# X-ray emission by an ultrarelativistic charge in a plate with allowance for multiple scattering

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A formula for the frequency-angular distribution of the total radiation emitted by an ultrarelativistic charge in a plate of arbitrary thickness is derived with allowance for multiple scattering. A formula for the frequency spectrum of the radiation is also obtained by integrating the frequency-angular distribution over the emission angle. A detailed analysis of, and a numerical computation based on, the obtained formulas are carried out. It is shown that, besides giving rise to bremsstrahlung emission along the entire path of the charge inside the matter, multiple scattering leads to some smoothing out of the interference oscillations in the frequency spectrum of the transition radiation (if they exist) and, in the case of a sufficiently large value of the Lorentz factor for the charge, to the enrichment of this spectrum in the high-frequency region right up to frequencies proportional to the square of the Lorentz factor for the fast charge.

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Pomeranchuk, Pafomov, and one of the present authors[1,2] have noted that multiple scattering should, under certain conditions, exert an appreciable influence on the emission of x-ray transition radiation by an ultrarelativistic charge. The quantitative theory of this phenomenon in the case of one boundary was developed with the use of the techniques of the kinetic equation[3,4] in[5–7]. Notice that there is ambiguity in the indicated papers about the question of the separation of the edge effect. The problem of x-ray transition and bremsstrahlung emissions in a plate was first considered by Ternovskiĭ,[6] using the method of the quantum kinetic equation, and by Pafomov[8] in the classical approximation. However, the first paper lacks a detailed analysis of the results, and the second contains inaccuracies.

In the present paper we derive with the aid of the techniques of the kinetic equation[3,4] a formula for the frequency-angular distribution of the radiation that can be emitted in a plate of arbitrary thickness. By means of an explicit analytic integration of the indicated formula over the emission angle, we are also able to obtain an expression for the frequency spectrum. A detailed analysis of this expression and a corresponding numerical computation are presented.

## 1. DERIVATION OF THE BASIC FORMULAS

Let an ultrarelativistic charged particle move in a vacuum with velocity $\mathbf{v}_0$ along the $z$ axis from $-\infty$, and at the point $\mathbf{r}_0=0$ perpendicularly enter a plate of thickness $a$ and permittivity $\varepsilon=\varepsilon'+i\varepsilon''$. The distribution of the intensity of the emitted radiation at points far behind the plate is determined by the formula (see[9])

$$dW(\theta_0,\omega)=\frac{e^2\omega^2}{4\pi^2c^3}|\mathbf{A}|^2\,d\mathbf{\theta}_0\,d\omega, \tag{1}$$

where $d\mathbf{\theta}_0=\theta_0 d\theta_0 d\varphi$ is an element of solid angle in the direction of propagation of the radiation relative to the initial direction of motion of the particle.

The quantity $\mathbf{A}$ consists of the term $\mathbf{A}_1$ due to the motion of the charge in the plate and the term $\mathbf{A}_0$ connected with the motion of the charge in the vacuum:

$$\mathbf{A}=\mathbf{A}_1+\mathbf{A}_0; \tag{2}$$

$$\mathbf{A}_1=\exp[i(\lambda-\lambda_0)a]\int_0^T[\mathbf{n}\times\mathbf{v}]\exp\{i(\omega t-\mathbf{kr})\}dt, \tag{3}$$

$$\mathbf{A}_0=\frac{-2i[\mathbf{n}\times\mathbf{v}_0]\exp[i(\lambda-\lambda_0)a]}{\omega(\gamma^{-2}+\theta_0^2)}+\int_T^\infty[\mathbf{n}\times\mathbf{v}_1]\exp\{i[\omega t-\mathbf{k}_0(\mathbf{r}_1+\mathbf{v}_1(t-T))]\}dt. \tag{4}$$

In these formulas $\gamma=(1-\beta^2)^{-1/2}$ is the Lorentz factor for the particle; $\mathbf{n}$ is the unit vector in the direction of emission; $\mathbf{r}$ and $\mathbf{v}$ are the coordinate and velocity of the particle in the plate at an arbitrary moment of time $t$; $\mathbf{r}_1$ and $\mathbf{v}_1$ are the same quantities at the moment $T$ the particle emerges from the plate. The two-dimensional vector $\boldsymbol{\theta}_0$ lies in a plane perpendicular to the direction, $\mathbf{n}$, of emission of the photon, and is determined by the relation

$$v_0\boldsymbol{\theta}_0=\mathbf{n}(\mathbf{n}\mathbf{v}_0)-\mathbf{v}_0. \tag{5}$$

Furthermore,

$$\lambda=\frac{\omega}{c}(\varepsilon-\sin^2\theta_0)^{1/2}=\lambda'+i\lambda'',\quad \lambda_0=\frac{\omega}{c}\cos\theta_0,$$

and the vectors $\mathbf{k}$ and $\mathbf{k}_0$ have the same components in the $xy$ plane, equal to $\varkappa$ ($\varkappa=(\omega/c)\sin\theta_0$), and longitudinal components equal respectively to $\lambda$ and $\lambda_0$. The quantity $\exp[i(\lambda-\lambda_0)a]$ is the coefficient of transmission of the radiation through the plate.[10]

The charged particle moves in the plate along some trajectory resulting from multiple scattering by the atoms of the material. Here we shall assume that the absolute value $v$ of the velocity remains unchanged and that the changes in the direction of the velocity are small.[3] After the emergence from the plate, the charge moves again in a vacuum with a constant velocity $\mathbf{v}_1$.

