Tilted vortices in type II superconductors

R. S. Thompson

L. D. Landau Institute of Theoretical Physics
(Submitted July 28, 1975)
Zh. Eksp. Teor. Fiz. **69**, 2249–2266 (December 1975)

Static and dynamic properties of a flat superconductor placed in a magnetic field inclined with respect to the surface are calculated. An exact solution is obtained for small angles of inclination, which corrects the work of Saint-James for the upper critical field and the work of Maki on the impedance, giving a consistent flux-flow characteristic. The possibility of sharp vortex-lattice structural phase transitions at intermediate angles of inclination and the variation of the upper critical field near the perpendicular orientation are also investigated.

PACS numbers: 74.40.–j, 74.20.Gh

1. INTRODUCTION

The problem of the angular dependence of the critical magnetic field of a flat type II superconductor was solved by Tinkham[1] for a thin film. Further proposals [2,3,4] made for thicker films are incorrect. Maki[5,6] investigated the impedance of a thick film in an inclined field. However, his approximate calculations lead in some cases to an infinite conductivity. Actually the conductivity should be finite due to flux flow of the vortex lines formed for any finite inclination of the static magnetic field with respect to the surface. Significant improvement in calculating these static and dynamic responses of a superconductor with tilted vortices is obtained in the present work.

Previous work is reviewed in detail and some better variational limits are obtained in Sec. 2. Exact solutions for the superconducting wave functions and eigenvalues for all film thicknesses near the parallel orientation are presented in the third section. The variation in the critical field near the perpendicular orientation is discussed in Sec. 4. The results of Sec. 3 are applied to calculate the impedance near the parallel orientation in Sec. 5. Higher order effects in perturbation theory, leading to possible phase transitions in the structure of the vortex lattice at intermediate orientations, are indicated in Sec. 6.

2. PREVIOUS WORK AND VARIATIONAL LIMITS

The value of the upper critical magnetic field H_u of a dirty type-two superconductor and the spatial form of the superconducting order parameter (or wave function) Δ near H_u are determined by first minimizing the functional λ.

$$\lambda=\int d^3r|(i\nabla+2e\mathbf{A})\Delta|^2\left(2eB_e\int d^3r|\Delta|^2\right)^{-1} \tag{1}$$

A is the vector potential corresponding to the uniform externally-applied magnetic field $\mathbf{B}_e$. Then the critical field is found by setting λ equal to H_{C2}/B_e. H_{C2} is determined by

$$\ln(T/T_c)+\psi(1/2+\rho)-\psi(1/2)=0,$$

where ψ is the digamma function. $\rho = 2eH_{C2}D/4\pi T$, T_C is the critical value of the temperature T, and D is the diffusion constant ($k_B = \hbar = c = 1$).

Variation of λ with respect to Δ gives an eigenvalue equation which determines Δ:

$$(2eB_e)^{-1}(i\nabla+2e\mathbf{A})^2\Delta\equiv H\Delta=\lambda\Delta. \tag{2}$$

The boundary condition requires the normal component of the gauge-invariant derivative $(i\nabla + 2eA)\Delta$ to vanish at the sample surfaces. When λ is set equal to H_{C2}/H_u, (2) is the linearized Ginzburg-Landau[8] equation.

In general one cannot solve this non-separable equation simply to find the exact value of the critical field. The variational functional (1) is useful then, since a trial function, which does not necessarily solve (2) or the boundary condition, may be used to estimate λ. The estimated value of λ must exceed or equal the true minimum value of λ. In addition to the minimum value of λ there are other stationary values of λ, which are solutions of (2) corresponding to excited states. These states may be excited by applying an electro-magnetic field.

The first work on the dependence of the critical field on the angle of inclination of the magnetic field with respect to the sample surface was by Tinkham[1], who considered the case of a thin film with a thickness d much less than the coherence length $\xi = (2eH_{C2})^{-1/2}$. Using a coordinate system where the x-y coordinates are in the plane of the film and the field is in the x-z plane, the vector potential may be chosen to have only a y-component $A_y = B_e(x \sin\theta - z \cos\theta)$, where θ is the angle between the field and the plane of the film. The z-dependence of Δ is unimportant for a thin film. A good approximation is to write Δ as the Abrikosov[9] solution Δ_0, a linear combination of Gaussians, for the perpendicular component of the field $B_e \sin\theta$. Then $\lambda = eB_e \times (d\cos\theta)^2/6 + \sin\theta$. If $\theta = 1/2\pi$ one gets $H_u = H_{C2}$. Similarly if $\theta = 0$, H_u is $H_{C\parallel}$, the parallel critical field. In terms of H_{C2} and $H_{C\parallel}$, H_u satisfies

$$1=(H_u\cos\theta/H_{c\parallel})^2+H_u\sin\theta/H_{c2}. \tag{3}$$

Tinkham[2] further suggested that (3) might be valid also for thick films $d \gg \xi$ when the appropriate value for $H_{C\parallel}$, $H_{C3} = 1.69H_{C2}$ is substituted[10]. However, the formula is not correct when the z-dependence of Δ is important.

Saint-James[3] investigated the variation of H_u near the parallel orientation, evaluating $dH_u/d\theta$ at $\theta = 0$. He separated the operator H in (2) into two parts, a separable H_0 and a perturbation P which has mixed coordinates. The y-dependence of Δ is a trivial exponential, and one has $\Delta = \Delta(x, z)e^{iky}$. Then

$$\begin{aligned}(2eB_e)H_0&=-\nabla_z^2+[2eB_e\cos\theta(z-z_0)]^2-\nabla_x^2+[2eB_e\sin\theta(x-x_0)]^2,\\(2eB_e)P&=-2(2eB_e)^2(z-z_0)(x-x_0)\sin\theta\cos\theta,\end{aligned} \tag{4}$$

where $k = 2eB_e(x_0 \sin\theta - z_0\cos\theta)$. The solution for H_0 is separable into the product of a surface state S(z) and an Abrikosov harmonic oscillator state $A(x - x_0)$. The value

Copyright © 1976 American Institute of Physics

of x_0 does not affect the eigenvalue of H_0, and z_0 is chosen to minimize it. The lowest eigenvalue of H_0, λ_{00}^0 is then

$$\lambda_{00}^0=\mu_0\cos\theta+\sin\theta, \tag{5}$$

$\mu_0 = 0.5901$ for a thick film, $\mu_0 = eB_e\cos\theta\, d^2/6$ for a thin film, and μ_0 is calculated numerically for intermediate thicknesses[10]. For the normalized slope

$$\beta=\frac{1}{H_u}\frac{dH_u}{d\theta}\bigg|_{\theta=0}=-\left(1+H_u\frac{1}{\lambda}\frac{d\lambda}{dB_e}\right)^{-1}\frac{1}{\lambda}\frac{d\lambda}{d\theta}\bigg|_{\theta=0}$$

this simple formula gives $\beta = -\mu_0^{-1} = -1.69$ for thick films. Tinkham's (3) gives half this value. Saint-James used perturbation theory to second order in P and calculated an "exact" value for β, supposing that the higher order terms of order P^{2n} would only give corrections of order θ^n to the eigenvalue. For a thick film he got $\beta = -1.35$. He got an interesting sharp minimum in β at the critical thickness $d = 1.812\xi$, which would not be expected from Tinkham's arguments, but which was observed experimentally[11]. Actually his analysis was wrong, and the terms P^{2n} all give linear contributions of order θ, which must be summed to obtain the correct slope β. This is done here in Sec. 3.

