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Sound oscillations in a plasma with "magnetic filaments"

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In laboratory and astrophysical conditions one meets with a situation when the magnetic flux is concentrated in the plasma in narrow tubes ("magnetic filaments") which are lying far from one another. We study on the basis of the magnetohydrodynamic equations the long-wavelength sound oscillations of such a system. We show that even if there are no dissipative processes (viscosity, thermal conductivity, ohmic losses) the sound oscillations are absorbed because of an effect that is in a certain sense analogous to Landau damping in a rarefied plasma and which consists in the resonance excitation of flexure waves that move along the magnetic filament. We find the contribution to the damping from the scattering of the sound wave by the filaments and we indicate the conditions under which the scattering is unimportant. We consider the damping of a small-(but finite-) amplitude monochromatic sound wave.

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1. INTRODUCTION

In the present paper we derive equations describing the propagation of waves in a plasma which contains a system of "magnetic filaments" (tubes which include inside them magnetic flux). Apparently, such a kind of structure exists in the solar chromosphere.[1,2] They can also occur near the boundary between a plasma and a magnetic field when flute instability develops.

We shall assume that the tube radius a is small compared to the average distance between tubes l (small tube concentration). We also assume that all characteristic dimensions of the problem are sufficiently large so that we can use single-fluid magnetohydrodynamics. The magnetic field outside the tubes is assumed to be small. Just such a situation exists apparently in some regions of the solar chromosphere.

It was noted in[3] that this kind of system has an interesting property: long-wavelength ($\lambda \gg l$) sound oscillations are damped in it even if there are no dissipative processes (such as viscosity, thermal conductivity or ohmic losses) whatever, due to an effect which is similar to Landau damping and which consists of the following.

Bending oscillations with phase velocity u which is equal to (see below) $H/[4\pi(\rho_e+\rho_i)]^{1/2}$, where H is the magnetic field strength inside the tube, while ρ_i and ρ_e are, respectively, the density of matter inside and outside the tube, can propagate along a separate filament. If a sound wave propagates in the plasma at an angle θ to the direction of the tubes, when

$$v_s=u\cos\theta, \quad (1)$$

where v_s is the sound velocity in the liquid, there occurs a resonance transfer of wave energy into the energy of the oscillations of the filament. Since, generally speaking, the density of matter and the magnetic field strength are different inside different filaments, the velocity u varies from filament to filament. Correspondingly one can find for each angle of propagation θ filaments for which the condition (1) is satisfied[1)] and which, therefore, remove energy from the sound wave.

The plan for solving the problem is the following. First of all, we consider the motion of a separate tube relative to the liquid and find the force of the interaction between the tube and the liquid. After that, by averaging over a volume through which many tubes pass (but which is small compared to the wavelength), we get an expression for the volume force acting on the liquid due to the tubes and, accordingly we can write down macroscopic equations of the liquid. We also study the dispersion characteristics of the system and find the damping rate of the sound wave.

2. EQUATIONS OF MOTION

We take the initial direction of the tube as the z axis. We shall characterize the displacement of the tube from its equilibrium position by a vector $\boldsymbol{\xi}(z, t)$ which is at right-angles to the z-axis. Since the relative velocity of the motion of the liquid and of the tube is a first-order quantity, we can neglect the change in the size and shape of the cross section of the tube when evaluating the interaction force between the tube and the liquid, and assume it to be a circle of radius a. Taking all this into account we can write down the following equation for the bending oscillations of the tube:

$$\rho_i \pi a^2 \frac{\partial^2 \xi}{\partial t^2} = \rho_e \pi a^2 \frac{\partial \mathbf{v}_\perp}{\partial t} + \rho_e \pi a^2 \left(\frac{\partial \mathbf{v}_\perp}{\partial t} - \frac{\partial^2 \xi}{\partial t^2} \right) + \frac{H^2}{4\pi} \pi a^2 \frac{\partial^2 \xi}{\partial z^2}. \quad (2)$$

We elucidate the meaning of the different terms on the right hand side of this equation: $\mathbf{v}_\perp$ is the perpendicular component of the macroscopic velocity of the liquid; the first term describes the "extruding force" acting on a tube in the liquid which is moving with an acceleration; the second term takes into account the effect of the virtual mass (see, e.g., [4])[2]); finally, the last term is a rotating force which acts on the tube due to the magnetic field included in it. The longitudinal component of the velocity of the liquid does not enter into Eq. (2), since the longitudinal motion of the liquid does not cause an interaction between the liquid and the tubes.