In order to obtain the observable radiation intensity, it is necessary to average the expression (1) over all possible trajectories of the charge:

$$\langle dW(\theta_0,\omega)\rangle=\frac{e^2\omega^2}{4\pi^2c^3}\{\langle|\mathbf{A}_1|^2\rangle+\langle|\mathbf{A}_0|^2\rangle+2\,\mathrm{Re}\langle\mathbf{A}_0^*\mathbf{A}_1\rangle\}d\mathbf{\theta}_0\,d\omega. \tag{6}$$

Using the procedure employed by Migdal in his paper, [3] we have

$$\langle|\mathbf{A}_1|^2\rangle=2v^2\exp(-2\lambda''a)\,\mathrm{Re}\iint_0^T w(\mathbf{r}_0,\mathbf{v}_0;\mathbf{r},\mathbf{v};t)\,\theta\theta'\,d\mathbf{r}\,d\theta\,dt$$
$$\times\int_t^T w(\mathbf{r},\mathbf{v};\mathbf{r}',\mathbf{v}';t'-t)\exp\{i[\omega(t'-t)-\mathbf{k}\mathbf{r}'+\mathbf{k}^*\mathbf{r}]\}\,d\mathbf{r}'\,d\theta'\,dt'. \qquad (7)$$

Here $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{r}'$, $\mathbf{v}'$ are the coordinates and velocities of the charge at the moments of time $t$ and $t'$ respectively. The two-dimensional vectors $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ are connected with $\mathbf{v}$ and $\mathbf{v}'$ by relations similar to (5). The quantity $w(\mathbf{r}_0, \mathbf{v}_0; \mathbf{r}, \mathbf{v}; t)$ is the conditional probability that the coordinate and velocity of the particle have at the moment of time $t$ the values $\mathbf{r}$, $\mathbf{v}$ if at the initial moment of time $t=0$ they had the values $\mathbf{r}_0$, $\mathbf{v}_0$.

The second term of the formula (6) has the form

$$\langle|\mathbf{A}_0|^2\rangle=\frac{4v^2}{\omega^2}\exp(-2\lambda''a)\frac{\theta_0^2}{(\gamma^{-2}+\theta_0^2)^2}+v^2\iint_0^\infty\int_0^\infty\theta_1^2 w(\mathbf{r}_0,\mathbf{v}_0;\mathbf{r}_1,\mathbf{v}_1;T)$$
$$\times\exp\{i(\omega-\mathbf{k}_0\mathbf{v}_1)(t-t')\}\,d\mathbf{r}_1\,d\theta_1\,dt\,dt'+\frac{4v^2}{\omega}\exp(-2\lambda''a)\,\mathrm{Re}\,i$$
$$\times\iint_0^\infty\frac{\theta_0\theta_1}{\gamma^{-2}+\theta_0^2}w(\mathbf{r}_0,\mathbf{v}_0;\mathbf{r}_1,\mathbf{v}_1;T)\exp\{i[\omega T-\mathbf{k}\mathbf{r}_1+(\omega-\mathbf{k}_0\mathbf{v}_1)t]\}\,d\mathbf{r}_1\,d\theta_1\,dt. \qquad (8)$$

Here the two-dimensional vector $\boldsymbol{\theta}_1$ is related to the vector $\mathbf{v}_1$. Furthermore, in the integrals figuring in the formula (8) we have, for simplicity of notation, changed the reference time point.

Finally, the third term of the formula (6) can be written in the form

$$2\,\mathrm{Re}\langle\mathbf{A}_0^*\mathbf{A}_1\rangle$$
$$=\frac{4v^2}{\omega^2}\exp(-2\lambda''a)\,\mathrm{Re}\,i\iint_0^T\frac{\theta_0\theta}{\gamma^{-2}+\theta_0^2}w(\mathbf{r}_0,\mathbf{v}_0;\mathbf{r},\mathbf{v};t)\exp[i(\omega t-\mathbf{k}\mathbf{r})]\,d\mathbf{r}\,d\theta\,dt$$
$$+2v^2\,\mathrm{Re}\exp[i(\lambda-\lambda_0)a]\iint_0^T dt\int_0^\infty\theta\theta_1\exp\{i[\omega(t-t'-T)$$
$$-\mathbf{k}\mathbf{r}+\mathbf{k}_0(\mathbf{r}_1+\mathbf{v}_1t')]\}\,w(\mathbf{r}_0,\mathbf{v}_0;\mathbf{r},\mathbf{v};t)\,w(\mathbf{r},\mathbf{v};\mathbf{r}_1,\mathbf{v}_1;T-t)\,d\mathbf{r}\,d\theta\,d\mathbf{r}_1\,d\theta_1\,dt'. \qquad (9)$$

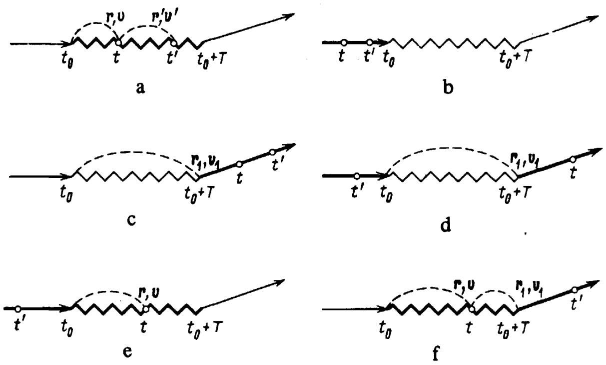

FIG. 1. Diagrams corresponding to the various terms of the formula for the radiation arising during the passage of a charge through a plate. The straight lines correspond to the particle motion in the vacuum; the zigzag, in the plate in the presence of multiple scattering. The heavy parts of the diagrams indicate the ranges of integration over the moments of time marked by the small circles. The dashed lines correspond to relative probabilities with integration over the arguments indicated at their right ends. The diagram a corresponds to the formula (7) the diagrams b, c, and d, to the three terms of the formula (8); the diagrams e and f, to the terms of the formula (9).