Following Saint-James' work Yamafugi, Kusayanagi, and Irie[4] proposed a new formula for $H_u(\theta)$ for thick films which was in agreement with Saint-James' result for small θ, $\beta = -1.35$, and agreed much better with experimental data than Tinkham's formula. However, their reasoning was mathematically wrong, since they worked with (2) but ignored the boundary condition. In fact if one uses a separable trial function as they did, the minimum value of λ which can be obtained is just that given in (5), which is very different from their formula. The formula for H_u resulting from the use of (5) has the bad result that $H_u < H_{C2}$ for $29° < \theta < 90°$. The choice of a separable trial function is quite poor in this range, since H_u must not be less than H_{C2}. The value $H_u = H_{C2}$ can always be achieved by using the simplest tilted-vortex trial solution, Δ proportional to $A_0(x\sin\theta - z\cos\theta)$. The resulting value for λ is just 1, independent of θ, which is smaller than that given by (5) when $H_u < H_{C2}$. The actual variation of H_u with $\varphi = 1/2\pi - \theta$ for small φ is neither flat according to this trial function, nor proportional to φ^2 as in the empirical formulae of Tinkham and of Yamafugi, et al., as shown in Sec. 4. Our best variational estimate in this section is proportional to φ^4, but φ^3 is also possible.

The tilted vortex character of the solution for Δ in an inclined field was emphasized by Kulik[12]. The vortex character of the solution can be verified experimentally by applying an electric field parallel to the sample surface. A finite zero-frequency conductivity should be observed due to flux flow, instead of the infinite conductivity usually characteristic of superconductors due to the Meissner effect.

In a set of two papers Maki[5,6] has attempted to calculate the angular dependence of the impedance of a sample in an inclined field. In the first paper he used the simple separable trial function. He found a flux-flow characteristic in the longitudinal orientation, where the electric field E is parallel to the surface component of the static magnetic field B_{ex}. However, in the transverse orientation, E perpendicular to B_e, a Meissner effect resulted at all angles θ. In the second paper Maki used a trial function which mixed the ground and first excited states of the separated functions S(z) and A(x). The results for H_u were a little improved: the unacceptable region where $H_u < H_{C2}$ was slightly reduced to $38° < \theta < 90°$, and now $\beta = -1.35$. The transverse orientation still showed a Meissner effect, now with the opposite sign, but, what was worse, even the longitudinal orientation acquired a Meisner effect in order θ^2. These unphysical results are due to the inaccuracy of the trial functions used by Maki. The exact functions for small θ found in Sec. 3 when applied to impedance calculations in Sec. 5 always show a finite flux-flow-type conductivity, which is anisotropic. Experimental data[13] show reasonable agreement with the predicted angular dependence for flux-flow and disagree with the Meissner-type behavior.

Better approximate ground-state trial functions may be constructed based on the idea of tilted vortices. First suppose that $\Delta(x, z)$ may be written as a product $\Psi(x-\gamma z)\times\Phi(z)$, where γ is the cotangent of the angle of tilting and Φ must not grow as $z\to\infty$. When this trial function is substituted into (1) and λ is varied with respect to Ψ and Φ, separate equations for Ψ and Φ result. The separate solutions are $\Phi = S_0(z_n)$ and $\Psi = A_0(x_n - \gamma z_n)$, where the rescaled variables are

$$x_n=(x-x_0)\,(1+\gamma^2)^{1/4}(2eB_e\sin\theta)^{-1/2},$$
$$z_n=z[2eB_e(\cos\theta-\gamma\sin\theta)]^{-1/2}.$$

Minimizing λ with respect to γ for a thick film where μ_0 is independent of γ gives

$$\gamma=\mu_0(1-\mu_0^2)^{-1/2},$$
$$\lambda=(1-\mu_0^2)^{1/2}\sin\theta+\mu_0\cos\theta. \tag{6}$$

The formula for H_u obtained from (6) by setting $\lambda = H_{C2}/H_u$ was proposed previously by Tilley and Ward[14] but not justified by a variational method by them. For z_n to be real one must have $\cot\theta \leq \gamma$, so the solution given in (6) is only valid when $\theta \leq 53.8° = \arccos\mu_0$. For larger θ the solution is fixed at $\gamma = \cot\theta$, $\lambda = 1$, $H_u = H_{C2}$. This simple tilted vortex solution has a wider range where $H_u > H_{C2}$ than Maki's approximations and never has $H_u < H_{C2}$. The value of $\beta = -1.37$ is slightly worse than his second approximation.

Further improvement in the ground-state eigenvalue estimate for thick films is obtained by letting the angle of tilting of the vortex vary with z, using a trial function $\Psi(x - f(z))\Phi(z)$. The equation for Ψ separates out as before and is again a harmonic oscillator equation, $\Psi(x) = A_0(ax)$. The equations for f and Φ are coupled and nonlinear, not having a known solution. However, if the form of Φ is taken to be $S_0(bz)$, then the equation for f is soluable and f has the form

$$f(z)=\sum c_n S_n(bz)/S_0(bz).$$

The resulting expression for λ is

$$\lambda={}^1/_2\sin\theta\,(a^2+a^{-2})+{}^1/_2\cos\theta\,(b^2+b^{-2})\,\mu_0$$
$$-\frac{\sin\theta\cos\theta}{a^2b^2}\sum_{n\neq0}\frac{\langle S_n(z)|\sqrt{2}z'|S_0(z)\rangle^2}{b^2(\mu_n-\mu_0)\cos\theta+2a^{-2}\sin\theta}, \tag{7}$$

where μ_n are the excited state eignevalues. $z' = (2eB_e)^{1/2}z$.

Analytic results are obtained near the parallel and perpendicular directions. For small θ one gets

$$a=(1-2\alpha_0)^{1/4},\quad b=1,$$
$$\lambda=\mu_0+\theta(1-2\alpha_0)^{1/2},\quad \alpha_0=\sum_{n\neq0}\langle S_n|\sqrt{2}z'|S_0\rangle^2(\mu_n-\mu_0)^{-1}. \tag{8}$$

These results agree with the exact results obtained in the next section and give the true value of $\beta = -(1 - 2\alpha_0)^{1/2}/\mu_0 = -1.30$.

Near the perpendicular direction a formal expansion in powers of $\tan\varphi$ would give a series of sums

$$S_m = \sum_{n\neq 0} \langle S_n | \sqrt{2} z' | S_0 \rangle^2 (\mu_n - \mu_0)^m.$$

Due to the presence of the surface only $S_0 = \mu_0$ and $S_1 = 2$ converge. Using these values (7) can be rearranged

$$2\lambda \sec\varphi = a^2 \sec^2\varphi + a^{-2} + b^2 \mu_0 \operatorname{tg}\varphi - b^{-2} \operatorname{tg}\varphi \sum_{n\neq 0} \frac{\langle S_n | \sqrt{2} z' | S_0 \rangle^2 [{}^1/_2 a^2 b^2 (\mu_n - \mu_0) \operatorname{tg}\varphi]^2}{1 + {}^1/_2 a^2 b^2 (\mu_n - \mu_0) \operatorname{tg}\varphi}. \qquad (9)$$

The most important contributions to the sum come from terms with large n where the two terms in the denominator are comparable in magnitude. Unfortunately we do not know the required matrix elements and eigenvalues for the optimum choice of $z_0' = (0.5901)^{1/2}$. However, if instead $z_0 = 0$ is chosen the wave functions are just the even harmonic oscillator ones. (9) is still valid using $\mu_n = 4n + 1$, and the matrix element

$$\langle S_n | \sqrt{2} z' | S_0 \rangle^2 = 2[(2n)!]^{1/2} [\pi (2n-1) 2^n n!]^{-1},$$

which for large n is asymptotically $1/2\pi^{-3/2} n^{-5/2}$. Then for $z_0 = 0$ and small φ

$$2\lambda \sec\varphi = a^2 \sec^2\varphi + a^{-2} + b^2 \operatorname{tg}\varphi - {}^1/_2 b^{-2} \operatorname{tg}\varphi\, \pi^{-1/2} (2a^2 b^2 \operatorname{tg}\varphi)^{3/2}. \qquad (10)$$

Minimizing (10) gives

$$a^2 = \cos\varphi [1 + (3/2\pi) \sin^4\varphi], \qquad b = (2\pi)^{-1/2} \sin^{3/2}\varphi, \quad \lambda = 1 - (4\pi)^{-1} \sin^4\varphi. \qquad (11)$$

Therefore H_u begins to increase near $\varphi = 0$ at least as fast as φ^4. The choice of a finite z_0 would only change the coefficient of the φ^4 variation but not the power. The alternate choice $\Phi(z) = e^{-bz}$ gives a slightly better limiting expression $\lambda = 1 - \varphi^4/8$. We show in Sec. 4 that the increase for a thick film cannot be as fast as φ^2. The lower bound is especially valuable since the exact variation is not found; it could be either φ^3 or φ^4.