When we talk about the macroscopic velocity of the liquid we must bear in mind that the velocity of the liquid in the immediate vicinity of a filament (at a distance of the order of a) differs appreciably, by a quantity of the order of unity, from the volume-averaged (macroscopic) velocity which in general we must take into account when calculating the latter. However, since the concentration of the filaments is small, we can identify, apart from terms of first-order in the concentration, the macroscopic velocity with the velocity of the liquid at distances from the filaments which are large compared to a (but small compared to l), as we have indeed done above.

When using the virtual mass concept we assumed that the liquid was incompressible. This is valid since the velocities $\partial \xi / \partial t$ and $\mathbf{v}_\perp$ are small compared to the sound velocity while the period of the long-wavelength oscillations considered by us is large compared to the natural periods of the radial oscillations of the tube.

Each separate tube is characterized by a radius a, by the density ρ_i (we shall in what follows use instead of ρ_i the dimensionless parameter $\eta = \rho_i/\rho_e$) of the matter included inside it, and the temperature of that matter T_i; these parameters change in general from tube to tube.

We shall assume for the sake of simplicity that the matter inside the tubes is cold, $T_i \ll T_p$, and hence we shall neglect the gas-kinetic pressure p_i inside the tubes. This assumption is not at all one of principle but allows us to avoid an excessive complication of the equations. We have then in the unperturbed state

$$H^2/8\pi = p_e = \rho_e v_s^2/\gamma,$$

where p_e is the pressure outside the tubes while γ is the adiabatic index, i.e., we can rewrite Eq. (2) in the form

$$\frac{\partial^2 \xi_\eta}{\partial t^2} - \frac{2}{\gamma} \frac{v_s^2}{1+\eta} \frac{\partial^2 \xi_\eta}{\partial z^2} = \frac{2}{1+\eta} \frac{\partial \mathbf{v}_\perp}{\partial t}. \quad (3)$$

Here and henceforth we label the displacement ξ with an index η to reflect the fact that the displacement of the tubes depends on η.

We introduce for the tubes a distribution function with respect to the parameters a and η, which we define by the formula

$$d\alpha = f(a, \eta)\, da\, d\eta,$$

where $d\alpha$ is that fraction of the volume which is occupied by tubes with values of the parameters a and η in the intervals $(a, a+da)$, $(\eta, \eta+d\eta)$. The normalization of the function f, defined in such a way, is clearly the following:

$$\alpha = \int_0^\infty \int_0^\infty da\, d\eta\, f(a, \eta).$$

Here α is the total fraction of the volume occupied by tubes (in accordance with what we said earlier, $\alpha \sim a^2/l^2 \ll 1$). For what follows it is also useful to introduce the function

$$g(\eta) = \int_0^\infty f(a, \eta)\, da. \quad (4)$$

It is clear from (2) that the force due to a tube (per unit tube length) acting on the liquid is equal to

$$-\pi a^2 \rho_e \left(2 \frac{\partial \mathbf{v}_\perp}{\partial t} - \frac{\partial^2 \xi_\eta}{\partial t^2} \right).$$

Hence we get for the force acting on a unit volume of the liquid due to the tubes which pass through that volume

$$\mathbf{F} = -\rho_e \int \left(2 \frac{\partial \mathbf{v}_\perp}{\partial t} - \frac{\partial^2 \xi_\eta}{\partial t^2} \right) g(\eta)\, d\eta. \quad (5)$$

The macroscopic equation of motion of the liquid can now be written in the form

$$\rho_e \frac{\partial \mathbf{v}}{\partial t} = -v_s^2 \nabla \delta\rho + \mathbf{F}, \quad (6)$$

where $\delta\rho$ is the perturbation of the liquid density and is connected with $\mathbf{v}$ by the continuity equation:

$$\frac{\partial}{\partial t} \delta\rho + \rho_e \operatorname{div} \mathbf{v} = 0. \quad (7)$$

When writing down the equation of continuity and the first term on the right-hand side of Eq. (6) we restricted ourselves to the zeroth approximation in the parameter α, since taking into account first- (and higher-) order terms would only lead to an insignificant charge in the propagation velocity of the sound oscillations. On the other hand, taking the force $\mathbf{F}$ (which is also of order α) into account in Eq. (6) leads to a qualitatively new effect which is connected with the flexural elasticity of the tubes.

In contrast to the well-known problem of oscillations in a liquid with gas bubbles (see, e.g., [5]) taking the compressibility of the tubes into account is unimportant in the system considered. The reason is that in our case the compressibilities of the tubes and of the surrounding medium are of the same order of magnitude.