It is convenient to represent the terms of the formulas (7)–(9) in the form of the diagrams shown in Fig. 1. As follows from the general procedure used in Migdal's paper, [3] and as can be seen from these diagrams, averaging in all the terms that have been written down is carried out in such a way that the particle's coordinates and velocities $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{r}'$, $\mathbf{v}'$ at the moments of time $t$ and $t'$ (later than the moment of time $t_0$ corresponding to the entry of the particle into the plate) are successively related by the conditional probabilities to the value of the particle coordinate $\mathbf{r}_0$ and velocity $\mathbf{v}_0$ at the initial moment of time $t_0$.

It can be seen from the formulas (7)–(9) that for their explicit evaluation we should have the expressions for the quantities

$$u(\theta_0,\theta;t)=\int w(\mathbf{r}_0,\mathbf{v}_0;\mathbf{r},\mathbf{v};t)\exp\{i[\omega t-\mathbf{k}(\mathbf{r}-\mathbf{r}_0)]\}\,d\mathbf{r}, \qquad (10)$$

$$u_1(\theta_0,\theta;t)=\int w(\mathbf{r}_0,\mathbf{v}_0;\mathbf{r},\mathbf{v};t)\exp(2\lambda''z)\,d\mathbf{r}. \qquad (11)$$

The quantity (10) was obtained by Gol'dman[4] (see also[6,7]):

$$u(\theta_0,\theta;t)=\frac{\sigma}{\pi\,\mathrm{sh}\,x}\exp\left\{-\sigma\left[\theta^2-\frac{2\theta\theta_0}{\mathrm{ch}\,x}+\theta_0^2\right]\mathrm{cth}\,x-\sigma g x\right\}, \qquad (12)$$

where

$$\sigma=(-i\omega/8q)^{1/2},\quad x=(-2i\omega q)^{1/2}t,$$
$$q=\left(\frac{E_s}{mc^2}\right)^2\frac{c}{4L\gamma^2},\quad g=g'-i\frac{2\lambda''c}{\omega},\quad g'=\gamma^{-2}+\frac{\omega_0^2}{\omega^2},$$

$m$ is the rest mass of the traversing particle, $L$ and $\omega_0$ are the radiation unit of length and the plasma frequency of the material of the plate, and $E_s = 21$ MeV.

As to $u_1(\theta_0, \theta; t)$, a similar quantity was computed by Pafomov. [11] However, below we shall, for simplicity, neglect absorption, assuming the plate to be completely transparent in the region of x-ray frequencies under consideration ($g=g'$). Then

$$u_1(\theta_0,\theta;t)=u_0(\theta_0,\theta;t)=\frac{\exp\{-(\theta-\theta_0)^2/4qt\}}{4\pi qt}. \qquad (13)$$

Substituting (12) and (13) into the formulas (7)–(9), we obtain

$$\langle|\mathbf{A}_1|^2\rangle=2v^2\,\mathrm{Re}\int_0^T dt\int_0^{T-t}d\tau\int\theta\theta' u_0(\theta_0,\theta;t)\,u(\theta,\theta';\tau)\,d\theta\,d\theta', \qquad (14)$$

$$\langle|\mathbf{A}_0|^2\rangle=\frac{4v^2}{\omega^2}\frac{\theta_0^2}{(\gamma^{-2}+\theta_0^2)^2}+v^2\int_0^\infty dt\int_0^\infty dt'$$
$$\times\int\theta_1^2\exp\left[\frac{i\omega}{2}(\gamma^{-2}+\theta_1^2)(t-t')\right]u_0(\theta_0,\theta_1;T)\,d\theta_1$$
$$+\frac{4v^2}{\omega}\mathrm{Re}\,i\int_0^\infty dt\int\frac{\theta_0\theta_1}{\gamma^{-2}+\theta_0^2}\exp\left[\frac{i\omega t}{2}(\gamma^{-2}+\theta_1^2)\right]u(\theta_0,\theta_1;T)\,d\theta_1. \qquad (15)$$

$$2\,\mathrm{Re}\langle\mathbf{A}_0^*\mathbf{A}_1\rangle=\frac{4v^2}{\omega}\mathrm{Re}\,i\int_0^T dt\int\frac{\theta\theta_0}{\gamma^{-2}+\theta_0^2}u(\theta_0,\theta;t)\,d\theta+2v^2\,\mathrm{Re}\int_0^T dt\int_0^\infty dt'\int\theta\theta_1$$
$$\times\exp\left[-\frac{i\omega t'}{2}(\gamma^{-2}+\theta_1^2)\right]u_0(\theta_0,\theta;t)\,u^*(\theta,\theta_1;T-t)\,d\theta\,d\theta_1. \qquad (16)$$

In the formulas (14)–(16) it is necessary to carry out the integration over the intermediate angles $\boldsymbol{\theta}$, $\boldsymbol{\theta}'$, and $\boldsymbol{\theta}_1$. (For example, $d\boldsymbol{\theta}=\theta\,d\theta\,d\varphi$, $0\leq\varphi\leq 2\pi$, $0\leq\theta<\infty$.)