3. EXACT SOLUTION NEAR PARALLEL ORIENTATION

The exact solution is obtained by summing perturbation theory to all orders, splitting the operator H in (2) into $H_0 + P$ as in (4). We have only managed to find a complete exact solution for the eigenvalues and wave functions near the parallel orientation, for small θ.

According to perturbation theory the exact eigenvalue λ_i numbered i of H can be expressed as a series in terms of the eigenvalues λ_i^0 of H_0 and the matrix elements $P_{nm} = \langle n | P | m \rangle$ of the operator P calculated between the normalized states of H_0. We have

$$\lambda_i = \lambda_i^0 + P_{ii} - \sum_{n\neq i} \frac{|P_{in}|^2}{\lambda_n^0 - \lambda_i} + \sum_{m,n\neq i} \frac{P_{im} P_{mn} P_{ni}}{(\lambda_m^0 - \lambda_i)(\lambda_n^0 - \lambda_i)} - \ldots \qquad (12)$$

The solutions of H_0 are separable into Abrikosov harmonic oscillator states $A_i(x)$ and surface states $S_j(z)$, so it is convenient to regard the index i in (12) as a double index, the first index referring to the x-eigenvalue and the second to the z-eigenvalue. Looking for the solution for small θ, we replace $\sin\theta$ by θ and $\cos\theta$ by 1 in (4). To make the θ dependence explicit and ease the calculation of matrix elements it is convenient to make a scale change in x and make x and z dimensionless. Let $z' = (2eB_e)^{1/2} z$ and $x' = (2eB_e\theta)^{1/2}(x - x_0)$ so that

$$H_0 = -\nabla_{z'}^2 + (z' - z_0')^2 + \theta(-\nabla_{x'}^2 + x'^2), \qquad P = -2\theta^{1/2} x' (z' - z_0'). \qquad (13)$$

The value of z_0' is chosen equal to the ground-state expectation value of z'. This choice minimizes the ground-state eigenvalue of the surface state μ_0. Then the eigenvalues $\lambda_{ij}^0 = (2i + 1)\theta + \mu_j$, where μ_j are calculated numerically. For example when $d \to \infty$, $\mu_1 = 5.62\mu_0$, etc. [3,15,16]. Only terms involving an even number of matrix elements P^{2n} contribute to (12) because P connects only even harmonic oscillator states to odd ones.

Let us first discuss the ground-state eigenvalue λ_{00} which determines the critical field. $\lambda_{00}^0 = \mu_0 + \theta$. The only non-vanishing matrix element of x' with the ground-state harmonic oscillator is $\langle A_0 | \sqrt{2} x' | A_1 \rangle = 1$. Thus to second order in perturbation theory we get Saint-James' result $\lambda_{00} = \mu_0 + (1 - \alpha_0)\theta$ with α_0 as in (8). Numerical values of α_0 may be taken from the graph of the function $A \equiv 1 - 2\alpha_0$ in [16]. In particular for $d \to \infty$, $A = 0.586$ and $\alpha_0 = 0.207$. If the calculation is stopped at this point, as Saint-James did, one gets $\beta = -(1 - \alpha_0)/\mu_0 = -1.344$, which except for a numerical uncertainty of a part in a thousand is the result quoted by him.

However, in spite of the fact that P^{2n} is of order θ^n, it is not correct to cut off the perturbation theory at second order. An example is the term of fourth order in (12). The numerator is of order θ^2. The first and last factors in the denominator cannot be infinitesimally small since the matrix elements of P to the ground state vanish unless the surface states are excited. However, the middle factor is small, of order θ, for those intermediate states which are a combination of the surface ground state and excited Abrikosov states. This factor of θ in the denominator compensates one factor of θ in the numerator, leaving an overall contribution of order θ. Similar cancellations occur in all higher order even terms, giving an infinite series of contributions of order θ. Writing the exact eigenvalue $\lambda_{00} = \mu_0 + s\theta$, where s is the slope to be determined, the series to order θ is

$$\mu_0 + s\theta = \mu_0 + \theta - \alpha_0\theta - \alpha_0^2\theta \sum_{n\neq 0} \frac{\langle A_0 | 2x'^2 | A_n \rangle^2}{2n + 1 - s} - \alpha_0^3\theta \sum_{m,n\neq 0} \frac{\langle A_0 | 2x'^2 | A_m \rangle \langle A_m | 2x'^2 | A_n \rangle \langle A_n | 2x'^2 | A_0 \rangle}{(2m + 1 - s)(2n + 1 - s)}. \qquad (14)$$

Cancelling the constant term μ_0 and dividing (14) by θ the series has the same form as (12). Therefore, s is the ground-state eigenvalue of an operator H' which is the sum of H_0' and P' where $H_0' = \nabla_{x'}^2 + x'^2$ and $P' = -2\alpha_0 x'^2$. H' is just a harmonic oscillator hamiltonian, which is easily solved by letting $x'' = A^{1/4} x'$, so $s = A^{1/2} = (1 - 2\alpha_0)^{1/2}$.

This corrected exact value for the slope agrees with Saint-James' result only when $\alpha_0 \to 0$, which does occur in the Tinkham thin film limit. In the thick film limit the corrected value for β is $\beta = -A^{1/2}/\mu_0 = -1.30$, which is only 3.5% smaller in magnitude than Saint-James' value. The maximum error made by him occurs at the critical thickness $d = 1.812\xi$ where $s = 0$, whereas he got a finite value $1/2$. The actual vanishing of the slope at this thickness, rather than its having a finite minimum value was evidently missed in the experiments[11] because only the difference between H_u at a finite angle $\theta = 2.5°$ and $H_{C\|}$ was measured.

For further calculations H is more conveniently separated into $H_0'' + P''$.

$$H_0''=-\nabla_{z'}^2+(z'-z_0')^2+\theta A^{1/2}(-\nabla_{x''}^2+x''^2),$$
$$P''=-2\theta^{1/2}A^{-1/4}x''(z'-z_0')+2\theta\alpha_0 A^{-1/2}x''^2. \tag{15}$$

Using this separation $\lambda_{00} = \lambda_{00}^0$, and all the higher terms in (12) cancel to order θ. The normalized wave functions u_i are then constructed from the unperturbed wave functions u_i^0 according to the general perturbation theory formula.

$$u_i=c\left[u_i^0-\sum_{n\neq i}\frac{u_n^0 P_{ni}}{\lambda_n^0-\lambda_i}+\sum_{m,n\neq i}\frac{u_m^0 P_{mn}P_{n0}}{(\lambda_m^0-\lambda_i)(\lambda_n^0-\lambda_i)}-\dots\right]. \tag{16}$$

