27) into (2.26). The total density matrix of the system of particles and gravitational field in this approximation is determined by the expressions

$$\langle a^{+}_{\mathbf{k}'\omega'}a_{\mathbf{k}\omega}\rangle=N_{\mathbf{k}}\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')\delta(\omega-\omega_k),\qquad(2.29)$$

$$\langle c^{+}_{\mathbf{p}'\nu'}c_{\mathbf{p}\nu}\rangle=\delta(\mathbf{p}-\mathbf{p}')\delta(\nu-\nu')\frac{9}{4}(\varkappa\hbar c)^2\sum_{\mathbf{k},\mathbf{q}}\frac{(\omega_k\omega_q+\mathbf{kq})^2}{p^4\omega_k\omega_q}N_{\mathbf{k}}N_{\mathbf{q}}$$
$$\times\{\delta(\mathbf{k}+\mathbf{q}-\mathbf{p})\delta(\omega_k+\omega_q-\nu)+\delta[\mathbf{p}+(\mathbf{k}-\mathbf{q})\,\mathrm{sign}(\omega_k-\omega_q)]$$
$$\times\delta[\nu+(\omega_k-\omega_q)\,\mathrm{sign}(\omega_k-\omega_q)]\},\qquad(2.30)$$

$$\langle f^{+}_{\mathbf{p}'\nu'}f_{\mathbf{p}\nu}\rangle=\delta(\mathbf{p}-\mathbf{p}')\delta(\nu-\nu')(\varkappa\hbar c)^2\sum_{\mathbf{k},\mathbf{q}}\left[\frac{(\omega_k\omega_q+\mathbf{kq})^2}{2p^4\omega_k\omega_q}\right.$$
$$\left.+\frac{4(\omega_k{}^2\omega_q{}^2-(\mathbf{kq})^2)}{p^2(p^2-\nu^2)\omega_k\omega_q}+\frac{4(\omega_k\omega_q-\mathbf{kq})^2}{(p^2-\nu^2)^2\omega_k\omega_q}-\frac{(\omega_k{}^2\omega_q{}^2-(\mathbf{kq})^2)^2}{(p^2-\nu^2)^2\omega_k{}^3\omega_q{}^3}\right]N_{\mathbf{k}}N_{\mathbf{q}}$$
$$\times\{\delta(\mathbf{k}+\mathbf{q}-\mathbf{p})\delta(\omega_k+\omega_q-\nu)+\delta[\mathbf{p}+(\mathbf{k}-\mathbf{q})\,\mathrm{sign}(\omega_k-\omega_q)]$$
$$\times\delta[\nu+(\omega_k-\omega_q)\,\mathrm{sign}(\omega_k-\omega_q)]\}.\qquad(2.31)$$

Equations (2.30)–(2.31) can be obtained directly from (2.12) and (2.13) by multiplying these equations by their Hermitian conjugates and averaging. The operators of the photon field were ordered in the calculation.

§3. COSMOLOGICAL SOLUTION

A cosmological solution taking into account the above quantum gravitational effect in the photon gas can be found from Eq. (2.3). Substituting the expansions (2.7) and (2.9) into the right-hand side of (2.3), we obtain (3.1)

$$6\frac{a''}{a^3}=\frac{1}{a^6}\sum_{\mathbf{p},\nu;\mathbf{p}',\nu'}(\mathbf{pp}'-\nu\nu')\left[\frac{1}{3}\langle c^{+}_{\mathbf{p}'\nu'}c_{\mathbf{p}\nu}\rangle+\frac{1}{2}\langle f^{+}_{\mathbf{p}'\nu'}f_{\mathbf{p}\nu}\rangle\right]e^{i(\mathbf{p}-\mathbf{p}')\mathbf{x}-i(\nu-\nu')\eta}.$$

In the following calculations, we use (2.30) and (2.31) and go over from summation to integration. Equation (3.1) takes the form

$$6\frac{a''}{a^3}=\frac{(\varkappa\hbar c)^2}{a^6}\int W(\mathbf{k},\mathbf{q})N_{\mathbf{k}}N_{\mathbf{q}}\frac{d^3k\,d^3q}{(2\pi)^6}.\qquad(3.2)$$

In (3.2) the kernel $W(\mathbf{k},\mathbf{q})$ of the integral is equal to the kernel (2.22) in the expression for the photon energy spectrum. This result is a consequence of the self-consistency of the Einstein and Maxwell equations. Using the fact that the integral in (3.2) does not depend on the time in this approximation, we integrate this equation once. We have

$$a'^2=R_0{}^2-\frac{1}{6}\frac{(\varkappa\hbar c)^2}{a^2}\int W(\mathbf{k},\mathbf{q})N_{\mathbf{k}}N_{\mathbf{q}}\frac{d^3k\,d^3q}{(2\pi)^6}\qquad(3.3)$$