This integration is performed in all the formulas by the same method; to wit, we complete in each exponent the square involving the angle over which the integration is performed. As a result, taking (6) into account, we obtain

$$\langle dW(\theta_0, \omega)\rangle = (W_1+W_2)\,d\zeta\, d\omega; \tag{17}$$

$$W_1 = -\frac{2e^2}{\pi c}\operatorname{Re}\left\{\sigma\int_0^{x_0} dx \int_0^{x_0-x} M(x,y)\,dy\right\}, \tag{18}$$

$$W_2 = \frac{e^2}{\pi c}\operatorname{Re}\left\{\frac{\zeta}{(\gamma^{-2}+\zeta)^2} + \sigma\int_0^\infty dx \int_0^\infty \frac{x_0+\sigma\zeta/[1+x_0(x+y)]}{[1+x_0(x+y)]^2} \times \exp\left\{-\sigma(x+y)\left[\gamma^{-2}+\frac{\zeta}{1+x_0(x+y)}\right]\right\}dy \right.$$
$$-\frac{2\zeta\sigma}{\gamma^{-2}+\zeta}\int_0^\infty \frac{\exp\{-\sigma[\gamma^{-2}x+gx_0+\zeta(x+\operatorname{th} x_0)/(1+x\operatorname{th} x_0)]\}}{(1+x\operatorname{th} x_0)^2\operatorname{ch}^2 x_0}dx$$
$$-\frac{2\zeta\sigma}{\gamma^{-2}+\zeta}\int_0^{x_0}\frac{\exp\{-\sigma(gx+\zeta\operatorname{th} x)\}dx}{\operatorname{ch}^2 x}$$
$$\left. -2\sigma\int_0^{x_0} dx\int_0^\infty \frac{(\xi(x_0-x)+\sigma\zeta)\exp\{-\sigma[gx+\gamma^{-2}y+\zeta(\xi-1)/\xi(x_0-x)]\}dy}{\xi^3(\operatorname{ch} x+y\operatorname{sh} x)^2}\right\}; \tag{19}$$

where

$$\zeta=\theta_0^2,\quad x_0=(-2i\omega q)^{1/2}T,\quad \xi=1+\frac{(x_0-x)(\operatorname{sh} x+y\operatorname{ch} x)}{\operatorname{ch} x+y\operatorname{sh} x},$$
$$M(x,y)=\left(x+\frac{\sigma\zeta}{1+x\operatorname{th} y}\right)\frac{1}{(1+x\operatorname{th} y)^2\operatorname{ch}^2 y}\exp\left\{-\sigma\left[yg+\frac{\zeta\operatorname{th} y}{1+x\operatorname{th} y}\right]\right\}. \tag{20}$$

In the formula (17) the term $W_1$ corresponds to the radiation in the plate (the diagram a, Fig. 1). As to the term $W_2$, the first three of its addends correspond to radiations emitted before the plate (the diagram b), after the plate (c), and the interaction between them (d); the fourth and fifth addends correspond to the interference between the radiation emitted in the plate and the radiations emitted before and after the plate (the diagrams e and f).

In the analogous formula in Pafomov's paper, [8] the terms corresponding to the diagrams c and f are inaccurate. In the first term the path of integration in the complex $y$ plane is incorrectly chosen. In the computation in Pafomov's paper[8] of the term corresponding to the diagram f, the conditional probabilities for the values of the particle coordinates and velocities at the moments of time $t_0$, $t$, and $t+T$ (see Fig. 1) were not successively related to each other.

The formula (17) has been derived in the approximation when the mean square of the angle of multiple scattering of the particle in the plate is small, i.e., when

$$\langle\theta_m^2\rangle = 4qT = x_0/\sigma \ll 1. \tag{21}$$

In contrast to the usual transition radiation without allowance for multiple scattering, in the case under consideration the radiation intensity per unit solid angle does not tend to zero at small angles ($\zeta\to 0$), owing to the diagrams a, c, and f, i.e., because of the bremsstrahlung.

## 2. ANALYSIS OF THE FORMULA FOR THE FREQUENCY-ANGULAR DISTRIBUTION

Let us investigate the formula (17) in the case when the plate thickness is sufficiently small, so that the following two conditions are fulfilled;

$$|x_0|\ll 1,\quad x_0\gamma^2/\sigma\ll 1. \tag{22}$$

Then we can expand the integrands in (18) and (19) in series in powers of the small quantities, and restrict ourselves to the lowest terms.

After the integration we obtain as the dominant terms

$$\langle dW(\theta_0,\omega)\rangle_0 = \frac{4e^2}{\pi c}\left\{\zeta\left(\frac{1}{\gamma^{-2}+\zeta}-\frac{1}{g+\zeta}\right)^2\sin^2\frac{X_a}{2}+qT\frac{\gamma^{-4}+\zeta^2}{(\gamma^{-2}+\zeta)^4}\right\}d\zeta\, d\omega, \tag{23}$$

$$X_a = i\sigma x_0(g+\zeta) = {}^1\!/_2\,\omega T(g+\zeta). \tag{24}$$

The first term in (23) corresponds to the usual transition radiation produced in a plate by a uniformly and rectilinearly moving particle (i.e., without allowance for multiple scattering). The second term corresponds to bremsstrahlung, also without allowance for the influence of multiple scattering. Such a result is not fortuitous. Indeed, the conditions (22) imply that: 1) the plate thickness is much smaller than the dimension of the bremsstrahlung-formation zone, $z_{\text{brems}} = c(q\omega)^{-1/2}$ (see[5], as well as[12]); 2) the mean square of the angle of multiple scattering of the particle in the plate is much smaller than the square of the emission angle, $\gamma^{-2}$. It is clear that under these conditions multiple scattering should not significantly affect the spectrum of the radiation, which, in the present case, is a sum of the usual transition radiation and bremsstrahlung.

