Spectrum of states in a resonance-particle system

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A resonance-particle interaction mechanism is indicated which, depending on the width of the initial resonance and the particle-mass ratio, can lead to the appearance of an anomalous spectrum, namely, a large number of resonance-particle bound states with a small binding energy. Exact formulas, which give the locations of the bound states in the limiting case when the spectrum becomes infinite with a condensation point near the mass threshold for the resonance and the particle, are presented; in the general case the position of the levels can be determined by solving an integral equation. A modification of the same mechanism is investigated for the cases of interaction between a resonance and a complex system and interaction between a weakly-bound two-particle state and a third particle or system. The results are discussed in applications to atomic-molecular and nuclear systems and to the isobar-nucleon system.

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1. We shall investigate here a system consisting of three particles with masses $m_1 = m_2 = \mu$ and $m_3 = m$, when there exists a resonance or a bound state in the subsystem m and μ, and establish the most general properties of the spectrum of the three-particle system. An important mechanism in the interaction between the resonance and the particle μ is the decay-product exchange mechanism depicted in Fig. 1. This exchange mechanism has been considered in a number of papers [1–5], and its most important property consists in the fact that the exchange interaction corresponding to it decreases with distance like $1/r$. From this quasi-Coulomb behavior follows [2–5] the possibility of the appearance of near-threshold three-particle resonances that decay mainly into a resonance and a particle, a fact which has been demonstrated in its general form in concrete computations for the isobar-nucleon system [3].

In the present paper we study the two-stage decay-product exchange mechanism (the diagram in Fig. 2). In the framework of the N/D method such a mechanism should be considered together with the exchange mechanism because in this method the dynamics is defined by the sum of the singularities of all the diagrams in the t- and u-channels and the singularities of the two-stage exchange diagram do not coincide with those of the single-stage exchange diagram (Fig. 1). And what is more, these singularities may lie arbitrarily close to the physical region, which explains their long-range nature. Physically, we can explain the two-stage exchange mechanism (we shall henceforth call it anomalous), if we can compute the potential corresponding to the diagram in Fig. 2. It turns out that the potential has the form $-C/r^2$, which, for a sufficiently large constant C, leads to the appearance of an infinite spectrum of levels with a point of accumulation near the threshold; we shall call such a spectrum an anomalous spectrum. Notice that in solving the Faddeev equation it is not necessary to use as the kernel of the equation both diagrams in Figs. 1 and 2, it being sufficient to consider only the exchange diagram; but for the investigation of the anomalous spectrum it is more convenient to take the once-iterated equation whose kernel is the diagram in Fig. 2. Notice that the anomalous spectrum can exist even if the interaction $-C/r^2$ occurs in the region $r > r_0$ [6] and other mechanisms corresponding to diagrams neglected by us here obtain in the region $r < r_0$. It is obvious from physical considerations that the particle-

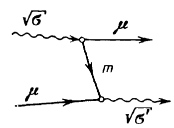

FIG. 1

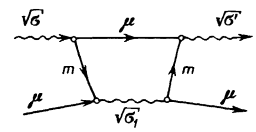

FIG. 2

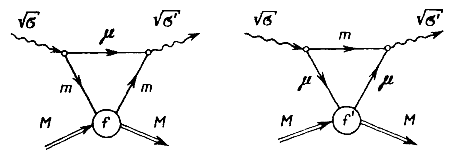

FIG. 3

resonance interaction cannot be infinitely long ranged, but must die off over distances $R_{int} \sim v\tau$, where v is the velocity of the exchangeable particle and $\tau = 1/\Gamma$ is the life-time of the resonance. Therefore, the anomalous interaction will die off over long distances $\sim R_{int}$, and, in this case, the point of accumulation in the three-particle spectrum vanishes, but there may remain a large number of levels, which is easy to estimate.

In an interaction of the resonance with a complex system of mass M there can arise a coherent enhancement of the effect (of the constant C) in comparison with the interaction with the single particle μ. The corresponding anomalous mechanism is depicted in Fig. 3. The necessary condition for such a coherent enhancement is $R_{int} \gtrsim R_S$, where R_S is the radius of the system M.

We also show that the anomalous spectrum in the three-particle system remains and corresponds to the so-called Efimov effect [7] when the resonance is replaced by a weakly-bound state. Furthermore, the effect of coherent interaction intensification is considered for weakly-bound states.