For small θ (16) gives to order $\theta^{1/2}$

$$u_{00}(x'',z')=A_0(x'')S_0(z')+\frac{\theta^{1/2}}{A^{1/4}}\sqrt{2}\,x''A_0(x'')\sum_{n\neq0}\frac{S_n(z')}{\mu_n-\mu_0}\langle S_n|\sqrt{2}\,z'|S_0\rangle$$
$$+\frac{\theta^{1/2}\beta_0}{A^{5/4}}S_0(z')\sum_{m\neq0}\frac{A_m(x'')}{2m}\langle A_m|(\sqrt{2}\,x'')^3|A_0\rangle, \tag{17}$$

$$\beta_0=\sum_{k,l\neq0}\frac{1}{(\mu_k-\mu_0)(\mu_l-\mu_0)}\langle S_0|\sqrt{2}\,z'|S_k\rangle\langle S_k|\sqrt{2}(z'-z_0')|S_l\rangle\langle S_l|\sqrt{2}\,z'|S_0\rangle.$$

The excited eigenvalues λ_{i0} are easily constructed following the above procedure, giving $\lambda_{i0} = \mu_0 + (2i + 1) \times A^{1/2}\theta$. The wave functions u_{i0} are similar to (17), the only difference being the replacement of A_0 by A_i and of m by m − i. In order to obtain convergent perturbation theory series for the general excited-state eigenvalues λ_{ij} when $\theta \neq 0$ it is necessary to rearrange the terms in H_0'' and P'' separately for each value of j into H_j'' and P_j''. The result has the same form as (15) but z_0' must be replaced by z_j'. This change is compensated for by replacing x' by $x' + (z_0' - z_j')\theta^{-1/2}$. z_j' is chosen such that the average value of z' in the state j is z_j'. This choice minimizes μ_j^j, the j[th] eigenvalue of the operator $-\nabla_{z'}^2 + (z' - z_j')^2$ as illustrated by Fink[15]. This choice is necessary so that the quantity α_j which now replaces α_0 should not have any contribution to the sum from n = j where the denominator vanishes. α_j is defined like α_0 in (8) except that the term n = j is omitted instead of n = 0, the functions S_n^j with eigenvalues μ_n^j calculated with z_0' replaced by z_j' are used, and S_0 and μ_0 are replaced by S_j^j and μ_j^j. If this choice is not made the anomalously large contribution to the sum from n = j prevent the renormalization of H into the simple form (15). The reason is related to the fact that the harmonic oscillator has only positive energies, and it is therefore necessary to start at the bottom of the band in order to construct all the states by adding variously excited harmonic oscillator energies. After the change is made the previous procedure can be repeated and one finds results having the same form as before but with appropriate changes of 0 to j and in x'. The results for the eigenfunctions and eigenvalues will be used in Sec. 5 to calculate the impedance.

4. CRITICAL MAGNETIC FIELD NEAR PERPENDICULAR ORIENTATION

For a film of finite thickness perturbation theory may be applied as before to find the variation of H_u for small angles $\varphi = 1/2\pi - \theta$. Proceeding as in Sec. 3, one immediately finds that the linear variation of λ with φ vanishes for all thicknesses, since the quantity α_0 is here evaluated for simple harmonic oscillator functions and equals 1/2. This result also suggests as before that the terms in H be regrouped from (4) to $H = H_0' + P'$.

$$H_0'=\cos\varphi\,(-\nabla_{x'}^2+x'^2)-\nabla_{z'}^2,$$
$$P'=-\sin\varphi\cos^{1/2}\varphi\cdot 2x'(z'-z_0')+\sin^2\varphi\,(z'-z_0')^2 \tag{18}$$

where $z' = (2eB_e)^{1/2}z$ as before and now $x' = (2eB_e\cos\varphi)^{1/2} \times (x - x_0)$. Then perturbation theory gives, to order φ^2

$$\lambda_{00}=1-{}^1/_2\varphi^2+\varphi^2\sum_{m\neq0}\langle0|z'|m\rangle^2[1-2(2+\varepsilon_m)^{-1}]. \tag{19}$$

The eigenfunctions in the z-direction are $\cos(m\pi z/d)$, so

$$\varepsilon_m=(m\pi/d)^2(2eB_e)^{-1},\ \text{and}\ \langle0|z'|m\rangle^2=8d^2(m\pi)^{-4}(2eB_e)$$

for odd m and vanishes for even m. The sum (19) is easily performed

$$\lambda_{00}=1-\varphi^2(\xi/d\sqrt{2})\,\text{th}\,(d/\xi\sqrt{2}). \tag{20}$$

In the thin film limit this result agrees with Tinkham's formula (3), $\lambda = 1 - 1/2\varphi^2 + \varphi^2 d^2/(12\xi^2)$. For thick films $\lambda = 1 - \varphi^2\xi/(d\sqrt{2})$. Comparison of higher terms in the perturbation series indicates that the expansion parameter is $\tan\varphi\,\sin\varphi\,d^2 2eB_e$. The validity of (20) is thus restricted for thick films to the range $\varphi \ll \xi/d$, and so it does not apply to semi-infinite films where $d \to \infty$. The transition from the small angle to the finite angle regime evidently occurs when the film thickness equals the size of the surface sheath, which is therefore approximately ξ/φ. At this angle (20) should join smoothly on to the result for a semi-infinite film, which would require $\lambda - 1$ to be proportional to φ^3. This rough argument gives a faster variation to the size of the surface sheath and λ than our variational solution of Sec. 2.

To solve the case of a semi-infinite film we arrange H so that there is only one expansion parameter, $\tan\varphi$, by dividing it by $\cos\varphi$: now

$$H''=H/\cos\varphi,\quad H_0''=H_0'/\cos\varphi,\quad P''=P'/\cos\varphi.$$

We are also lead to rescale z again: $z'' = z'\tan^{1/2}\varphi\,\sin^{1/2}\varphi$, since as above we expect this combination to be of order unity.

$$H_0''=-\nabla_{x'}^2+x'^2-\text{tg}^2\varphi\,\nabla_{z''}^2,$$
$$P''=-2x'(z''-z_0'')+(z''-z_0'')^2. \tag{21}$$

The perturbation expansion (12) for $\lambda_{00}'' = \lambda_{00}/\cos\varphi$ is

$$\lambda_0''=1+\sum_m\langle0|(z''-z_0'')|m\rangle^2\left(1-\frac{2}{2+\varepsilon_m\,\text{tg}^2\varphi-\delta\lambda''}\right)$$
$$+\sum_{lmn}\langle0|(z''-z_0'')|l\rangle\langle l|(z''-z_0'')|m\rangle\langle m|(z''-z_0'')|n\rangle\langle n|(z''-z_0'')|0\rangle$$
$$\times\left\{-\left(1-\frac{2}{2+\varepsilon_l t^2-\delta\lambda''}\right)\frac{\Theta_{m0}}{\varepsilon_m t^2-\delta\lambda''}\left(1-\frac{2}{2+\varepsilon_n t^2-\delta\lambda''}\right)\right.$$
$$\left.+\frac{2}{2+\varepsilon_l t^2-\delta\lambda''}\left(1-\frac{4}{4+\varepsilon_m t^2-\delta\lambda''}\right)\frac{1}{2+\varepsilon_n t^2-\delta\lambda''}\right\}+\dots, \tag{22}$$

where $\delta\lambda'' = \lambda'' - 1$, which is of order $t^2 = \tan^2\varphi$, and Θ_{m0} is a projection operator which means to omit the m = 0 term. Keeping terms of order $\tan^2\varphi$, this series can be written in the same form as (12), defining $a = \delta\lambda''/\tan^2\varphi$.

$$a=p_{00}-\sum_{m\neq0}p_{0m}^2/(\varepsilon_m-a)+\dots,$$
$$p=2^{-1/2}(z''-z_0'')(\overleftarrow{\nabla}_{z''}\overrightarrow{\nabla}_{z''}-a)\cdot2^{-1/2}(z''-z_0'')+$$
$$+[2^{-1/2}(z''-z_0'')]^2(\overleftarrow{\nabla}_{z''}\overrightarrow{\nabla}_{z''}-a)[2^{-1/2}(z''-z_0'')]^2/2!+\dots \tag{23}$$