The influence of the multiple scattering of the particle is expressed in the following terms of the expansion:

$$\langle dW(\theta_0,\omega)\rangle_{\text{corr}} = \frac{8e^2}{\pi c}qT\left(\frac{1}{\gamma^{-2}+\zeta}-\frac{1}{g+\zeta}\right)\left\{\left(\frac{\sin X_a}{X_a}-1\right)\right.$$
$$\times\frac{3\zeta^3+(2\gamma^{-2}-g)\zeta^2-g\gamma^{-2}\zeta+\gamma^{-2}g^2}{(\gamma^{-2}+\zeta)(g+\zeta)^3}+\zeta(\cos X_a-1)\left(\frac{2\gamma^{-2}}{(\gamma^{-2}+\zeta)^3}-\frac{1}{(g+\zeta)^2}\right)$$
$$\left.+\frac{\zeta X_a\sin X_a}{(g+\zeta)}\left(\frac{\gamma^{-2}}{(\gamma^{-2}+\zeta)^2}-\frac{g}{(g+\zeta)^2}\right)+\frac{X_a^2\zeta^2\cos X_a}{3(g+\zeta)^2}\left(\frac{1}{\gamma^{-2}+\zeta}-\frac{1}{g+\zeta}\right)\right\}d\zeta\, d\omega. \tag{25}$$

Notice that the conditions (22) can be fulfilled not only on account of a reduction in the plate thickness, but also as a result of the weakening of the scattering process itself, i.e., for $q\to 0$ (for example, if we consider a particle of large mass for the same value of $\gamma$). In the latter case, as was to be expected, the dominant term in (23) turns out to be the first term, which describes the transition radiation in the plate.

Let us now consider the case of large thicknesses. First of all, notice that for a sufficiently thick plate part of the radiation intensity, directly proportional to the plate thickness, should separate out. This part corresponds to bremsstrahlung, which is emitted along the entire length of the particle's path inside the plate. Such a separation will occur under conditions when the plate thickness is much greater than the dimension of the bremsstrahlung-formation zone when allowance is made for multiple scattering, i.e., provided

$$|x_0|\gg 1. \tag{26}$$

The part proportional to the thickness $a$ should separate from $W_1$ (the diagram a). If in the expression for $W_1$ we replace the upper integration limit $x_0-x$ by infinity, then we can obtain a quantity corresponding to bremsstrahlung emitted over a distance $a$ in an infinite medium.[3] The remaining part of the expression for $W_1$ will have the form

$$W_1' = \frac{2e^2}{\pi c}\operatorname{Re}\sigma\int_0^\infty dx\int_{x_0-x}^\infty M(x,y)\,dy. \tag{27}$$

The radiation due to the edge effects will, consequently, be described by the formula

$$\langle dW(\theta_0,\omega)\rangle' = (W_1'+W_2)\,d\zeta\,d\omega. \tag{28}$$

If in this case the mean square of the angle of multiple scattering over the distance $a$ is nevertheless smaller than the square, $\gamma^{-2}$, of the emission angle (i.e., if the second of the conditions (22) is fulfilled), then we can show that the expression (28) reduces (for $\zeta \gtrsim \gamma^{-2}$) to the formula for the frequency-angular distribution of the normal transition radiation produced in the plate.

## 3. THE FREQUENCY SPECTRUM OF THE RADIATION

To obtain the frequency spectrum, it is necessary to integrate the formulas for the angular spectrum over the emission angle. This can be done in the formulas (14)–(16), as well as in the formulas (18) and (19). Bearing in mind that

$$\int u_0(\theta_0,\theta;t)\,d\theta_0 = 1,$$

we can easily verify that the first two terms in the formula (15) coincide after they have been integrated over $\theta_0$, as well as over $t$ and $t'$. Also coincident will be the two terms of the formula (16) if they are integrated over $\theta_0$ and $t'$. As a result, the total radiation will consist of four terms which are equal to the corresponding terms of the formula (22) of Ternovskiĭ's paper[6] in the classical limit, i.e., when the energy of the emitted photons is much lower than the energy of the primary charge.

After this, it is not difficult to integrate over each intermediate angle in the same manner as was done in the first section. We obtain

$$\langle dW(\omega)\rangle = \frac{2e^2}{\pi c}\operatorname{Re}\left\{\int_0^\infty \frac{\zeta\,d\zeta}{(\gamma^{-2}+\zeta)^2} - \sigma\int_0^\infty \frac{\exp[-\sigma(\gamma^{-2}x+gx_0)]\,dx}{\operatorname{ch}^2 x_0(1+x\operatorname{th}x_0)^2}\right.$$
$$\times\int_0^\infty \frac{\zeta\exp\{-\sigma\zeta(1+x\operatorname{cth}x_0)/(x+\operatorname{cth}x_0)\}}{\gamma^{-2}+\zeta}\,d\zeta - 2\sigma\int_0^{x_0}\frac{\exp(-\sigma gx)\,dx}{\operatorname{ch}^2 x}$$
$$\left.\times\int_0^\infty\left[\frac{1}{\gamma^{-2}+\zeta}+\frac{\sigma(x_0-x)}{2}\right]\zeta\exp(-\sigma\zeta\operatorname{th}x)\,d\zeta\right\}d\omega. \tag{29}$$

In this formula, a direct integration over $\zeta$ to an infinitely large upper limit leads to the following difficulties. First, a logarithmic divergence immediately appears in the first term. Second, there arise in the third term integrals that diverge at the lower limit in the case of the integration over $x$. To avoid these difficulties, we shall integrate over $\zeta$ to some maximum value $\zeta_m$ (see[4]). Bearing in mind that the value $\zeta_m$ should eventually be allowed to go to infinity, let us separate out in the integrands of the diverging integrals the parts containing $\exp(-\sigma x\zeta_m)$. Then let us expand the expressions attached to this quantity in powers of small values of $x$, and then carry out the subsequent integration. After this, let us allow $\zeta_m$ to go to infinity. The logarithmic divergences arising here in all the terms will cancel each other out. As a result, we obtain