The constant of integration $R_0{}^2$ is found by comparing (3.3) with the (0,0) component of the Einstein equations. Obviously, its value is determined by the principal term in the energy—momentum tensor of the photons. We therefore rewrite Eq. (3.3) as follows:

$$R_0{}^0-\frac{1}{2}R=3\frac{a'^2}{a^4}=\varkappa\varepsilon,\qquad(3.4)$$

where

$$\varepsilon=\frac{2\hbar c}{a^4}\int kN_{\mathbf{k}}\frac{d^3k}{(2\pi)^3}-\frac{1}{2}\frac{\varkappa\hbar^2c^2}{a^6}\int W(\mathbf{k},\mathbf{q})N_{\mathbf{k}}N_{\mathbf{q}}\frac{d^3k\,d^3q}{(2\pi)^6}\qquad(3.5)$$

is the energy density of the photon gas with allowance for the gravitational exchange interaction. It is clear from the calculations that the ordering of the operators in (2.30), (2.31), and (3.1) eliminates the self-interaction energy of the photons and the gravitational energy of the vacuum.

Calculation of the integrals in (3.5) gives

$$2\int kN_{\mathbf{k}}\frac{d^3k}{(2\pi)^3}=\frac{\pi^2k_0^4}{15}, \tag{3.6}$$

$$\frac{1}{2}\int W(\mathbf{k},\mathbf{q})N_{\mathbf{k}}N_{\mathbf{q}}\frac{d^3k\,d^3q}{(2\pi)^6}$$
$$=\frac{k_0^6}{32\pi^4}\int_0^\infty\int_0^\infty\frac{(x^4+6x^2y^2+y^4)\ln|(x+y)/(x-y)|-2xy(x^2+y^2)}{(e^x-1)(e^y-1)}dx\,dy$$
$$=1.68\pi^2\cdot10^{-3}k_0^6.$$

The energy density ε can now be represented as a function of the particle density $n=0.244k_0^3/a^3$:

$$\varepsilon=4.31\hbar cn^{4/3}-0.282\varkappa\hbar^2c^2n^2.$$

Thus, we have calculated the first term of the Sakharov expansion.[5]

The cosmological solution is obtained by integrating (3.4). Using the numerical values of (3.6), we write the solution in the form

$$a^2=\frac{\pi^2k_0^4\varkappa\hbar c}{45}\eta^2+a_0^2,$$

where

$$a_0=0.16\,k_0(\varkappa\hbar c)^{1/2}. \tag{3.7}$$

Going over to the cosmological time

$$t=\frac{1}{c}\int a\,d\eta,$$

we obtain

$$\frac{2\pi k_0^2(\varkappa\hbar c)^{1/2}}{3\sqrt{5}}ct=a(a^2-a_0^2)^{1/2}+a_0^2\ln\frac{a+(a^2-a_0^2)^{1/2}}{a_0}. \tag{3.8}$$

It is interesting to note that the cosmological solution (3.8), which takes into account the gravitational exchange interaction, does not formally contain a singularity.

§4. DISCUSSION OF RESULTS

Our results enable us to discuss the question of whether allowance for quantum gravitational effects in a many-particle system resolves the singularity problem. The approximate solution (3.8) is valid, as follows from (2.2), when $a^2\gg a_0^2$. Values $a\sim a_0$ are at the limit of applicability of the approximation, and therefore, on the basis of (3.8), one can say that there is a tendency for the singularity to disappear in the quantum theory. The extremal parameters of the photon gas when[3)] $a=a_0=10^{-4}$ cm calculated from our equations:

$$n_0=59.6(\varkappa\hbar c)^{-3/2}=0.47(G\hbar/c^3)^{-3/2},$$
$$T_0=6.25(\hbar c/\varkappa)^{1/2}=1.25(\hbar c^5/G)^{1/2}.$$

agree with the usually quoted dimensional agruments.

In reality, the problem is complicated by the fact that near the extremal state one can no longer keep the picture of macroscopic spacetime as a continuous manifold nor of photons as objects existing in this spacetime. Even from Eq. (3.8) it follows that the transition from contraction to expansion occurs over the time "quantum" $t\sim l_g/c=t_g$, and strictly speaking it cannot therefore be considered in the theory formulated here. The inadequacy of the description of photons follows from (2.27) and (2.28)—it is easy to see that when $a\sim a_0$ we have $\omega^2<0$, i.e., there are no solutions that represent the electromagnetic field as a collection of particles. It would seem that we can get away from these difficulties only in a theory that combines the physics of quantum gravitational processes with canonical quantization of spacetime.

However, despite its limitations, the theory of quantum gravitational processes in a self-consistent "reference" spacetime does enable us, we feel, to discuss some of the characteristic features of physical phenomena at extreme densities and curvatures.

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1)This will be seen in the actual calculations.

2)The terms corresponding to nonbound second-order processes cancel in the derivation of Eq. (2.23) for the reason given at the end of Sec. §1 (see Remark 1).

3)According to the data on the microwave background $k_0\approx10^{27}$ in (2.26).

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