It is important to note that these mechanisms, i.e., the exchange and anomalous mechanisms, are distinguished by their long-range character: All the other interactions corresponding to more complex diagrams fall off much more rapidly with distance, and do not lead to any qualitative phenomena.

The method used in the present paper is strictly quantitative in nature, and is based on the use of the analyticity and unitarity properties of the particle-resonance amplitude, the unitarity relations being solved here in the framework of the generalized N/D method [2,5]. As the initial quantities, we use only the particle masses and the width and location of the two-particle resonance or the binding energy for the bound state of the two particles; the formulas obtained in the present paper allow us to find the locations of the levels of the entire three-particle system. The method and formulas are applicable to atomic-molecular and nuclear systems, as well as to elementary-particle systems.

In Sec. 2, where the resonance-particle amplitude is discussed, we give expressions for the discontinuities across the unitary and dynamic branch cuts. With the aid of the N/D method we write down an integral equation for the denominator D of the amplitude, whose zeros determine the locations of the three-particle levels. In Sec. 3 we give an expression for the anomalous discontinuity of the diagram in Fig. 2 which determines the anomalous interaction mechanism. In the limiting case when $R_{int} \to \infty$ the equation can be solved exactly, and an analytic expression is obtained for the spectrum. In Sec. 4 we investigate the real case of finite R_{int} and the anomalous interaction between the resonance and a complex system M (Fig. 3a). In Sec. 5 the interaction of a weakly-bound system with a third particle (μ) or a system (M) is considered. In Sec. 6 the obtained formulas are illustrated by applying them to atomic-molecular systems (the molecular ion H_2^-), nuclear systems (the nucleon-deuteron and ^{8}Be-α systems), and the isobar-nucleon system.

2. Let the two-particle resonance in the system of particles with masses m and μ have the mass $\sqrt{\sigma}$ and excitation energy ϵ:

$$\sqrt{\sigma}=m+\mu+\varepsilon. \tag{1}$$

The amplitude $C(\sigma, \sigma'; s, t)$ of the interaction of this resonance with a third particle μ is defined as a set of Feynman diagrams in which the initial and final masses of the resonance are assumed to have the same value (1), but with different imaginary corrections: $\sigma = \sigma_0 - i\delta$, $\sigma' = \sigma_0 + i\delta$ [4].

It is convenient to introduce the dimensionless energy variable

$$y=[s-(\sqrt{\sigma}+\mu)^2]/2\varepsilon(\sqrt{\sigma}+\mu) \tag{2}$$

and determine the partial wave $M_L(y)$ of the amplitude of the resonance-particle interaction, a wave which is equal (up to kinematic factors) to the partial wave given by the sum of the Feynman diagrams: the amplitude $C(\sigma, \sigma'; s, t)$. The point y = 0, as can be seen from (2), corresponds to the threshold energy in the resonance-plus-particle system.

The amplitude $M_L(y)$ as a function of y has branch cuts in the complex y plane: a unitary branch cut $[0, \infty)$, a right cut $[y_1, y_2]$ corresponding to the exchange diagram in Fig. 1, and left cuts $[y_3, y_4]$ arising from the diagram in Fig. 2 and, in principle, from other more complex diagrams (see Fig. 4). Then the general expression for the discontinuity in the amplitude $M_L(y)$ has the form

$$\frac{1}{2i}\Delta M_L(p)=\sqrt{y}\,M_L(y_+)M_L(y_-)\theta_1+\xi(y)\theta_2+\tilde{\xi}(y)\theta_3, \tag{3}$$

where

$$y_\pm=y\pm i\delta,\quad \theta_1=\theta(y),\quad \theta_2=\theta(y-y_1)\theta(y_2-y),$$
$$\theta_3=\theta(y-y_3)\theta(y_4-y).$$

Here $\xi(y)$ is the amplitude discontinuity across the dynamic cut arising from the exchange diagram. Its ex-

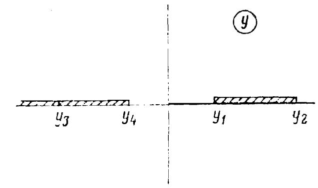

FIG. 4

plicit expression is given together with the values of y_1 and y_2 in [2,3].