$$\langle dW(\omega)\rangle = \frac{2e^2}{\pi c}\operatorname{Re}\Big\{\ln(g\gamma^2) - \operatorname{ci}(X_g) + X_g\operatorname{si}(X_g) + \cos(X_g) - 2$$
$$-\sigma\gamma^{-2}\frac{\exp(-\sigma gx_0)}{\operatorname{ch}^2 x_0}\int_0^\infty \frac{\exp(-\sigma\gamma^{-2}x)}{(1+x\operatorname{th}x_0)^2}\left[\exp(\alpha)\operatorname{Ei}(-\alpha)+\frac{1}{\alpha}\right]dx$$
$$+2\int_0^{x_0}\exp(-\sigma gx)\left[\frac{1}{x}-\frac{1}{\operatorname{sh}x\operatorname{ch}x}-\frac{\sigma\gamma^{-2}\exp(\sigma\gamma^{-2}\operatorname{th}x)}{\operatorname{ch}^2 x}\operatorname{Ei}(-\sigma\gamma^{-2}\operatorname{th}x)\right.$$
$$\left.+\frac{(x_0-x)}{2}\left(\frac{1}{x^2}-\frac{1}{\operatorname{sh}^2 x}\right)\right]dx\Big\}d\omega, \tag{30}$$

where $\operatorname{ci}(x)$, $\operatorname{si}(x)$, and $\operatorname{Ei}(x)$ are the integral cosine, integral sine, and the exponential integral function (see, for example, [13]), while the quantities $X_g$ and $\alpha$ are determined by the formulas

$$X_g = |\sigma gx_0| = \frac{\omega T}{2}g, \quad \alpha = \frac{\sigma\gamma^{-2}(x+\operatorname{th}x_0)}{1+x\operatorname{th}x_0}. \tag{31}$$

We can also arrive at the formula (30) if we integrate the formulas (18) and (19) over $\zeta$ from zero to infinity in the manner indicated above. We then find again that the expressions corresponding to the diagrams b and c, as well as those corresponding to e and f, coincide in pairs after the integration over $\zeta$ has been performed.

Let us now analyze the formula (30). Let us first consider the case when the conditions (22) are fulfilled. Then we can show that the relation

$$\langle dW(\omega)\rangle = \langle dW(\omega)\rangle_{\text{tran}} + \langle dW(\omega)\rangle_{\text{brems}}$$

where $\langle dW(\omega)\rangle_{\text{tran}}$ is the frequency spectrum of the usual transition radiation produced in the plate without allowance for the influence of multiple scattering,[14] follows from (30). The quantity $\langle dW(\omega)\rangle_{\text{brems}}$ describes the frequency spectrum of the bremsstrahlung in the plate with allowance for the polarization of the medium. If $X_g \gg 1$, then from (30) we obtain

$$\langle dW(\omega)\rangle_{\text{brems}} = \frac{8e^2qT}{3\pi cg}\,d\omega. \tag{32}$$

We can also arrive at the same result if we integrate the appropriate terms in the formulas (23) and (25) over the angle $\zeta$. If, on the other hand, $X_g \ll 1$, then we obtain a formula similar to (32) with $g$ replaced by $\gamma^{-2}$. The same formula can be obtained if the second term of the formula (23) is integrated over $\zeta$. In order for this formula to coincide with the Bethe-Heitler formula, the quantity $q$ must be decreased by a factor of two (see *apropos* of this[15]).

Now let the plate thickness be sufficiently large, so that the condition (26) is fulfilled. It follows at once from this that the first integral in (30), which corresponds to the diagram d, vanishes. Furthermore, we can separate from (30) the part

$$\langle dW_{\text{M}}(\omega)\rangle=\frac{2e^2}{\pi c}\operatorname{Re}\left[x_0\int_0^\infty \exp(-\sigma g x)\left(\frac{1}{x^2}-\frac{1}{\operatorname{sh}^2 x}\right)dx\right]d\omega, \tag{33}$$

which, as was noted in the preceding section, corresponds to bremsstrahlung emitted over a distance $a$ in an infinite medium.[3] Let us denote the remaining part, which describes the effect of the boundaries on this radiation, by $\langle dW(\omega)\rangle'$:

$$\langle dW(\omega)\rangle'=\langle dW(\omega)\rangle-\langle dW_{\text{M}}(\omega)\rangle. \tag{34}$$

Let us consider the following two cases.

a) Let the bremsstrahlung-formation zone $z_{\text{brems}}$ be much smaller than the transition-radiation-formation zone $z_{\text{tran}}=2c/\omega g$ in the material:

$$z_{\text{brems}}\ll z_{\text{tran}} \quad \text{i.e.} \quad |\sigma g|\ll 1. \tag{35}$$

Then there exists a quantity $x'$, such that, on the one hand, $|x'|\gg 1$ and, on the other, $|\sigma g x'|\ll 1$. Let us split up the second integral of the formula (30) into two parts: from zero to $x'$ and from $x'$ to $x_0$. In the first part the common exponential function can be dropped, after which the integral can easily be evaluated. In the second part we can drop all the terms having $\sinh x$ or $\cosh x$ in the denominators, after which the integral can also be evaluated. As a result, we obtain

$$\langle dW(\omega)\rangle'=\frac{2e^2}{\pi c}\left\{\ln\frac{1}{2|\sigma\gamma^{-2}|}+X_g\operatorname{si}(X_g)+\cos(X_g)-C-2\right\}d\omega, \tag{36}$$

where $C=0.5772$ is the Euler constant.

b) Now let

$$z_{\text{brems}}\gg z_{\text{tran}} \quad \text{i.e.,} \quad |\sigma g|\gg 1. \tag{37}$$

Then in the second integral of the formula (30) the important values of $x$ are the small values. Successively expanding the integrand in powers of $x$, and performing the corresponding integration, we obtain

$$\langle dW(\omega)\rangle'=\frac{2e^2}{\pi c}\left\{\left[\left(\frac{1+p}{1-p}\right)\ln\left(\frac{1}{p}\right)-2\right]+f\right\}d\omega; \tag{38}$$

$$p=\frac{1}{g\gamma^2}=\frac{1}{1+(\omega_0\gamma/\omega)^2}, \tag{39}$$

$$f=\frac{1}{|\sigma g|^4}\left\{\frac{10}{3}+\frac{296}{15}p+\frac{40}{3}p^2-\frac{32}{5}pF(1,5;6,1-p)\right.$$
$$\left.-\frac{56}{3}p^2F(1,6;7,1-p)-\frac{40}{7}p^3F(1,7;8,1-p)\right\}. \tag{40}$$

Here $F(\alpha, \beta; \gamma, z)$ is the hypergeometric series (see, for example, [13]).