The derivative operators are shown acting in different directions since this choice does not require any boundary terms resulting from the partial integrations performed to eliminate the intermediate states. Letting the differential operators act either on the powers of $z'' - z_0''$ or outside, the exponential series can be summed.

$$p=\exp[{}^1/_4(z''-z_0'')^2]\overleftarrow{\nabla}_{z''}\overrightarrow{\nabla}_{z''}\exp[{}^1/_4(z''-z_0'')^2] \\ -\overleftarrow{\nabla}_{z''}\overrightarrow{\nabla}_{z''}+{}^1/_2\exp[{}^1/_2(z''-z_0'')^2]-a\{\exp[{}^1/_2(z''-z_0'')^2]-1\}. \quad (24)$$

The solution of (23) is that a is the eigenvalue of $h = \overleftarrow{\nabla}_{z''}\overrightarrow{\nabla}_{z''} + p$. The eigenfunctions are proportional to $\exp(-1/4(z''-z_0'')^2)\cos(kz'')$, and the resulting values of a are $a = 1/2 + k^2$. The ground-state value is $a = 1/2$. With this value $\lambda = (1 + 1/2\varphi^2)\cos\varphi$ and is just a constant $\lambda = 1$ to order φ^2, agreeing with the tendency of (20).

We have not actually calculated the coefficient of the φ^3 term in the variation of λ. However, we can point out how such terms arise, in spite of an apparent expansion parameter of $\tan^2\varphi$ in (21) and (22). The matrix element $\langle 0|z''|m\rangle$ is proportional to m^{-2} and ϵ_m to m^2. Thus although a sum like we used above

$$\sum\langle 0|z''|m\rangle^2\varepsilon_m\,\mathrm{tg}^2\,\varphi$$

converges, the next term in the direct expansion in powers of $\tan^2\varphi$,

$$\sum\langle 0|z''|m\rangle^2\varepsilon_m^2\,\mathrm{tg}^4\,\varphi$$

does not converge, and factors of $\epsilon_m\tan^2\varphi$ must be left in the denominator. Then the contribution is of order φ^3. Other terms act similarly. Also we note that a contribution of order $\tan^2\varphi$ in p could lead to a contribution of order $\tan^3\varphi$ in λ. For example, if

$$\tan^2\varphi(z''-z_0'')^2\exp[{}^1/_2(z''-z_0'')^2],$$

is added to p, a becomes $1/2 + \tan\varphi$, which makes $\lambda = 1 + \tan^3\varphi$. Although the terms of order $\tan^3\varphi$ do not appear when there is no boundary, it seems unlikely that their sum vanishes when the boundary conditions are used. However, only our establishment that H_u varies at least like φ^4 in Sec. 2 is rigorous.

5. IMPEDANCE

The response of a superconductor to an external electromagnetic field when it is also in an inclined static magnetic field B_e is calculated using the formalism developed in [16]. One needs to know the response function Q relating the time-dependent part of the current $j\omega$ to the time-dependent part of the vector potential $A\omega$ according to $j\omega = -QA\omega$. Slightly generalizing the results of [17] and expanding to keep all terms of order $\omega/2\pi T$ we find for a dirty superconductor with B_e near H_u that $Q = Q_n + Q'$, where $Q_n = -i\omega\sigma$ is the normal state response and Q' is the first correction due to superconductivity.

$$Q'=|\Delta|^2\sigma\frac{\psi'({}^1/_2+\rho)}{\pi T}\Big[1-\frac{1}{2}\frac{i\omega}{\varepsilon_0-i\omega}-\frac{i\omega}{8\pi T}\frac{\psi''({}^1/_2+\rho)}{\psi'({}^1/_2+\rho)}\Big(3-\frac{i\omega}{\varepsilon_0-i\omega}\Big) \\ -4\sum_n\frac{D|\langle 0|\hat{e}(i\nabla+2e\mathbf{A})|n\rangle|^2}{\varepsilon_n-\varepsilon_0-i\omega}\Big(1-\frac{i\omega}{4\pi T}\frac{\psi''({}^1/_2+\rho)}{\psi'({}^1/_2+\rho)}\Big)\Big]; \quad (25)$$

ω is the angular frequency of the electro-magnetic field. σ is the normal state conductivity. ψ' and ψ'' are the first and second derivatives of the digamma function ψ. $\epsilon_n = 2eB_eD\lambda_n$. $\hat{e}$ is a unit vector along the direction of the electric field.

The zero-frequency limit of Q' is especially important. If $Q'(\omega = 0)$ is finite the response to a static electric field E is infinite since $i\omega A_\omega = E$. $Q'(\omega = 0)$ must vanish for the d.c. conductivity to be finite. In order for $Q'(\omega = 0)$ to vanish the sum S(0) must equal 1, where

$$S(\omega)=4\sum_n D|\langle 0|\hat{e}(i\nabla+2e\mathbf{A})|n\rangle|^2/[2eB_eD(\lambda_n-\lambda_0)-i\omega]. \quad (26)$$

Most of the variation of the impedance at microwave or smaller frequencies occurs for small angles. In this limit we can get exact expressions for Q' using the eigenvalues and eigenfunctions calculated in Sec. 3. In the longitudinal orientation, l, we need to use only u_{00}, u_{10}, λ_{00} and λ_{10}. Writing $\omega' = \omega/2eB_eD$,

$$S(\omega)_l=2A^{1/2}\theta/(2A^{1/2}\theta-i\omega'). \quad (27)$$

If $\theta \neq 0$, $S(0)_l = 1$. Equation (27) differs from the corresponding result of Maki's first approximation only by the presence of the factors $A^{1/2}$. In the transverse orientation, t, we also need λ_{ij} and u_{ij} for $j \neq 0$, but only to order θ^0 to get the same accuracy, so we need not shift z_0' yet.

$$S(\omega)_t=\frac{2A^{3/2}\theta}{2A^{1/2}\theta-i\omega'}+2\sum_n\frac{\langle S_0|\sqrt{2}(z'-z_0')|S_n\rangle^2}{\mu_n-\mu_0-i\omega'}. \quad (28)$$

Again for $\theta \neq 0$, $S(0)_t = A + 2\alpha_0 = 1$ as required.

The impedance is obtained by substituting $S(\omega)$ from (27) or (28) back into (25). For a sample thinner than the skin depth $\delta_0 = (2\pi\omega\sigma)^{-1/2}$ or for a thicker sample when $\xi \ll \delta_0$, which is the same as $\omega' \ll \kappa^2$, where κ is the Ginzburg-Landau parameter, one only needs the spatially averaged, $\langle\rangle$, value of Q'. For small but finite θ the average of $|\Delta|^2$ is almost the same as calculated in [16]. However, here the Abrikosov vortex structure in the x-y plane must be averaged over in addition to the average on the surface sheath structure in the z-direction, which reduces $\langle|\Delta|^2\rangle$ by a factor of $\beta_A = 1.16$. To lowest order

$$\langle|\Delta|^2\rangle\sigma\psi'({}^1/_2+\rho)/\pi T=2\varepsilon_0L/\beta_A, \\ L=4\sigma N(1.20\varkappa)^2(H_u-B_e)/[(\varkappa_2^2-J_2)H_u], \quad (29)$$

N and J_2 are functions of film thickness plotted in [16]. The temperature and film-thickness dependence of κ_2 is discussed in [18].