3. DAMPING OF THE SOUND WAVES

Equations (3) and (5)–(7) form a closed set that describes the linear oscillations of the medium. We consider the eigensolutions of this set in the form of traveling waves, i.e., we assume that $\boldsymbol{\xi}_\eta$, $\mathbf{v}$ and $\delta\rho_e$ change in proportion to $\exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})$. Using Eq. (2) to express ξ_η in terms of $\mathbf{v}_\perp$

$$\boldsymbol{\xi}_\eta = \frac{2i\omega \mathbf{v}_\perp}{\omega^2(1+\eta) - 2\gamma^{-1}v_s^2k^2\cos^2\theta}, \tag{8}$$

and substituting the result in (5) we get:

$$\mathbf{F} = i\omega\rho_e \mathbf{v}_\perp I(\omega, \mathbf{k}),$$
$$I(\omega,\mathbf{k}) = 2\int_0^\infty d\eta\, g(\eta)\left[1 - \left(\eta + 1 - \frac{2v_s^2k^2\cos^2\theta}{(\omega+i0)^2}\right)^{-1}\right]. \tag{9}$$

We recognize that as $t\to-\infty$ the perturbation must vanish and hence we replace ω by $\omega + i0$ in the denominator of the integrand.

The effect of the magnetic filaments on the oscillations of the medium enters into the problem through the intergral I which is of order of magnitude of α and hence is small compared to unity. Using this fact we can easily obtain through (6), (7), and (9) a dispersion relation

$$\omega \approx kv_s\left[1 - \frac{\sin^2\theta}{2} I(\omega, \mathbf{k})\right]$$

(for the sake of argument, we consider the solution with $\mathrm{Re}\,\omega > 0$).

To evaluate I we can set the frequency equal to the solution of the dispersion relation for a "pure" (tubeless) liquid: $\omega = kv_s$. We must also note that taking into account the real part of I leads only to an insignificant change in the frequency of the oscillations, so that it is sufficient to evaluate merely the imaginary part of I. Using the well-known equation

$$\mathrm{Im}\frac{1}{x+i0} = -i\pi\delta(x),$$

we easily find the following expression for the damping rate $\nu \equiv -\mathrm{Im}\,\omega$:

$$\nu = \frac{kv_s\sin^2\theta}{2}\,\mathrm{Im}\,I = \pi kv_s\sin^2\theta \begin{cases} g(\eta_0), & \eta_0 > 0 \\ 0 & \eta_0 < 0 \end{cases},$$
$$\eta_0 = \frac{2\cos^2\theta}{\gamma} - 1. \tag{10}$$

Of course, we could arrive at the same result also by a more formal method, considering the solution of the Cauchy problem for the set (3) to (7) using a Laplace transformation—exactly in the same way as is done in Landau's paper[6] in application to the Langmuir oscillations problem.

If we make the obvious assumption that the width of that region of η-values where the distribution function is appreciably different from zero is of the order of unity, i.e., that the matter density inside the tubes does not change by more than a factor of order unity with respect to ρ_e, we have $g(\eta_0) \sim \alpha$ and we can write the following estimate for the damping rate:

$$\nu/kv_s \sim \alpha. \tag{11}$$

This estimate, of course, "works" only in that region of θ values where $\eta_0 > 0$, i.e., when $\cos\theta > (\gamma/2)^{1/2}$. For a monatomic gas the corresponding region is rather narrow, $\theta \leq \arccos(5/6)^{1/2} \sim 5°$, so that it has sense to take into account the possibility that separate filaments are not collinear, and we have done that in what follows.

We can characterize the direction of a separate filament by a single vector $\mathbf{n}$ in the direction of the filament axis. We denote the distribution function of the filaments with respect to their directions by $h(\mathbf{n})$. For simplicity we assume then that all filaments have the same radius and the same density. The normalization of the function h is the following:

$$\alpha = \int h(\mathbf{n})\,do,$$

where α is the fraction of the volume occupied by the tubes and do an element of solid angle. The component of the macroscopic velocity at right angles to the vector $\mathbf{n}$ is clearly equal to $\mathbf{v} - \mathbf{n}(\mathbf{n}\cdot\mathbf{v})$. We have thus instead of (8)

$$\boldsymbol{\xi}_\mathbf{n} = -\frac{2i\omega\,[\mathbf{v} - \mathbf{n}(\mathbf{n}\cdot\mathbf{v})]}{\omega^2(1+\eta) - 2\gamma^{-1}v_s^2(\mathbf{kn})^2}$$