When $\omega\lesssim\omega_0\gamma$, the dominant quantity in the formula (38) is the quantity standing inside the square brackets. As is well known, this quantity corresponds to the frequency spectrum of the usual transition radiation produced at the two independent medium-vacuum interfaces. When, on the other hand, $\omega\gg\omega_0\gamma$, then the formula (38) can be written in the form

$$\langle dW(\omega)\rangle'=\frac{2e^2}{\pi c}\left\{\frac{1}{6}\left(\frac{\omega_0\gamma}{\omega}\right)^4-\frac{8}{21}\left(\frac{4q\gamma^4}{\omega}\right)^2\right\}d\omega. \tag{41}$$

The formula (34) determines the effect of the boundaries on the production of radiation in a thick plate, when the plate thickness is much greater than the dimension of the bremsstrahlung-formation zone (the condition (26)). If in this case the plate thickness is also much larger than the formation zone for the transition radiation, i.e., if

$$a/z_{\text{tran}}=X_g\gg 1, \tag{42}$$

then it is to be expected that the edge effects from each of the two plate boundaries will additively add up.

Indeed, it can be shown that the formula (34) (with allowance for (30) and (33)) coincides, when the condition (42) is fulfilled, exactly with two times the formula (7) of[5], a formula which describes the edge effect in the case of a semi-infinite medium. It does not, however, coincide with the analogous formula of the paper[7]. Thus, we can consider it to be established that the method used in[4,5] to separate the edge effect is correct.

It should be noted that the separation of the edge effect was accomplished back in[6] on the basis of an analysis of the limiting case of a thick plate. It can be verified that the formula (16) obtained there in the classical limit also coincides with two times the formula (7) of[5].

## 4. DISCUSSION OF THE RESULTS

In the preceding section we derived the formulas (36) and (38) for certain particular cases by means of a formula analysis of the general formula (34), which describes the influence of the boundaries on the frequency spectrum of the radiation in a thick plate. These particular cases canbe understood on the basis of the following obvious arguments. Underlying the formulas (30) and (34) are two radiation-production mechanisms: transition and braking. On the other hand, each of these radiation-production mechanisms is characterized by a corresponding formation zone. It is clear that of these radiations the mechanism that is faster, i.e., whose formation zone is shorter, will predominate.

In Fig. 2 the continuous curves give the size of the formation zone of the transition radiation, while the dashed curves give that of the bremsstrahlung. In those frequency regions where the continuous curve lies far below the dashed curve (for a given value of $\gamma$), the dominant contribution to the edge effect will be made by the transition radiation, and the influence of multiple scattering can be neglected. In those regions where the opposite picture obtained, however, the radiation is produced mainly on account of the multiple-scattering mechanism. In this case the dominant contribution to the net boundary effect will be made by the boundary effect in the bremsstrahlung, the contribution of the transition radiation being negligible.

The maximum of the formation-zone curve for the transition radiation occurs at the frequency $\omega_{\text{lim}}=\omega_0\gamma$,

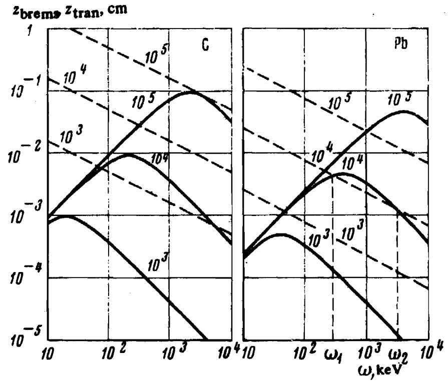


FIG. 2. Dependence of the production zones of the transition radiation (solid curves) and the bremsstrahlung (dashed lines) on the frequency $\omega$ for carbon and lead. The numbers on the curves indicate the values of the Lorentz factor for the electron.

and is equal to $c\gamma/\omega_0$. The length of the formation zone of the bremsstrahlung for $\omega=\omega_{1lm}$ is equal to $c(\gamma/q_0\omega_0)^{1/2}$, where $q_0=q\gamma^2=(E_s/mc^2)^2c/4L$. The solid curve will lie wholly below the dashed curve upon the fulfillment of the condition

$$c\gamma/\omega_0<c(\gamma/q_0\omega_0)^{1/2},$$

i.e., when $\gamma<\gamma_{cr}$, where

$$\gamma_{cr}\approx\omega_0/q_0. \quad (43)$$

When, however, $\gamma>\gamma_{cr}$, these curves will intersect. It is not difficult to verify that the intersection will occur at frequencies (see Fig. 2)

$$\omega_1\approx(\omega_0^4\gamma^2/q_0)^{1/3}, \quad \omega_2\approx q_0\gamma^2. \quad (44)$$

These formulas are correct if $\omega_1\ll\omega_0\gamma\ll\omega_2$, i.e., when $\gamma\gg\gamma_{cr}$.

Thus, the expression (34) should possess different frequency dependences in different regions of $\gamma$ values. If $\gamma\ll\gamma_{cr}$, then the expression (34) goes over into (38),

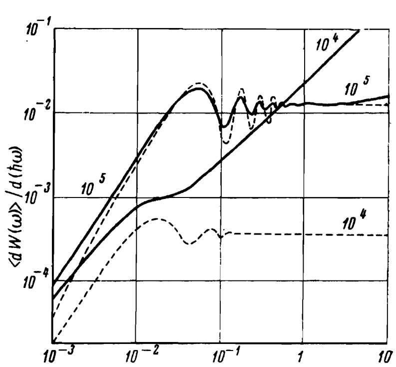


FIG. 3. Dependence of the radiation intensity at the frequency $\omega=200$ keV on the plate thickness $a$. The solid curves correspond to the total radiation (the formula (30)). The dashed curves correspond to ordinary transition radiation without allowance for the effect of multiple scattering. The numbers on the curves indicate the values of the Lorentz factor for a fast electron traversing the plate.