The results for Q' simplify somewhat if we are not too close to T_C, so that $\omega' \ll 1$, which is the same as $\omega \ll \pi(T_C - T)$. Then we get

$$\mathrm{Re}\langle Q_l'\rangle=2\varepsilon_0L\frac{1}{\beta_A}\frac{\omega'^2}{4A\theta^2+\omega'^2}, \\ \mathrm{Im}\langle Q_l'\rangle=-\omega L\frac{1}{\beta_A}\Big[\frac{4\mu_0A^{1/2}\theta}{4A\theta^2+\omega'^2}+1+3\rho\frac{\psi''({}^1/_2+\rho)}{\psi'({}^1/_2+\rho)}\Big], \\ \mathrm{Re}\langle Q_t'\rangle=2\varepsilon_0L\frac{1}{\beta_A}\frac{A\omega'^2}{4A\theta^2+\omega'^2}, \\ \mathrm{Im}\langle Q_t'\rangle=-\omega L\frac{1}{\beta_A}\Big[\frac{4\mu_0A^{3/2}\theta}{4A\theta^2+\omega'^2}+1+B+(1+2A)\rho\frac{\psi''({}^1/_2+\rho)}{\psi'({}^1/_2+\rho)}\Big], \quad (30)$$

$B = 4S_{-2}$. For $\theta = 0$ these results agree with those of [16] when β_A is replaced by 1. This corresponds to the complete absence of vortices, which requires $\theta \ll (\xi/\text{sample length})^2$, a ratio too small to be met in practice.

The impedance Z is calculated as for example in [16]. For a very thick film with $d \gg$ both ξ and δ_0 only one surface is excited, and

$$Z=2\pi\omega\delta_0(1-i+2\pi\delta_0\langle Q'\rangle d).$$

$R = \mathrm{Re}Z$ is proportional to $\mathrm{Re}Q'$. Using the identity $16\pi\sigma D(1.20\kappa)^2 = 1$, the limiting expression for L gives

$$2\varepsilon_0 L = 2.63(H_{c3} - B_e)/[2\pi d\xi H_{c3}(2\varkappa_2^2 - 0.328)]. \tag{31}$$

For films thinner than δ_0 a common experimental arrangement is to evaporate the film on one side of a thin insulating substrate, the other side of which is then stuck to the metal wall of a microwave cavity. Let the index of refraction of the substrate be n and its thickness l. Let $k_f = (1 - i)/\delta_0$ for the film, and $k_c = (1 - i)/\delta_{0c}$ for the cavity, where δ_{0c} is the cavity skin depth. Then when the film is in the normal state the general result for Z is

$$Z = -\left(\frac{4\pi i\omega}{k_f}\right)\frac{k_f(n\omega + k_c t) + n\omega(k_c - n\omega t)\,\mathrm{th}}{k_f(n\omega + k_c t)\,\mathrm{th} + n\omega(k_c - n\omega t)}, \tag{32}$$

where $t = \tan(n\omega l)$ and $\mathrm{th} = \tanh(k_f d)$. Assuming $n\omega l \ll 1$ and $d \ll \delta_0$ and using $\omega n\delta_{0c} \ll 1$,

$$Z = -4\pi i\omega[1 + k_c(l + d)]/(k_c + k_f^2 d + k_c k_f l d). \tag{33}$$

If it is further assumed that $l \ll \delta_{0c}$ and $l \ll d$ the result is the same as was derived in [16] for a film on a metal neglecting the substrate.

However, these last two conditions on l are usually fulfilled with the inequalities reversed, so that it is not justified to neglect the substrate. Then with $l \gg \delta_{0c}$ and $l \gg d$ we get a new expression

$$Z = -4\pi i\omega l/(1 + k_f^2 l d). \tag{34}$$

For a typical situation $\delta_0 \sim 10^{-4}$ cm, $l \sim 10^{-2}$ cm, and $d \sim 10^{-5}$ cm. Then the second term in the denominator is ~10 and $Z = (\sigma d)^{-1}$, now independent of the exact properties of the substrate and cavity. Superconductivity of the film is included as in [16] by replacing σ by $\sigma + \sigma'$ and expanding in σ' to first order. $Z = (\sigma d)^{-1}(1 - \sigma'/\sigma)$, where $\sigma' = \langle Q'\rangle/(-i\omega)$. Therefore under these conditions R is proportional to $\mathrm{Im}\langle Q'\rangle$, unlike the previous cases considered.

6. POSSIBLE PHASE TRANSITIONS OF THE VORTEX LATTICE STRUCTURE

By using the wave functions calculated to order $\theta^{1/2}$ near the parallel orientation the essential flux-flow character of the impedance has been obtained in Sec. 5. Besides having an overall form in agreement with these calculations, the available experimental data[19] for the slope $\partial R/\partial B_e$ very near H_u has a fine structure which is irregular, deviating significantly above and below the smoothed average slope obtained somewhat below H_u. The geometrical considerations suggested in [19] have no apparent validity since they do not enter into (25). However, the sharp changes in this fine structure may possibly be due to sudden changes in the equilibrium vortex-lattice structure, which would give rise to a sudden change in the slope $\partial\langle|\Delta|^2\rangle/\partial B_e$ similar to that observed in tunneling experiments in parallel field near the critical thickness[18].

The free energy difference between the superconducting and normal states as a function of the external field B_e is F_{sn}.

$$F_{sn} = -\langle|\Delta|^2\rangle^2 V_s I/64\pi e^2\lambda^2\xi^2, \tag{35}$$

where

$$I = \left(\langle|\Delta|^4\rangle - 8e^2\lambda^4\varkappa^{-2}\int B_s^2\,dV/V_s\right)\Big/\langle|\Delta|^2\rangle^2,$$

V_s is the volume of the superconductor. B_s is the magnetic field generated by the current j_s flowing in the superconductor. The total magnetic field $\mathbf{B}$ is $\mathbf{B}_s + \mathbf{B}_e$. $\lambda = \kappa\xi$ is the London penetration depth in this section. $\langle|\Delta|^2\rangle$ calculated in the same way as originally done by Abrikosov[9] is proportional to $H_u - B_e$ and is inversely proportional to I. All the dependence on the vortex-lattice structure in F_{sn} and $\langle|\Delta|^2\rangle$ is included in the factor I. The structure having the smallest I has the most negative free energy difference F_{sn} and is the favored equilibrium structure. If the lattice structure changes continuously as a function of θ, no sharp changes in I, and therefore in $\partial\langle|\Delta|^2\rangle/\partial B_e$ and $\partial R/\partial B_e$, will occur. A sharp change in the slope $dI/d\theta$ would occur if two solutions, both of which are locally stable with respect to local lattice distortions, existed, and their respective curves $I(\theta)$ crossed at a finite value of θ with different slopes. The equilibrium $I(\theta)$ would have a sharp peak and $\partial R/\partial B_e$ a corresponding sharp minimum at this angle.

The second contribution to I which involves B_s is smaller than the first one involving $\langle|\Delta|^4\rangle$ by the factor κ^{-2}. In the experiments $\kappa \approx 3$, so the first term should be more important and will be calculated first. For very small θ the solution for the normalized function is

$$\Delta = u_{00}(x'', z')e^{iky} = A_0(x'')S_0(z')\exp(ix_0''y''),$$

where $x_0'' = (2eB_e\theta)^{1/2}A^{1/4}x_0$, and $y'' = (2eB_e\theta)^{1/2}A^{-1/4}y$ is chosen so that $x_0''y'' = 2eB_e\theta x_0 y$. The phase factor in z_0 has been eliminated for convenience by making a gauge transformation. In terms of the variables x'' and y'' the x-y dependence of the solution is the same as Abrikosov's, so the equilibrium solution for very small angles has a lattice of equilateral triangles in x'' and y'' with no preferred orientation. In real space the unit cell is longer than the Abrikosov value in the direction of the surface component of B_e by the factor $(H_{c2}/B_e\theta)^{1/2}A^{-1/4}$ and in the surface direction perpendicular to B_e by $(H_{c2}/B_e\theta)^{1/2}A^{1/4}$. The area per vortex in the surface is increased by $H_{c2}/B_e\theta$. Thus there is just one quantum of flux of the perpendicular component of the magnetic field for each vortex.