We have labeled in this case the displacement $\boldsymbol{\xi}$ by the index $\mathbf{n}$, since $\boldsymbol{\xi}$ clearly depends on the orientation of the filaments (as to the parameter η, we assume it here to be fixed). The volume force acting on the plasma can by analogy with (9) be written in the form

$$F_\alpha = -i\omega\rho_e K_{\alpha\beta} v_\beta,$$

where

$$K_{\alpha\beta} = -2\int do\, h(\mathbf{n})\,(\delta_{\alpha\beta} - n_\alpha n_\beta)\left[1 - \left(1 + \eta - \frac{2v_s^2(\mathbf{kn})^2}{\gamma(\omega+i0)^2}\right)^{-1}\right].$$

Once we have this expression for the force we can easily use Eqs. (6) and (7) to write down the dispersion relation and, bearing in mind that $K_{\alpha\beta}$ is small, find the small imaginary correction to the frequency:

$$\frac{\nu}{kv_s} = \frac{k_\alpha k_\beta}{2k^2}\,\mathrm{Im}\,K_{\alpha\beta} = \int do\, h(\mathbf{n})\left[1 - \frac{(\mathbf{kn})^2}{k^2}\right]\delta\left[1 + \eta - \frac{2(\mathbf{kn})^2}{\gamma k^2}\right].$$

In the important particular case of an isotropic filament distribution we have $h(\mathbf{n}) = \alpha/4\pi$, and the expression for the damping rate can be written down especially simply:

$$\frac{\nu}{kv_s}=\frac{\sqrt{2}}{8}\frac{\alpha\gamma[2-\gamma(1+\eta)]}{\sqrt{\gamma(1+\eta)}}$$

(it is understood that $\eta<2/\gamma-1$).

So far when considering the equation of motion of a separate filament (see (2)) we have completely neglected the compressibility of the medium surrounding the filament; this is justified by the smallness of the frequency of the oscillations considered by us, $\omega=kv_s$, as compared to v_s/a. Taking the compressibility into account (i.e., in actual fact taking into account higher-order terms in the parameter $\omega a/v_s\sim ka$) leads to the appearance of a new effect: namely, to the emission of sound waves when the filaments perform bending oscillations. The radiation damping of the bending oscillations caused by this effect has a rate $\nu_{\text{rad}}\sim\omega k^2a^2$ (see the Appendix).

Of course, the consideration given here is valid only if the damping rate ν of the sound wave found earlier is large compared to ν_{rad}, i.e., if

$$(ka)^2<\alpha. \qquad (12)$$

If this condition is satisifed, the energy of the sound wave is transferred in a time on the order of ν^{-1} to the energy of the filament oscillations, after which, over a considerably longer time of the order of ν_{rad}^{-1}, the filaments give off their energy through secondary sound waves. Bearing in mind that the distance l between filaments is of the order of $a/\sqrt{\alpha}$ (according to the definition of α) we can write inequality (12) in the form $kl\ll 1$. In other words, under the conditions when the macroscopic description is applicable ($k\ll l^{-1}$) the inequality (12) is satisfied automatically.

We note that for the calculation of the (weak) damping of the sound wave a macroscopic approach is in general not necessary. It is sufficient to consider the driving of the oscillations of the separate filaments and also to use the energy conservation law to determine the rate of damping of the initial wave. However, the procedure used above has that advantage that it allows us better to exhibit the analogy with Landau damping.

4. DAMPING DUE TO SCATTERING BY FILAMENTS

In this section we shall make clear the relation between the Landau damping considered above and the damping connected with the scattering of a sound wave by the filaments.

We show in the Appendix that the energy Q of the secondary waves emitted per unit time and per unit length of filament is connected with the energy density W of the initial sound wave through the relation

$$Q=\beta(\eta,a,\omega)W, \qquad (13)$$

where

$$\beta=\beta_0+2\sum_{m=1}^{+\infty}\beta_m, \qquad (14)$$

$$\beta_0=\frac{v_s}{k}\frac{\pi^2}{16}(ka)^4\left(\sin^2\theta-\frac{\gamma}{2}\right),$$

$$\beta_m=\frac{v_s}{k}\left[\frac{\pi}{m!(m-1)!(1+\eta)}\right]^2\left(\frac{ka\sin\theta}{2}\right)^{4m}\frac{\Omega^2}{(\omega-\Omega)^2+\nu_{\text{rad}}^{(m)2}}$$

$$\Omega=k_zv_s\sqrt{2/\gamma(1+\eta)},\quad \cos\theta=k_z/k,$$

while $\nu_{\text{rad}}^{(m)}$ is given by Eq. (A9).

