The theory of resonance interaction of wave packets in nonlinear media

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We use the inverse scattering method to consider the (decay and explosive) resonance interaction of three one-dimensional wave packets in a nondissipative medium. We indicate an algorithm for constructing exact solutions and study in detail the scattering of wave packets due to their decay resonance interaction.

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INTRODUCTION

In various physical situations the problem arises of the interaction, in a nonlinear medium, of three wave packets with characteristic frequencies ω_1, ω_2, ω_3 and wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, satisfying the resonance conditions

$$\omega_1=\omega_2+\omega_3, \quad \mathbf{k}_1=\mathbf{k}_2+\mathbf{k}_3. \tag{1}$$

The physical nature of the packets may be completely different. The interaction characterized by the conditions (1) is the coherent decay of the quanta of wave 1 (we shall call it the pumping wave) into the quanta of waves 2 and 3 (which we shall call the secondary waves) and the reverse process of the fusion of quanta 2 and 3 into a quantum 1; it plays a basic role in induced Raman and Mandel'shtam-Brillouin scattering processes, in various processes of energy exchange between waves in a plasma, and so on (see [1-3]).

At the present time the theory of type (1) processes (which we shall call decay processes) is developed mainly in the approximation which is linear in the amplitudes of the secondary waves [1,3] and it describes the initial stage of the process: the transfer of energy from the pumping to the secondary waves. However, in many experimental conditions the linear approximation is not satisfied and the reaction of the secondary on the pumping wave plays an important role. To take this effect into account one must construct a consistent nonlinear theory. Such a theory was developed by Armstrong et al.[4] for the strongly idealized case of the interaction of three monochromatic waves. They showed in[4] that when there is no dissipation the development of the decay process (1) leads to a periodic exchange of energy between the pumping and the secondary waves; it is clear from the results of this paper that the problem of the finite intensities and the spectral characteristics of bounded wave packets which undergo a strong decay interaction is far from trivial.

In stable media which allow only waves with positive energy to propagate the type (1) decay processes are the only three-wave resonance processes. In unstable media in which waves with a negative energy are allowed to propagate (e.g., a plasma with a current) a three-wave interaction is possible which satisfies the resonance conditions:

$$\omega_1+\omega_2+\omega_3=0, \quad \mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0. \tag{2}$$

Conditions (2) can be satisfied only if at least one of the frequencies ω_i is negative and they describe the process of the "creation of three quanta from the vacuum"—an explosive instability of the medium. The explosive instability is a process which is by its very nature nonlinear. As in the decay situation one can easily construct a theory for it for the case of strictly monochromatic waves.[5] In this case the result of the development of the explosive instability is that after a finite time all three wave amplitudes become infinite. It is clear that when finite wave packets interact one must expect the appearance of singularities of their amplitudes which are localized in space.

We consider in the present paper (both decay and explosive) resonance interactions of three one-dimensional wave packets in a nondissipative medium. We apply to this problem a relatively new mathematical technique: the inverse scattering method. This method was first used in 1967 by Gardner, Greene, Kruskal, and Miura[6] and subsequently improved in[7-9]; it enables one to study efficiently several nonlinear equations without any assumptions whatever about the nonlinearity level of the form of the initial conditions. The inverse scattering method has been successfully applied to the Korteweg—de Vries equation,[6-9] to the nonlinear "parabolic" equation of stationary self-focusing and defocusing,[10-13] to the problem of the propagation of a pulse through a two-level medium,[14-16] to several systems with discrete degrees of freedom,[17,18] and so on.

When we use the inverse scattering method the solution of the initial problem reduces for the equation studied to the solution of the direct and the inverse spectral problems for some linear operator. In principle the application of this method must enable us to find the solution for any initial conditions and at all times. In reality, however, one has to be satisfied with less: a study of the asymptotic solutions as $t \to \pm\infty$ and also finding a set of separate exact solutions which enable us, however, to formulate rather complete ideas about the general properties of the solutions. In the present paper the use of the inverse scattering method enables us to study efficiently the scattering of wave packets due to their decay resonance interaction and in the case of "long" (i.e., strongly nonlinear) packets to find explicit expressions for the finite wave amplitudes. It then turns out that the physical picture of the interaction depends in a fundamental way on the ratio of the velocities of the pumping and the secondary waves: if the group velocity of the pumping wave lies between the velocities of the secondary waves a long pumping wave packet is unstable and decays practically completely when colliding with arbitrarily small secondary wave packets. The exact solutions which we have found refer to this situation. If, however, the velocity of the pumping packet is the largest or the smallest, the disintegration of the pumping wave is possible only when it collides with sufficiently strong secondary wave packets.