The corrections to Δ of order $\theta^{1/2}$ are given in (17). They give no correction to $\langle|\Delta|^4\rangle$ to order $\theta^{1/2}$, so the first corrections to $\langle|\Delta|^4\rangle$ must be found in order θ. To calculate them one must first calculate the corrections to u_{00} of order θ, $u_{00}^{(1)}$:

$$\begin{aligned}
u_{00}^{(1)}(x'', z') = &-\theta A^{-1/2}\delta S_0(z')\sum_{m\neq 0}\frac{A_m(x'')\langle A_m|2x''^2|A_0\rangle}{m}\\
&+\frac{2\theta x''^2 A_0(x'')}{A^{1/2}}\sum_{m,n\neq 0}\frac{S_n(z')\langle S_n|\sqrt{2}(z'-z_0')|S_m\rangle\langle S_m|\sqrt{2}z'|S_0\rangle}{(\mu_n-\mu_0)(\mu_m-\mu_0)}\\
&+\theta A^{-1/2}(\gamma-\alpha_0\delta)S_0(z')\sum_{m\neq 0}\frac{A_m(x'')\langle A_m|4x''^4|A_0\rangle}{2m}\\
&+\theta A^{-1/2}\beta_0\sum_{m\neq 0}\frac{\sqrt{2}x''A_m(x'')\langle A_m|(\sqrt{2}x'')^3|A_0\rangle}{2m}\\
&\times\sum_{n\neq 0}\frac{S_n(z')\langle S_n|\sqrt{2}z'|S_0\rangle}{\mu_n-\mu_0}+\theta A^{-5/2}\beta_0^2 S_0(z')\\
&\times\sum_{l,m\neq 0}\frac{A_m(x'')\langle A_m|(\sqrt{2}x'')^3|A_l\rangle\langle A_l|(\sqrt{2}x'')^3|A_0\rangle}{4lm}
\end{aligned}\tag{36}$$

where

$$\delta = S_{-2} = \sum_{m\neq 0}\langle S_m|\sqrt{2}z'|S_0\rangle^2(\mu_m-\mu_0)^{-2},$$

$$\gamma = \sum_{l,m,n\neq 0}\frac{4\langle S_0|z'|S_l\rangle\langle S_l|(z'-z_0')|S_m\rangle\langle S_m|(z'-z_0')|S_n\rangle\langle S_n|z'|S_0\rangle}{(\mu_l-\mu_0)(\mu_m-\mu_0)(\mu_n-\mu_0)}.$$

Since we are using perturbation theory we must consider solutions which are near to the original triangular one. Thus for Δ we take a periodic superposition of our solutions with different values of x_0'' just like Abrikosov's with two vortices per unit cell. Our variable parameters are the ratio of the dimensions of the unit cell $L_{x''}/L_{y''} = r$ and rotation of the unit cell with respect to the magnetic field direction by the angle η.

The various combinations of products of excited and ground-state wave functions are worked out in terms of derivatives of the ground-state ones. For example, the sum of all combinations arising in $\langle|\Delta|^4\rangle$ when two Δ's are proportional to A_1 and the other two to A_0 is proportional to $2\beta_A + l^2_{x''} - l^2_{y''}$ where $l^2_{x''} = -\langle|\Delta_0|^2\nabla^2_{x''}|\Delta_0|^2\rangle$. Using the explicit form of the wave functions, an identity is obtained that $l^2_{x''} + l^2_{y''} = \beta_A$ regardless of r and η. Higher excited states give rise to higher derivatives. Denoting as c_{ij} the combination of i^{th} derivatives which transform with $\cos(j\eta)$ under rotation, we start with $c_{00} = \beta_A$. Further combinations are

$$c_{22}=l_{x''}^2-l_{y''}^2,$$
$$c_{40}=l_{x''}^4+2l_{x''}^2l_{y''}^2+l_{y''}^4, \quad (37)$$
$$c_{44}=l_{x''}^4-6l_{x''}^2l_{y''}^2+l_{y''}^4.$$

The missing combinations c_{20} anc c_{42} are related to the other functions by identities. All of them grow proportional to $r^{1/2}$ for large r. β_A, c_{40}, and c_{44} are even functions of ln(r), while c_{22} is odd. β_A has a maximum of 1.18 at r = 1 and a minimum of 1.16 at $r = \sqrt{3}$. $c_{22} = 2d\beta_A/d(\ln(r))$ has zeroes at r = 1 and $r = \sqrt{3}$ with a minimum of -0.12 between the zeroes. c_{40} has a minimum of 8.01 at r = 1, a maximum of 8.45 at $r = \sqrt{3}$, and a minimum of 8.01 at r = 1, a maximum of 8.45 at $r = \sqrt{3}$, and a minimum of about 6 for a larger r. c_{44} has a single minimum of -5.65 at r = 1 and vanishes at $r = \sqrt{3}$. The zeroes of these functions are required by symmetry so that $c_{ij}\cos(j\eta)$ should be invariant under rotation of $1/2\pi$ for the square symmetry at r = 1 and of $\pi/3$ for the equilateral triangular symmetry at $r = \sqrt{3}$ and $1/\sqrt{3}$. The symmetry with respect to a change of sign of ln(r) is determined for all r by its equivalence to a rotation by $1/2\pi$.

The resulting expression for $\langle|\Delta|^4\rangle$ is

$$\langle|\Delta|^4\rangle=\langle|\Delta_0|^2\rangle^2\frac{1}{d}\int_0^d S_0^4\,dz[(1+c'\theta)c_{00}+\theta(ac_{44}\cos 4\eta-bc_{22}\cos 2\eta+cc_{40})]. \quad (38)$$

The variations of c_{40} and of its coefficient are small. The correction to the coefficient of c_{00} is also not important since θ must be small for perturbation theory in θ to apply. Therefore the contributions to (38) of the terms proportional to c anc c′ may be henceforth ignored.

For $\theta \neq 0$ (38) can be decreased by letting r differ from $\sqrt{3}$. Minimizing (38) with respect to η gives two solutions:

$$\sin 2\eta=0, \quad \cos 2\eta=bc_{22}/4ac_{44}.$$

The second solution only applies when the magnitude of the last ratio is less than 1. The first solution gives $\eta = 0$ and $\eta = 1/2\pi$, which are the same if $r \to 1/r$. There are usually two minima of (38) as a function of r: one with $|\ln(r)| < \ln\sqrt{3}$ and the other with a larger value of $|\ln(r)|$. The solution most favored for small variations starts near $r = 1/\sqrt{3}$ with $\eta = 0$ and r increasing toward 1, because this gives the maximum negative contributions to the terms proportional to θ for linear variations in r. Taking into account the curvatures of c_{22} and c_{44} there are two minimum solutions with $\eta = 0$. A sharp phase transition between these two solutions with different values of r is obtained in perturbation theory if $ac'_{44} \ll bc'_{22}$ for $\theta = 3ac'_{44}/(2b^2c'_{22})$. The derivatives are taken with respect to ln(r) and evaluated at $r = \sqrt{3}$:

$$c'_{44}=14.95, \quad c''_{44}=-7.46, \quad c'_{22}=0.95, \quad c''_{22}=3.74.$$

Perturbation theory in ln(r) is not useful otherwise. If $ac'_{44} = bc'_{22}$ perturbation theory leaves the second solution fixed at $r = \sqrt{3}$, while the first solution is attaining a lower value of $\langle|\Delta|^4\rangle$. The second solution becomes unstable with respect to rotation of η when $bc'_{22} \leq 4ac'_{44}$, and the second choice for η is favored. However, owing to the negative curvature of c_{44} versus the positive curvature of c_{22} perturbation theory does not favor a transition.