For the sake of argument, we discuss the first of the cases considered in Sec. 3—when all filaments are collinear. In that case the damping rate of the primary sound wave ν_{scatt} caused by the scattering can as follows be expressed in terms of the coefficient β and the distribution function $f(a,\eta)$ introduced above:

$$\nu_{\text{scatt}}=\int\frac{f(a,\eta)}{\pi a^2}\beta(\eta,a,\omega)\,d\eta\,da. \qquad (15)$$

The main contribution to the scattering comes from the dipole ($m=1$) term in β: the subsequent multipole ($m=2,3,\ldots$) terms contain higher powers of the small parameter ka, while in the $m=0$ term there is no resonance denominator. The presence of resonances in the scattering is connected with the existence of weakly damped eigenoscillations of the filaments (see the Appendix).

We must bear in mind that Eq. (14) is obtained for a strictly monochromatic primary wave. If, however, we are dealing with the damped wave where its damping rate ν is large compared to $\nu_{\text{rad}}^{(m)}$ (just such a situation occurs in the case $\alpha\gg(ka)^2$) one clearly needs not approach the resonance $\omega=\Omega$ at a distance less than ν and when evaluating the contribution of the scattering to the damping one must limit oneself to the region $|\omega-\Omega|\gtrsim\nu$. The integration over the corresponding region of η-values gives the following estimate for the damping rate due to scattering:

$$\nu_{\text{scatt}}/kv_s\sim(ka)^4/\alpha. \qquad (16)$$

It follows from a comparison of Eqs. (11) and (16) that we can neglect the scattering if condition (12) is satisfied.

We note that the damping rate of short-wavelength waves $a\ll\lambda\ll 1$ is completely determined by the scattering: it turns out to be (see above) small compared to ν_{rad} and this means that the damping proceeds through a direct scattering of the primary sound wave into secondary waves without a preliminary build-up of energy in the natural oscillations of the filaments. In other words, for short waves $\nu=\nu_{\text{scatt}}$, where ν_{scatt} is given by Eq. (15).

We noted above that it is sufficient to retain only the $m=1$ term in the coefficient β. As a narrow region of η-values of width $\Delta\eta\sim\nu_{\text{rad}}/\omega\ll 1$ around the resonance $\eta=\eta_0$ contributes to the integral (15) we can replace the function $f(a,\eta)$ by an η-independent factor $f(a,\eta_0)$ which we can take in front of the integral over η. After this we can easily integrate over η and a, which leads to the following result, which is formally the same as (10):

$$\nu=\nu_{\text{scatt}}=\pi kv_s\sin^2\theta g(\eta_0),\quad \eta_0=2\cos^2\theta/\gamma-1. \qquad (17)$$

We imply that $\eta_0>0$; in the opposite case the scattering becomes non-resonant and the damping rate decreases to a value $\nu\sim\alpha(kv_s)(ka)^2$.

5. DAMPING OF A MONOCHROMATIC SOUND WAVE

In the previous two sections we showed the the damping of long-wavelength ($\lambda\gg l$) sound waves is completely determined by an effect which is analagous to Landau damping. The corresponding analogy becomes particularly complete if we make non-linear estimates connected with the problem of the damping of a monochromatic sound wave of small (but finite!) amplitude.[3)]

To fix the ideas we shall talk about the case where all filaments have the same orientation. Let initially there be excited in the plasma a sound wave with particle displacement amplitude $\boldsymbol{\zeta}$ and wavevector $\mathbf{k}$ an an angle θ to the z-axis. We consider the motion of a separate filament in the field of such a sound wave. To do this we turn to Eq. (3) which we write in the form

$$\frac{\partial^2\xi_\eta}{\partial t^2}+\Omega^2(\eta)\xi_\eta=-\omega^2\zeta_\perp e^{-i\omega t}, \qquad (18)$$

where

$$\Omega(\eta)=k_z v_s\sqrt{2/\gamma(1+\eta)},\qquad \omega=kv_s$$

(for the present we assume that the sound wave amplitude does not change with time; see later in this section for an elucidation of this problem). We denote the resonance value of η by η_0, i.e., that value of η for which $\Omega(\eta)=\omega$. For those values of η which are sufficiently close to η_0 the amplitude of the oscillations of the filaments increases linearly in time and very rapidly becomes much larger than $\zeta_\perp$ (we can therefore neglect the initial value of ξ_η which one must assume to be equal in order of magnitude to $\zeta_\perp$; cf.[9]). A limit on the amplitude arises due to the non-linear frequency shift (see [10]) which we denote by $\Delta\Omega(\tilde{\xi}_\eta)$, where $\tilde{\xi}_\eta$ is that amplitude value of ξ_η. The first non-vanishing term in the expansion of $\Delta\Omega$ in powers of ξ_η has the form $\Delta\Omega=A\tilde{\xi}_\eta^2$) where the coefficient A is equal to ωk_z^2, as to order of magnitude. For values of η which are sufficiently close to η_0 the stationary value of the amplitude can be estimated by using a simple relation following from (18):

$$\tilde{\xi}_\eta\left|\frac{\partial\Omega}{\partial\eta}(\eta-\eta_0)-A\tilde{\xi}_\eta^2\right|\sim\zeta_\perp\omega. \qquad (19)$$