We shall seek the solution to Eq. (3) with the aid of a modified N/D method that takes into account the fact that the branch cuts $[y_1, y_2]$ and the unitary cut $[0, \infty)$ are superimposed on each other (see [2,5]). Then, separating out the clearly threshold behavior of the amplitude:

$$M_L(y)=y^L N_L(y)/D_L(y), \tag{4}$$

we can explicitly write down the solution to Eq. (3) for the denominator of the amplitude in the form

$$D(y)=\psi(-\sqrt{y})\varphi_0(-\sqrt{y}). \tag{5}$$

Here the function $\varphi_0(x)$ is determined by the discontinuity ξ across the right dynamic branch cut, and is given by the expression

$$\varphi_0(x)=\exp\left\{\frac{x}{2\pi i}\int_{x_1}^{x_2}\frac{\ln g(x')\,dx'}{(x'-x)x'}\right\}, \tag{6}$$

where

$$x_i=\sqrt{y_i},\quad i=1,2,\quad g(x)=1-4x\xi(x^2),$$

and the function $\psi(x)$ satisfies the integral equation

$$\psi(x)=p_L(x)+\frac{2ix}{\pi}\int_{x_3}^{x_4}\frac{dx'\tilde{\xi}(x'^2)\eta(x')\psi(x')}{(x'+x)}\left(-\frac{x}{x'}\right)^L, \tag{7}$$

where

$$x_k=-i\sqrt{|y_k|},\quad k=3,4,\quad \eta(x)=\varphi_0(x)\varphi_0^{-1}(-x);$$

for L = 0

$$p_0(x)=1;$$

for L = 1

$$p_1(x)=1-\frac{x}{2\pi i}\int_{x_1}^{x_2}\frac{\ln g(x')}{x'^2}dx'.$$

Thus, the problem of the determination of the spectrum in a three-particle (resonance plus particle) system reduces to one of finding the zeros of the function $\psi(x)$ if the discontinuity $\tilde{\xi}(x^2)$ for the diagram (or set of diagrams) corresponding to the mechanism in question is given.

3. The diagram in Fig. 2 was investigated in the decay situation (i.e., for the case when $\sqrt{\sigma} > m + \mu$) in [4], where it was shown that in the t-channel the amplitude acquires an anomalous singularity that depends on the imaginary corrections to the resonance masses ($\sigma = \sigma_0 - i\delta$, $\sigma' = \sigma_0 + i\delta$):

$$t_{an}=4m^2\sin^2\delta. \tag{8}$$

The position of the singularity approaches the origin as $\delta \to 0$, and depends on the distribution of the resonance masses σ and σ' around the values $\sigma = \sigma' = \sigma_0$. This fact is most simply taken into account by shifting σ and σ' to the locations of the resonance poles:

$$\sigma=\sigma_0-i\sqrt{\sigma_0}\,\Gamma,\quad \sigma'=\sigma_0+i\sqrt{\sigma_0}\,\Gamma \tag{9}$$

(other methods of taking the spread in the resonance mass into account lead essentially to the same results). On choosing (9), we obtain in place of (8) the expression

$$t_{an} \approx 4m^2\nu, \quad \nu = \frac{\Gamma^2(m+\mu+\varepsilon)^2}{\varepsilon(\varepsilon+2m+2\mu)(\varepsilon+2m)(\varepsilon+2\mu)}, \tag{10}$$

where we have used the approximation $\nu \ll 1$.

Another singular point of the diagram is the normal threshold in the t-channel:

$$t_{norm} = 4m^2, \tag{11}$$

and the dispersion representation of the diagram in Fig. 2 in the t-channel includes an integral over t from t_{an} to t_{norm} (the anomalous part of the diagram) and from t_{norm} to $t = \infty$ (the normal part of the diagram). Below we shall be interested only in the anomalous part of the diagram, which part corresponds to the forces having the longest range and leads to the appearance of an anomalous spectrum in the system. Indeed, the interaction radius is connected with the threshold singularities in the t-channel by the usual relation:

$$R_{int}^2 t_s \sim 1, \tag{12}$$

where t_s is equal either to t_{an} or t_{norm}. Then for t_{an} in the approximation $\epsilon \ll m$, μ we immediately obtain the expression

$$R_{int} = (2\bar{m}\varepsilon)^{1/2}/m\Gamma = v\tau, \quad \bar{m} = m\mu/(m+\mu), \tag{13}$$

the physical meaning of which was discussed in the Introduction. Thus, the longest radius corresponding to the lowest threshold in the t-channel has a simple physical meaning. The normal part of the dispersion integral corresponds to an interaction radius $R_n \sim 1/2m$, which, for all interesting physical systems, is much less than R_{int} and turns out to be of the same order of magnitude as the interaction radius for the complex diagrams discarded by us. Therefore, we retain only the anomalous part of the diagram in Fig. 2. The discontinuity corresponding to it is equal to