In order to apply this discussion to the actual situation we need to evaluate the coefficients appearing in (38). The values of α_0 and $\delta = 0.075$ were calculated numerically in [16]. We have only roughly estimated the others using the Gaussian approximation to S_0, finding a = 0.02, b = 0.2 and c = 0.004. Then $bc'_{22}/ac'_{44} = 1$. A phase transition is only predicted for an angle of approximately $1/2\pi$, which is far too large for the perturbation calculation in θ to be valid.

So far we have obtained only the first term contributing to I in (35). The term involving B_s remains to be calculated to be sure it is not large enough to overcome the factor κ^{-2} multiplying it. Using a Maxwell equation it can be converted into an integral involving only the sample volume.

$$\int B_s^2\,dV=4\pi\int \mathbf{j}_s\mathbf{A}_s\,dV. \quad (39)$$

The vector potential A_s satisfies $\nabla\times\nabla\times A_s = 4\pi j_s$. j_s is periodic in x and y and can be written

$$\mathbf{j}_s=\sum_l \mathbf{j}_{sl}(z)\exp(il_xx+il_yy).$$

Then A_s is obtained in terms of the Green's function for ∇^2_z and $l = (l^2_x + l^2_y)^{1/2}$.

$$\mathbf{A}_s=\sum_l 2\pi\int_0^d d\zeta\exp(-l|z-\zeta|+il_xx+il_yy)\mathbf{j}_{sl}(\zeta)/l,$$
$$\int B_s^2\,dV=A_{xy}\sum_l 8\pi^2\int_0^d dz\int_0^d d\zeta\exp(-l|z-\zeta|)\mathbf{j}_{s(-l)}(z)\mathbf{j}_{sl}(\zeta)/l; \quad (40)$$

A_{xy} is the surface area of the sample, j_s is calculated using the Ginzburg-Landau equation and (16)

$$\mathbf{j}_s=-\mathrm{Re}[\Delta^*(i\nabla+2e\mathbf{A})\Delta]/8\pi e\lambda^2,$$
$$j_{sx}=-A^{1/2}S^2_0(z)\nabla_y|\Delta_0|^2/16\pi e\lambda^2,$$
$$j_{sy}=[4eB_c(z-z_0)S_0^2(z)|\Delta_0|^2+A^{1/2}S_0^2(z)\nabla_x|\Delta_0|^2]/16\pi e\lambda^2, \quad (41)$$
$$j_{sz}=-\int_0^z d\zeta[\nabla_xj_{sx}(\zeta)+\nabla_yj_{sy}(\zeta)].$$

There are additional contributions to j_{sy} of order $\theta^{1/2}$ which are not written down because they average to zero when integrated over z and therefore do not contribute to I to order $\theta^{1/2}$. The leading term of order θ^0 is the first contribution to j_{sy}. Expanding $\exp(-l|z-\zeta|)$ in a power series in l, which is of order $\theta^{1/2}$, we obtain

$$16e^2\lambda^4\int B_s^2\,dV=A_{xy}\sum_l(|\Delta_0|_l^2)^2\left\{\int_0^d dz\left[\int_0^z 4eB_c(\zeta-z_0)S_0^2(\zeta)\,d\zeta\right]^2+\frac{1}{2}\left(Al-\frac{\mu_0^2l_x^2}{l}\right)\frac{1}{(2eB_c)^{1/2}}\right\}. \quad (42)$$

The integral on the right side of (42) has been evaluated numerically[16] and equals $0.328 \int_0^d dz\, S_0^4(z)$. Transforming to our previous variables $l_{x''}$ and $l_{y''}$:

$$l=(2eB_e\theta)^{1/2}(A^{1/2}l_{x''}^2+A^{-1/2}l_{y''}^2)^{1/2},$$

we see that l is not rotationally invariant. Fortunately for thick films $(A^{1/2} - A^{-1/2})/(A^{1/2} + A^{-1/2}) = -0.26$ is a small parameter, so we expand 1 in powers of it, obtaining

$$8e^2\lambda^4\int\frac{B_s^2\,dV}{V_s}=\frac{1}{2}\frac{\langle|\Delta_0|^2\rangle^2}{d}\int_0^d dz\,S_0^4(z) \tag{43}$$

$$\times[(0.328+0.39\theta^{1/2})c_{00}-0.083\theta^{1/2}c_{22}\cos 2\eta-0.003\theta^{1/2}(c_{44}+c_{40})(\cos 4\eta+1)].$$

The corrections of order θ to (43) approximately equal the corresponding terms in (38). The terms of order $\theta^{1/2}$ are larger or equal to them when $\theta \lesssim 0.1$. The magnetic field corrections are thus important when $2\kappa^2 \approx 1$. However, for the experimental $\kappa \approx 3$ they are not important.

The experimental data[19] show sharp minima in $\partial R/\partial B_e$ near the angles $\theta = 0.032$, 0.072 and 0.15. Our result to first order in θ could at most only produce one structural phase transition, so if three of them exist the higher order terms in θ must be important and strongly modify our first order result indicating no such transition. However, carrying the calculation further should only be made together with accurate numerical calculations of the coefficients.

7. CONCLUSION

We have calculated some properties of a flat type II superconductor in a magnetic field inclined with respect to its surface. Particular success has been obtained for small angles of inclination, where static and dynamic properties have been calculated exactly and interpreted in terms of flux flow of vortices. The results for larger angles of inclination are not so decisive or complete. We found that abrupt structural phase transitions of the vortex lattice are possible in principle but did not obtain agreement with the experimental observations. Near the perpendicular orientation we only found limits on the rate of variation of the upper critical magnetic field with the angle of inclination.

This work was begun at the author's permanent location, the University of Southern California, with the support of the National Science Foundation, U.S.A. The author's work in Moscow is in connection with the exchange program between the National Academy of Sciences, U.S.A. and the Academy of Sciences, U.S.S.R.

[1] M. Tinkham, Phys. Rev. **129**, 2413 (1963).
[2] M. Tinkham, Phys. Letters **9**, 217 (1964).
[3] D. Saint-James, Phys. Letters **16**, 218 (1965).
[4] K. Yamafugi, E. Kusayanagi, and F. Erie, Phys. Letters **21**, 11 (1966).
[5] K. Maki, Journal of Low Temp. Phys. **3**, 545 (1970).
[6] K. Maki, Tokyo Summer Lectures on Physics of Quantum Fluids (Syokabo, Tokyo, 1971), page 117.
[7] P. G. de Gennes, Phys. Kondens. Materie **3**, 79 (1964) and K. Maki, Physics (New York) **1**, 21 (1964).
[8] V. L. Ginzburg and L. D. Landau, Zh. Eksp. Teor. Fiz. **20**, 1064 (1950).
[9] A. A. Abrikosov, Zh. Eksp. Teor. Fiz. **32**, 1442 (1957) [JETP **5**, 1174 (1957)].
[10] D. Saint-James and P. G. de Gennes, Phys. Letters **7**, 306 (1963).
[11] J. P. Burger, G. Deutscher, E. Guyon, and A. Martinet, Phys. Letters **16**, 220 (1965).
[12] I. O. Kulik, Zh. Eksp. Teor. Fiz. **55**, 889 (1968) [Sov. Phys.-JETP **28**, 461 (1969)].
[13] Y. Brunet, P. Monoceau, and G. Waysand, Phys. Rev. B **10**, 1927 (1974).
[14] D. R. Tilley and R. Ward, J. of Phys. C **3**, 2119 (1970).
[15] H. J. Fink, Phys. Rev. **177**, 732 (1969).
[16] R. S. Thompson, Phys. Rev. B **3**, 1617 (1971).
[17] R. S. Thompson, Phys. Rev. B **1**, 327 (1970).
[18] E. Guyon, F. Menier, and R. S. Thompson, Phys. Rev. **156**, 452 (1967).
[19] P. Monceau, D. Saint-James, and G. Waysand, Studies of Parallel and Tilted Vortices in Type II Superconductors by Microwave Absorption, preprint (Grenoble, 1974).