As $\partial\Omega/\partial\eta\sim\omega$ (we assume that $\eta_0\sim1$), it follows from (19) that $\tilde{\xi}_\eta$ depends as follows on η:

$$\tilde{\xi}_\eta\sim\begin{cases}\dfrac{\zeta_\perp}{|\eta-\eta_0|}; & |\eta-\eta_0|\gg(\zeta_\perp k_z)^{2/3},\\ \zeta_\perp(\zeta_\perp k_z)^{-1/3}; & |\eta-\eta_0|\ll(\zeta_\perp k_z)^{2/3}\end{cases} \qquad (20)$$

(we take into account that $A\sim\omega k_z^2$). As the energy of the oscillations of the filaments, per unit of their length, is of the order of magnitude of $\rho_e a^2\tilde{\xi}^2\omega^2$, we can use (20) to estimate the energy which will be transferred by the filaments when the amplitude of the oscillations reaches saturation (we take the energy per unit volume of the medium):

$$W^*\sim\rho_e\omega^2\int g(\eta)\tilde{\xi}_\eta^2\,d\eta\sim\rho_e\omega^2\zeta_\perp^2(\zeta_\perp k_z)^{-2/3}g(\eta_0)\sim g(\eta_0)(\rho v_s^2/W)^{1/3}, \qquad (21)$$

where $W\sim\rho_e\omega^2\zeta^2$ is the energy density of the sound wave. Using the estimate $g(\eta_0)\sim\alpha$ we can rewrite the latter relation in the form

$$W^*/W\sim\alpha(\rho v_s^2/W)^{1/3}.$$

This estimate solves the problem of the damping of a finite-amplitude wave (cf.[9]): when $W/\rho v_s^2>\alpha^3$ the maximum energy which can be transferred by the filaments when there is resonance interaction between them and the sound wave is small compared to the energy of the wave, i.e., the wave transfers in that case only an insignificant fraction of its initial energy to the filaments, after which damping stops. The assumption made by us that the sound wave amplitude does not change is in that case valid.

On the other hand, if $W/\rho v_s^2<\alpha^3$ the energy of the filaments remains small even when the sound wave is completely absorbed, and non-linear effects (frequency shift) are unimportant. In that case the damping is described by the linear theory given in Sec. 3. The condition for the applicability of the linear approximation in the problem of the damping of a monochromatic sound wave with initial amplitude ζ has thus the form

$$k\zeta<\alpha^{3/2}. \qquad (22)$$

We neglected the non-linearity of the sound wave itself. It is well known that this non-linearity leads to a distortion of the sound wave profile and to the formation of discontinuities after a time $\tau\sim(\omega k\zeta)^{-1}$. Our non-linear estimates are valid if τ is large compared to the damping time of the wave in the case (22) or compared to the time for saturation of the amplitude in the opposite limiting case.

We note that the sound wave is ultimately absorbed due to the radiation damping of the filament oscillations and in the case $k\zeta>\alpha^{3/2}$ only the absorption time turns out to be appreciably longer than ν^{-1}. We can in that case write for the rate of dissipation of the sound wave energy the following estimate: $-\dot{W}\sim\nu_{\rm rad}W^*$, where W^* is estimated by Eq. (21), while $\nu_{\rm rad}$ is the radiation damping rate. We can rewrite this estimate differently:

$$-\dot{W}\sim\nu_{\rm rad}W\cdot\alpha(\rho v_s^2/W)^{1/3},$$

whence we find for the sound wave absorption time

$$\tau\sim\frac{1}{\nu_{\rm rad}\alpha}\left(\frac{W}{\rho v_s^2}\right)^{1/3}\gg\frac{1}{\nu_{\rm rad}}.$$

It is understood that $\nu_{\rm rad}\ll\Delta\Omega$ as otherwise non-linear effects become unimportant.