$$\tilde{\xi}(y) = \frac{1}{2i}\Delta_{an}M_L(y) = -\frac{\pi\lambda\chi_L(y)}{2\sqrt{-y}}\theta(y-y_{norm})\theta(y_{an}-y), \tag{14}$$

where we have introduced the constant

$$\lambda = \frac{4\Gamma^2(m+\mu+\varepsilon)^6}{\varepsilon^2(2\mu+m+\varepsilon)^2(2\mu+2m+\varepsilon)(2m+\varepsilon)^2(2\mu+\varepsilon)} \tag{15}$$

and the function $\chi_L(y)$, which is equal, for L = 0, to:

$$\chi_0(y) = 1-(y_{an}/y)^{1/2} \tag{16}$$

and, for L = 1, to:

$$\chi_1(y) = {}^1/_3 - (y_{an}/y)^{1/2}(1-2y_{an}/3y). \tag{17}$$

The values of y_{an} and y_{norm}, which we should identify respectively with y_4 and y_3 in (7), are obtained from t_{an} and t_{norm} by means of the usual transition from the t-channel to the s-channel:

$$y_{sing} = \frac{[(\mu^2-t_{sing}/4)^{1/2}+(\sigma-t_{sing}/4)^{1/2}]^2-(\sqrt{\sigma}+\mu)^2}{2\varepsilon(\sqrt{\sigma}+\mu)} \tag{18}$$

Hence for y_{an} we obtain:

$$y_{an} \approx -\frac{m^2}{2\varepsilon\mu}\frac{(\sqrt{\sigma}+\mu)}{\sqrt{\sigma}}\nu, \tag{19}$$

and y_{norm} is obtained from (18) by setting $t_{sing} = 4m^2$.

Let us investigate the most general features of the anomalous mechanism, substituting (14) into (17) and limiting ourselves to the L = 0 case. Then from (7) we have the equation

$$\psi(x) = 1-\lambda x\int_{x_{norm}}^{x_{an}}\frac{\psi(x')\eta(x')}{x'(x'+x)}\left(1-\frac{x_{an}}{x'}\right)dx', \tag{20}$$

$$x_{an} = -i|\sqrt{y_{an}}|, \quad x_{norm} = -i|\sqrt{y_{norm}}|.$$

Let us consider the solution to Eq. (20) in the limiting situation when $R_{int} \to \infty$. On account of (13), to this situation corresponds the case when $t_{an} \to 0$ and $y_{an} \to 0$. For $y_{an} = 0$ the discontinuity (14) corresponds to the potential λ/r^2 when $r > R_n = 1/2m$, and therefore we can expect the appearance of an infinite and crowding spectrum for sufficiently large values of λ. There is not, however, complete equivalence with the potential problem in our approach, which is based on the solution of N/D equations. This is reflected in, for example, the fact that the diagrams that are more complex than the diagram in Fig. 2 do not possess singularities of the type $(-y)^{1/2}$, whereas the iterations of the Born diagram for the potential r^{-2} possess such a singularity.

Equation (20) can be solved exactly in the limit when $y_{an} \to 0$. For this purpose, let us set $\eta(x) = 1$ in it (this is valid for $|x| \ll x_1, x_2$, where the phenomenon of anomalous spectrum will run high), and let us replace the equation in the form (20) by an equivalent homogeneous equation with a normalization condition of not $\psi(0) = 1$, which is explicitly allowed for in writing down (20), but imposed at another point, e.g., $\psi(x_{norm}) = 1$. Let us introduce the variable $u = x_3/x$. Then we obtain for the function $\bar{\psi}(u) = \psi(x)$ the equation

$$\bar{\psi}(u) = \lambda\int_1^\infty\frac{du'\,\bar{\psi}(u')}{u+u'}, \quad \bar{\psi}(1) = 1. \tag{21}$$

Equation (21) can be solved at once with the aid of the Fock-Mehler transformation:

$$\psi(x) = P_{-1/2+i\alpha}(x_{norm}/x), \tag{22}$$

where α is the root of the equation

$$1 = \lambda\pi/\operatorname{ch}\pi\alpha, \quad -1/2 < \operatorname{Im}\alpha < 1/2. \tag{23}$$