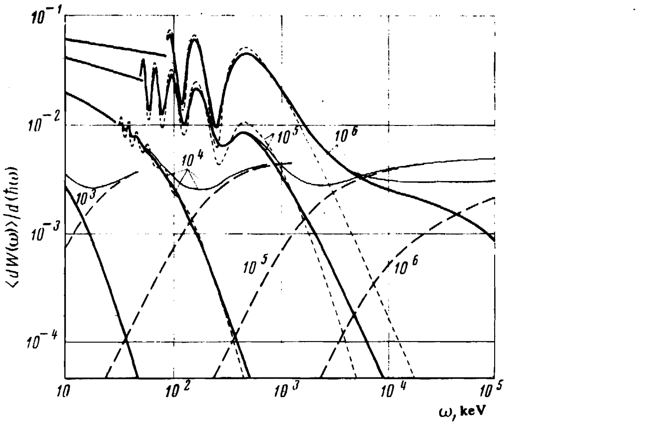


FIG. 4. The frequency spectra of the radiations arising in a mylar plate of thickness 1 mm (explanation in the text). In view of the strong oscillations in the curves at fairly low frequencies, we show at these frequencies only the averaged frequency spectra in the figure. The numbers on the curves indicate the $\gamma$ factor for the charge.

i.e., it basically coincides with the frequency spectrum of the usual transition radiation, but at high frequencies, when

$$\omega>\omega_3=7^{1/2}\omega_0^2/16q_0, \quad (45)$$

it becomes, according to (41), negative, being small in absolute value. The intensity of the total radiation remains, of course, positive.

If, on the other hand, $\gamma\gg\gamma_{cr}$, then the expression (34) coincides with the frequency spectrum of the usual transition radiation when $\omega\ll\omega_1$, goes over into (36) when $\omega_1\ll\omega\ll\omega_2$, and becomes negative (but small) when $\omega_2\ll\omega$, since in this case $\omega_2\gg\omega_3$.

Let us compare the expression (36) and the frequency spectrum of the usual transition radiation in the limiting-frequency region $\omega\gtrsim\omega_0\gamma$. In this region the frequency dependence of the expression (36) is made up of a logarithmic part and an oscillating (interference) part, while the frequency spectrum of the usual transition radiation decreases rapidly like $(\omega_0\gamma/\omega)^4$. Taking into account the fact that for $\gamma\gg\gamma_{cr}$ we have in the frequency region $\omega_1\ll\omega\ll\omega_2$ under consideration $|\sigma\gamma^{-2}|\ll1$ from the formula (36), we arrive at the conclusion that in this case multiple scattering leads to the enrichment of the spectrum right up to frequencies of the order of $\omega_2$. Consequently, the integrated—over all the frequencies—intensity of the bremsstrahlung connected with the edge effect will be approximately proportional to $\omega_2$, i.e., to $\gamma^2$. In this case the frequency spectrum of the edge effect in the plate may, in contrast to the case of the semi-infinite medium,[5] have oscillations in it.

As has already been noted, the analytic separation of the edge effect from the formula (30) is possible only in the case of a "thick" plate, when $a\gg z_{brems}$ (or, which is the same, $|x_0|\gg1$). For an arbitrary plate thickness, however, the radiation intensity should be computed from the formula (30). In Fig. 3 we present the results of such a numerical computation (the continuous curves), as well as the intensities of the ordinary transition radiation (the dashed curves) as functions of the plate thickness $a$ at a fixed frequency $\omega = 200$ keV. As the material of the plate we chose carbon, and the fast particle is the electron. It can be seen from the figure that for $\gamma = 10^5$, because of the fact that $z_{\text{brems}} \gg z_{\text{tran}}$ (see Fig. 2), the effect of multiple scattering is appreciable in the case of fairly small plate thicknesses ($a \lesssim 10^{-3}$ cm), when the transition radiation is negligible, as well as in the case of fairly large thicknesses ($a \gtrsim 10$ cm), when the dominant contribution is made by the bremsstrahlung produced along the entire thickness of the plate.[3] For intermediate thickness this effect is expressed in some smoothing out of the interference. In the $\gamma = 10^4$ case the linear growth of the radiation intensity sets in, because of the condition (26), significantly earlier than in the preceding $\gamma = 10^5$ case.

In Fig. 4 we give the results of a numerical computation, based on the same formula (30), of the spectra of the radiations arising in a mylar plate of thickness 1 mm. The thin continuous curves represent the total radiation computed from the formula (30); the heavy dashed curves represent the bremsstrahlung from 1 mm of path in an infinite medium[3]; the heavy continuous curves represent the difference between these quantities, i.e., the total boundary effect in the plate; the thin dashed curves represent the ordinary transition radiation (without allowance for multiple scattering) in the plate.[14] It can be seen that at relatively low frequencies the intensities of the total radiation, the total boundary effect, and the transition radiation differ little from each other. Furthermore, it can be seen that the effect of multiple scattering on the total radiation, besides the appearance of bremsstrahlung along the entire path of the charge in the plate, is expressed in some smoothing out of the interference oscillations in the spectrum of the transition radiation. The extent of this smoothing out decreases as the $\gamma$ factor increases. As to the total boundary effect, here the effect of multiple scattering leads also to the enrichment of the spectrum of the transition radiation in the high-frequency region (for sufficiently large values of the $\gamma$ factor of the charge) by new frequencies right up to frequencies $\sim q_0\gamma^2$.


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