APPENDIX

In the present Appendix we consider the exact theory of linear oscillations of a magnetic filament, taking into account the compressibility of the medium. The linearized set of magnetohydrodynamic equations for the plasma inside the filament has the form

$$\rho_i\frac{\partial \mathbf{v}}{\partial t}=\frac{1}{4\pi}[\operatorname{rot}\mathbf{h},\mathbf{H}],\quad \frac{\partial \mathbf{h}}{\partial t}=\operatorname{rot}[\mathbf{vH}],$$

where $\mathbf{v}$ and $\mathbf{h}$ are the perturbations of the velocity and of the magnetic field. As the gas kinetic pressure inside the filaments vanishes the equation of continuity is split off. One checks easily that in a cylindrical system of coordinates (r, φ, z) with a z-axis along the axis of the filament the velocity perturbation can be written in the form

$$v_r=-\frac{\partial\psi}{\partial r},\quad v_\varphi=-\frac{1}{r}\frac{\partial\psi}{\partial\varphi},\quad v_z=0, \tag{A.1}$$

where ψ is a function of the coordinates and the time. For perturbations which are proportional to $\exp(-i\omega t + ik_z z+im\varphi)$ where $m=0, \pm1, \pm2, \ldots, \psi$ satisfies the equation

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\psi}{\partial r}+\left(\frac{\omega^2}{v_A^2}-k_z^2-\frac{m^2}{r^2}\right)\psi=0, \tag{A.2}$$
$$v_A^2=H^2/4\pi\rho_i.$$

We can express the magnetic field perturbation in terms of the function ψ

$$h_r=\frac{k_zH}{\omega}\frac{\partial\psi}{\partial r},\quad h_\varphi=\frac{k_zH}{\omega r}\frac{\partial\psi}{\partial\varphi},\quad h_z=\frac{i\omega H}{v_A^2}\left(1-\frac{k_z^2v_A^2}{\omega^2}\right)\psi. \tag{A.3}$$

We assume the magnetic field to vanish in the region outside the filament. The linear equations in that region can thus be written as an equation for the velocity potential $\chi(\mathbf{v}=-\nabla\chi)$:

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\chi}{\partial r}+\left(\frac{\omega^2}{v_s^2}-k_z^2-\frac{m^2}{r^2}\right)\chi=0. \tag{A.4}$$

We can express the gas-kinetic pressure perturbation in terms of χ

$$\delta p=\frac{iv_s^2}{\omega\gamma}\rho_e\Delta\chi. \tag{A.5}$$

The quantities v_s^2 and v_A^2 are connected through the relation

$$v_A^2=\frac{2}{\gamma\eta}v_s^2. \tag{A.6}$$

On the surface of the filament $(r=a)$ we must satisfy the conditions for the continuity of the normal component of the velocity,

$$v_r|_i=v_r|_e,$$

and of the normal component of the momentum flux,

$$Hh_z/4\pi=\delta p.$$

Using Eqs. (A.1), (A.3), (A.5), and (A.6) we can reduce them to the following boundary condition for the logarithmic derivatives of χ and ψ:

$$\left(\eta-\frac{2}{\gamma}\frac{k_z^2v_s^2}{\omega^2}\right)\frac{\partial\ln\chi}{\partial r}=\frac{\partial\ln\psi}{\partial r}. \tag{A.7}$$

Inside the filament the solution is proportional to an m-th order Bessel function $\psi\propto J_m(q_ir)$, where $q_i=(\omega^2/v_A^2-k_z^2)^{1/2}$. On the outside the solution must have the form of outgoing waves: $\chi\propto H_m^{(1)}(q_er)$, where $H_m^{(1)}$ is a first-order Hankel function, and $q_e=(\omega^2/v_s^2-k_z^2)^{1/2}$ (we have chosen that branch of the root which corresponds to $\mathrm{Re}\,q_e>0$). As we consider oscillations with $k_za\ll1$ the arguments of the Bessel and Hankel functions in the boundary condition (A7) are small compared to unity. If we retain only the first non-vanishing terms in the expansion in this small parameter we get the following dispersion relation:

$$\omega_m=\Omega\equiv k_zv_s\sqrt{2/\gamma(1+\eta)}, \tag{A.8}$$

$m=\pm1, \pm2, \pm3, \ldots$. There is no radiation damping in this approximation. It is of interest to note that the natural frequencies are independent of the multipole order m of the mode (the dipole mode $m=\pm1$ corresponds to the bending oscillations). There are no weakly damped oscillations with $m=0$.