Let us investigate the properties of the obtained solution. For $\lambda < 0$ (repulsion) Eq. (23) has no roots. For $0 < \lambda < 1/\pi$ there is a root with $\operatorname{Re}\alpha = 0$ and $|\operatorname{Im}\alpha| < 1/2$, which can, in principle, lead to a finite number of zeros for the solution (22). The most interesting situation arises when $\lambda > \lambda_{cr} = 1/\pi$. Then Eq. (23) has a root with $\operatorname{Im}\alpha = 0$. and the solution corresponding to this root possesses in the interval $(x_{norm}, 0)$ an infinite number of zeros with an accumulation point at $x = 0$. These zeros correspond to $\operatorname{Im}\sqrt{y} > 0$ (let us recall that $D(y) \sim \psi(-\sqrt{y})$), i.e., there arises an infinite number of bound states when $\lambda > \lambda_{cr}$. Notice that the condition $\lambda > 1/\pi$ for the appearance of the anomalous spectrum in the system is very close to the analogous condition for the potential $\bar{\lambda}/r^2$: to wit, $\bar{\lambda} > 1/4$[6].

4. Let us now consider the real situation with a finite interaction radius R_{int} (i.e., with a finite y_{an}). Then the upper limit in Eq. (21) should be replaced by $u_\infty = (y_{norm}/y_{an})^{1/2}$. The resulting equation turns out to be a Fredholm equation, and its solutions vanish an infinite number of times. We can estimate the number of levels in the system if we compute the number of zeros in the interval $[x_{norm}, x_{an}]$ with the aid of the asymptotic formula for the solution (22). It turns out to be equal to

$$n \approx \frac{\alpha}{2\pi}\ln\frac{4}{\nu}. \tag{24}$$

This estimate is correct only in order of magnitude, since close to the upper limit the asymptotic solution (22) differs significantly from the exact solution to Eq. (20).

We can derive a sufficient condition for the appearance of a bound state at finite R_{int}. For this purpose, let us rewrite (20) in terms of the real variables $z = ix$ (and, for the sake of simplicity, set $\eta = 1$). We have

$$\psi(z)=1+\lambda z\int_{z_{an}}^{z_{norm}}\frac{\psi(z')\,dz'}{z'(z'+z)}\left(1-\frac{z_{an}}{z'}\right), \tag{25}$$

where $z_{norm} = |y_n|$ and $z_{an} = |y_{an}|$. From the positiveness of the kernel and of its derivative with respect to z, we obtain the condition for the function $\psi(z)$ to have a zero:

$$\lambda\geqslant\bar\lambda_{cr}=\left[2\ln\frac{2z_n}{z_n+z_{an}}+\frac{z_{an}}{z_n}-1\right]^{-1} \tag{26}$$

($z_n \equiv z_{norm}$). For $z_n \gg z_{an}$ the value $\bar\lambda_{cr} \approx 2.5$, which is much larger than the critical value $\lambda_{cr} = 1/\pi$, which obtains in the limiting situation when $z_{an} \to 0$. It can be seen from this that the requirement that $\lambda \geq \bar\lambda_{cr}$ may be too strong and, in any case, it is not necessary.

Let us now assume that λ is less than the critical value and that there is no level in the system of particles $(m\mu\mu)$. The question then arises whether there can, nevertheless, appear a bound state of the same resonance $(m\mu)$ with some other system of particles with a combined mass equal to M as a result of the anomalous mechanism depicted in Fig. 3a. We shall assume here that the particle m interacts with the system M with an amplitude of $f(\theta)$ that depends on the energy E_{cm} in the center-of-mass system and that is normalized by the condition

$$d\sigma_{cm}/d\Omega=|f(\theta)|^2.$$

It is easy to calculate the discontinuity of the Feynman diagram in Fig. 3a across the anomalous branch cut and determine from it the constant λ_{tr}, which should be substituted above in all the formulas for λ. It is equal to

$$\lambda_{tr}=\frac{2\Gamma E_{cm}f\sigma}{\sqrt{\varepsilon}(\sqrt{\sigma}+M)\left[(2\mu+2m+\varepsilon)(2m+\varepsilon)(2\mu+\varepsilon)\right]^{1/2}}. \tag{27}$$

Let us emphasize that the amplitude f in (27) is evaluated on the energy surface with the following value of the invariant square of the energy (when $\epsilon \ll m, \mu$):

$$P^2=(p_m+p_M)^2=(M+m+\varepsilon)^2+\frac{2m(M+m+\mu)}{m+\mu}(y-1)\varepsilon \tag{28}$$

and in the case when the momentum transfer $t \geq t_{an}$. Since $P^2 < (M + m + \epsilon)^2$, into (27) effectively enters the amplitude f of the forward scattering at energies lower than the "resonance" value $\sigma_r = (M + m + \epsilon)^2$. If for f in (27) we take the Breit-Wigner value, set $M = \mu$, and take the energy E_{cm} from (28), then we again arrive at the quantity (15) since under the above-cited conditions the triangular diagram in Fig. 3a reduces to the square diagram in Fig. 2. Notice also that another anomalous mechanism that corresponds to the diagram in Fig. 3b and that may turn out to be more important when $\mu < m$ and (or) $|f'| > f$ is, in principle, possible. The expression for the constant $\bar\lambda_{tr}$ corresponding to this diagram is obtained from (27) by making the substitution $\mu \to m$ and $f \to f'$.

The magnitude and sign of λ_{tr} depend on the magnitude and sign of f: Thus, if the particles m and M attract (repel) each other, then f is positive (negative), and the anomalous spectrum can (cannot) arise. If, on the other hand, for the energy (28) the particle m undergoes strong absorption in the system M, then according to the optical theorem $\mathrm{Im}\, f(0) = k\sigma_{tot}/4\pi$ and can be large because of the large value of $\sigma_{tot} \approx 2\pi R_S^2$. Then in the case when the real part Re f is small this leads to an anomalous spectrum that is shifted in the complex plane because, for a purely imaginary λ, Eq. (23) yields roots α with $|\mathrm{Im}\,\alpha| \sim 1/2$ and larger $\mathrm{Re}\,\alpha$. In this case the estimate (24) for the number of levels becomes inapplicable.

The question of the effective magnitude of λ_{tr}, or of the coherent enhancement of λ_{tr} in comparison with λ, (15), depends primarily on the admissibility of the choice of f in (27) in the form of a constant that coincides with the value of f at the end of the anomalous cut. For large dimensions of the system M, i.e., for $R_S \gg R_{int}$, the function f as a function of t decreases in the interval $\Delta t \sim 1/R_S^2 \ll t_{an}$, so that effectively the anomalous cut shrinks drastically, and the anomalous spectrum may disappear. In the opposite limiting case, when $R_S \ll R_{int}$, the amplitude f can be considered to be a constant quantity in the interval $[y_{an}, (R_S/R_{int})^2 y_{an}]$, which implies a coherent enhancement of the anomalous mechanism, since the amplitude f of the zero-angle scattering by the entire system M can significantly exceed the scattering amplitude for each of the components of the system M.

5. Now instead of a resonance let there exist in the system of particles m and μ a bound state with a small binding energy $\epsilon_b \ll m, \mu$; $\epsilon_b r_0^2 \ll 1$, where r_0 is the range of the forces in the system. Then the mass of the bound state $\sqrt{\sigma} = m + \mu - \epsilon_b$, while the dimensionless variable y is determined by the formula (2), in which it is only necessary to make the substitution $\epsilon \to \epsilon_b > 0$. Let us now consider the amplitude $M_L(y)$ for scattering of the bound state by a particle, and choose as the interaction mechanisms the same exchange and anomalous mechanisms (see Figs. 1 and 2). The equation (3) for the discontinuities will preserve its previous form. Also retained is the location of the unitary branch cut, but the cut arising from the exchange diagram will move from the right half of the complex y plane to the left half, retaining its logarithmic character. Let us recall that the exchange mechanism in this case can produce only a finite number of levels (one in the N/D method), and therefore for the phenomenon of interest to us here we may not consider it. The equation for the denominator of the amplitude then assumes, after (4) and (5) have been taken into account, exactly the form of (20) for L = 0, where it is necessary to set $\eta(x) = 1$ and redefine the quantities λ, x_{an}, and x_{norm}.