Generally speaking, in the interaction with a plane sound wave not only the dipole mode considered in Secs. 2 and 3, but also modes with $m=2, 3, \ldots$ are excited. However, their amplitude is very small (due to the fact that in the expansion of a plane wave in terms of cylindrical multipoles the amplitude of a mode with a given m is in the vicinity of the filament proportional to $(ka)^{|m|}$).

If we retain in the boundary condition (A.7) the next term in the expansion in powers of ka, it leads, on the one hand, to a small change in the real part of the frequency and, on the other hand, to the appearance of radiation damping. For the sake of simplicity we show only the formula for the radiation damping rate $\nu_{\mathrm{rad}}^{(m)}$:

$$\frac{\nu_{\mathrm{rad}}^{(m)}}{\Omega}=\frac{\pi}{|m|!(|m|-1)!(1+\eta)}\left(\frac{ak_z}{2}\right)^{2|m|}\left[\frac{2}{\gamma(1+\eta)}-1\right]^{|m|}. \tag{A.9}$$

We now consider the scattering problem. Let there propagate in the plasma a plane sound wave of unit amplitude:

$$\chi={}^1/_2\exp(-i\omega t+i\mathbf{kr})+\mathrm{c.c.} \tag{A.10}$$

where $\omega=kv_s$. When there are magnetic filaments present in the plasma the solution outside the filament will be a superposition of this plane wave (which in cylindrical coordinates can be written in the form $\frac{1}{2}\exp(-i\omega t+ik_zz+iq_er\cos\varphi)+\mathrm{c.c.}$) and outgoing cylindrical waves:

$$\chi=e^{-i\omega t+ik_zz}\left[\frac{1}{2}e^{-iq_er\cos\varphi}+\sum_{m=-\infty}^{+\infty}A_mH_m^{(1)}(q_er)e^{im\varphi}\right]+\mathrm{c.c.}$$

Inside the filament the solution has the form

$$\psi=e^{-i\omega t+ik_z z}\sum_{m=-\infty}^{+\infty}B_m J_m(q_i r)e^{im\varphi}+\text{c.c.}$$

Using the identity

$$e^{iq_e r\cos\varphi}=\sum_{m=-\infty}^{+\infty}i^m J_m(q_e r)e^{im\varphi}$$

and writing down the boundary condition (A. 7) for each azimuthal harmonic m we get the following expression for the coefficients A_m we are looking for:

$$A_m=-\frac{i^m}{2}\frac{\left(\eta-\frac{2}{\gamma}\frac{k_z^2v_s^2}{\omega^2}\right)q_iJ_m'(q_ia)J_m(q_ea)-q_eJ_m'(q_ea)J_m(q_ia)}{\left(\eta-\frac{2}{\gamma}\frac{k_z^2v_s^2}{\omega^2}\right)q_eH_m^{(1)\prime}(q_ea)J_m(q_ia)-q_iJ_m'(q_ia)H_m^{(1)}(q_ea)}.$$

Using the fact that the parameter ka is small we can for $m>0$ write A_m in the form

$$A_m=-\frac{\pi i^{m-1}}{2(1+\eta)m!(m-1)!}\left(\frac{ka\sin\theta}{2}\right)^{2m}\frac{\Omega}{\omega-\Omega+i\nu_{\text{rad}}^{(m)}},\quad A_{-m}=A_m^*.$$

where $\Omega(\eta)$ is given by Eq. (A8). When obtaining this expression we used the fact that A_m is appreciably different from zero when $\omega\approx\Omega$ and we have everywhere, except in the resonance denominator, replaced ω by Ω. In the wave zone when $q_e r\gg 1$ the asymptotic formula

$$H_m^{(1)}(q_er)\approx\sqrt{\frac{2}{\pi q_e r}}\exp\left\{i\left(q_e r-\frac{m\pi}{2}-\frac{\pi}{4}\right)\right\}$$

is valid. In small parts of space we can here assume the outgoing wave to be a plane wave which enables us to evaluate easily the density of the radial energy flux in the m-th mode:

$$q_m=|A_m|^2\rho v_s k/\pi r.$$

Per unit length of the filament a power $2\pi r q_m$ is emitted in the m-th mode. Bearing in mind that the energy density in the incident sound wave (A. 10) equals $\rho k^2/2$, we get thus the relation (14) with

$$\beta_m=4|A_m|^2v_s/k.$$

1) This condition is analogous to the Cerenkov resonance condition in the Landau damping theory.

2) One can easily check that the virtual mass per unit length of a cylindrical tube is equal to $\pi a^2\rho_e$.

3) Such kinds of estimates have been made in the theory of Langmuir oscillations by Mazitov[7] and O'Neil[8]; see also the review by Kadomtsev.[9]

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