Let us take into account the fact that the coupling constants at the vertices of the diagram in Fig. 2 are now expressible in terms of the two-particle binding energy:

$$g^2=\frac{16\pi\sigma}{\bar m}(2\bar m\varepsilon_b)^{1/2}. \tag{29}$$

Then the value of the coupling constant λ_b is obtained from (15) by making the substitution $\Gamma/\epsilon \to 4$, $\epsilon \to -\epsilon_b$:

$$\lambda_b=\frac{64(m+\mu-\varepsilon_b)^6}{(2\mu+m-\varepsilon_b)^2(2\mu+2m-\varepsilon_b)(2m-\varepsilon_b)^2(2\mu-\varepsilon_b)}. \tag{30}$$

The values of y_{an} and y_{norm} in this case are equal (we give their values for the diagram in Fig. 3a; for the diagram in Fig. 2 it is necessary to make the substitution $M \to \mu$) to:

$$y_{an}=-\frac{m}{\mu}\frac{(M+\sqrt{\sigma})}{M},\qquad y_{norm}=-\frac{m^2(M+\sqrt{\sigma})}{2\varepsilon_b(m+\mu)M}. \tag{31}$$

Let us redefine the variable $u = x_n/x$; then for the denominator of the amplitude for scattering of the bound system by a particle we obtain the equation

$$\tilde{\psi}(u)=1+\lambda_b\int_1^{u_b}\frac{du'\tilde{\psi}(u')}{u+u'}\left(1-\frac{u'}{u_b}\right), \quad (32)$$

where

$$u_b=[m\mu/2\varepsilon_b(m+\mu)]^{1/2}. \quad (33)$$

As $\epsilon_b \to 0$, the limit $u_b \to \infty$, and we again have the solution of the homogeneous equation in the form (22) with a condition for the roots of the form (23). Consequently, there also arises in the case of a weakly-bound state with $\epsilon_b \to 0$ an infinite spectrum of levels with a point of accumulation at the threshold—this corresponds to the so-called Efimov effect. We can also find the exact limiting position of the levels, since the roots of the conical function $P_{-1/2+i\alpha}(u_p)$ lie asymptotically ($u_p \gg 1$) at the points

$$u_p \approx \frac{1}{2}\exp\left\{\frac{1}{\alpha}\left[\frac{\pi}{2}(2p+1)-\arg\frac{\Gamma(i\alpha)}{\Gamma\left(\frac{1}{2}+i\alpha\right)}\right]\right\}, \quad (34)$$

and then the zeros of the denominator are given by

$$\sqrt{y_p}=i|y_{norm}{}^{1/2}|/u_p.$$

Notice that the condition for the appearance of the anomalous spectrum, $\lambda > \lambda_{cr} = 1/\pi$, is clearly satisfied for the value (30), since $\lambda_b \gtrsim 14$ for any m and μ. The sufficient condition for the appearance of a level here is also clearly fulfilled. This means that the forces binding the pair of particles are also always capable of binding a system of three particles among whom are two such pairs (we assume that the μ-μ particle interaction has been turned off). As in Sec. 3, the order of magnitude of the number of levels in the system can be estimated:

$$n_b \approx \frac{\alpha}{\pi}\ln 2u_b. \quad (35)$$

The condition (35) is similar in form to Eq. (17) in Efimov's paper[7]. The difference consists in the fact that instead of an effective radius r_0 in our expression figures the quantity $r_m = (m+\mu)/m\mu = 1/\overline{m}$, which is a consequence of the allowance for the contribution of the anomalous branch cut right up to the normal threshold t_{norm}. If, however, $r_0 \ll r_m$, then there appear at the vertices of the diagram in Fig. 2 form factors that can effectively cut off the integral (34) at a value u corresponding to r_0.

When the bound state interacts with a complex system of mass M, the anomalous interaction mechanism corresponding to the diagram in Fig. 3a gives the value

$$\bar{\lambda}_b=\frac{2(m+\mu)E_{cm}f}{\overline{m}(\sqrt{\sigma}+M)}(2\overline{m}\varepsilon_b)^{1/2},$$

where f enters at some energy value below the threshold. The effect of coherent enhancement of $\bar{\lambda}_b$ is not so important here, since λ_b, (30), is always so much larger than the critical value. However, in order that the system M might manifest itself as a whole and, consequently, in order that the amplitude f might not significantly cut down the integration domain in the integral (34), it is necessary, as in the resonance case, to impose on the interaction radius the condition:

$$R'_{int}=[\mu/2(m+\mu)m\varepsilon_b]^{1/2}\gg R_s,$$

where R_s is the radius of the system M. Notice that R'_{int} is of the order of the radius of the bound state of the particles μ and m (when $\mu \ll$ m, the diagram in Fig. 3b gives an interaction mechanism with radius

$$R''_{int}=[m/2(m+\mu)\mu\varepsilon_b]^{1/2}\gg R'_{int}).$$

6. The assumptions under which the results of the present paper were derived amount basically to the following: 1) we allow for interaction in only two of the three particle pairs and neglect the interaction between the particles μ and μ (μ and M); 2) we can neglect the nonresonance background in the two-particle system in comparison with the contribution of the resonance (correspondingly of the pole responsible for the bound state); 3) we can neglect the off-mass-shell corrections to the interaction between the particles m and μ; 4) the contribution of the neglected (by us here) diagrams corresponding to interactions of significantly shorter range than R_{int} is small compared to the considered contribution.

Physically, it is quite clear that the principal qualitative effect discovered in the present paper—the appearance of an anomalous spectrum with an accumulation point at the limit $y_{an} \to 0$ ($R_{int} \to \infty$) when $\lambda > \lambda_{cr}$—will not change if we take the contributions 2)–4) into account. The first limitation can be removed, and was used here only for the sake of simplicity. The quantitative conclusions, on the other hand, can, of course, change when proper allowance is made for the points 1)–4), and they strongly depend on the nature of the system in question.

Let us consider a few examples.

a) The resonances in the molecule H_2^-: Several groups of peaks are known in this system, in particular, near the energies ~11 and ~14 eV above the ground state[8]. At the same time, the ion H^- has a group of resonances between 9.558 and 12.35 eV above the energy of the ground state of the hydrogen atom[8]. If we assume (which is quite plausible) that the resonances in H^- are produced near the threshold for decay into the hydrogen atom with n = 2 (the principal quantum number and E(n = 2) = 10.204 eV), then the anomalous mechanism corresponding to the diagram in Fig. 2 with $m = m_e$ and $\mu = M_a$ can have a very large radius R_{int} and an "interaction constant"

$$\lambda=\frac{\Gamma^2}{16\varepsilon^2}\left(\frac{M_a}{m_e}\right)^2.$$

Even for the unfavorable case with Γ = 0.01 eV and $\epsilon \sim$ 0.5 eV (the experimental values for the widths are smaller than the instrumental widths and are not exactly known), $\lambda \approx 25 > \lambda_{cr}$. Therefore, one or several resonances of the molecule H_2^- should exist below the position of each resonance of the ion H^-. Allowance for the H-H interaction at such distances should shift these resonances downwards and, moreover, may lead to the appearance of rotational poles at each of the levels.

Notice that acting simultaneously is an exchange mechanism (Fig. 1) that, as shown in [2,3], can give peaks in the H_2^- spectrum precisely at the threshold (E = E(n = 2), E(n = 3), etc.) in the state with L = 0 and resonances above the threshold with L > 0. Thus, the exchange mechanism gives (in the terminology of [8,9]) resonances of the second kind above the position of the parent resonances in H^-, while the anomalous mechanism gives resonances of the first kind (Feshbach resonances) that, relative to the position of the resonances in H^-, are shifted downwards. The experimental picture is at present not clear enough for us to be able to uniquely identify the type of resonances observed near the 11-eV level in the H_2^- molecule, let alone establish their number.

b) In nuclear, as in atomic, systems, the most favorable case for the appearance of the anomalous mechanism is the case when $\mu \gg m$. There are many examples of levels located near the threshold for breakup into a resonance plus a cluster, but they lie in a region with a high excitation energy ($\gtrsim 8$ MeV), where the levels are many and the computation of the position of a level still does not in itself constitute proof of the mechanism producing it.

For $\epsilon < 0$ (a bound state + a system) there are several examples where the anomalous mechanism is exceptionally important. For example, for tritium composed of deuteron and a nucleon and the nucleus C^{12} composed of ^{8}Be (which is stable when the Coulomb forces are turned off) and an α-particle, the quantity $\lambda_b = 128/9 \gg \lambda_{cr}$. Then the binding energy of the whole system lies close to $4\epsilon_b$, which agrees quite well with the experimental value for tritium and the value obtained for ^{12}C(E = 12.9 MeV, $\epsilon_b \approx 3$ MeV) in exact computations with a given α-α interaction potential.

c) In the isobar Δ(1236)-nucleon system the quantity $\lambda = 1.84$, which is larger than $\lambda_{cr} = 1/\pi$, but smaller than the value 2.5 obtainable from the sufficient condition (26). Therefore, for an answer we must carry out a numerical solution of Eq. (25).

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