We also find for the case of an explosive instability a set of exact solutions which describe the occurrence of localized singularities of the wave amplitudes for bounded wave packets.

We must note that the mathematical facts used in the present paper are given without proper argumentation, which can be found in[19, 20]. We shall also omit many derivations which have by now become standard in the given framework. We proved in[21] that the inverse scattering method was in principle applicable to the problem considered. The results of the present paper were published earlier as a preprint.[19]

1. STATEMENT OF THE PROBLEM AND NECESSARY RESULTS FROM SCATTERING THEORY

The equations which describe the processes of interest to us are well known (see, e.g.,[1, 2]). We give them in an explicitly Hamiltonian form.[22] Let the $u_i(x, t)$ be the complex amplitudes of packets which have the appropriate dispersion laws, and v_i their group velocities. Then

$$\frac{\partial u_1}{\partial t}+v_1\frac{\partial u_1}{\partial x}=iqu_2u_3,\quad \frac{\partial u_2}{\partial t}+v_2\frac{\partial u_2}{\partial x}=iq^*u_1u_3^*,$$
$$\frac{\partial u_3}{\partial t}+v_3\frac{\partial u_3}{\partial x}=iq^*u_1u_2^* \tag{3}$$

in the case of the decay process (1) and

$$\frac{\partial u_1}{\partial t}+v_1\frac{\partial u_1}{\partial x}=iqu_2^*u_3^*,\quad \frac{\partial u_2}{\partial t}+v_2\frac{\partial u_2}{\partial x}=iqu_1^*u_3^*,$$
$$\frac{\partial u_3}{\partial t}+v_3\frac{\partial u_3}{\partial x}=iqu_1^*u_2^* \tag{4}$$

for the explosive instability. Here q is the interaction constant which can, without loss of generality, be assumed to be real.

We consider a pair of linear differential operators L and M:

$$L=iA\frac{d}{dx}+[A,Q], \tag{5}$$
$$M=iB\frac{d}{dx}+[B,Q], \tag{6}$$

where A and B are constant $N\times N$ diagonal matrices, $A_{ik}=a_i\delta_{ik}$, $a_1>a_2>\ldots>a_N$, $a_i\neq 0$, $B_{ik}=b_i\delta_{ik}$, and Q is also a square matrix with elements which are functions of x and, moreover, depend on the time t as a parameter. By a direct calculation one verifies easily that if we choose Q in the form

$$Q=\begin{pmatrix} 0 & p(a_1-a_2)^{-1/2}u_2, & p(a_1-a_3)^{-1/2}u_1 \\ -p(a_1-a_2)^{-1/2}u_2^*, & 0, & p(a_2-a_3)^{-1/2}u_3 \\ -p(a_1-a_3)^{-1/2}u_1^*, & -p(a_2-a_3)^{-1/2}u_3^*, & 0 \end{pmatrix} \tag{7}$$

or

$$Q=\begin{pmatrix} 0 & p(a_1-a_2)^{-1/2}u_2, & p(a_1-a_3)^{-1/2}u_1^* \\ p(a_1-a_2)^{-1/2}u_2^*, & 0, & p(a_2-a_3)^{-1/2}u_3 \\ -p(a_1-a_3)^{-1/2}u_1, & p(a_2-a_3)^{-1/2}u_3^*, & 0 \end{pmatrix}, \tag{8}$$

we can write the set (3) (or (4), respectively) in the form

$$\partial L/\partial t=i[L, M],$$

which enables us to apply to the sets considered the inverse scattering method. The velocities v_i are then expressed in terms of the coefficients of the operators L and M as follows:

$$v_1=\frac{a_1b_3-a_3b_1}{a_1-a_3},\quad v_2=\frac{a_1b_2-a_2b_1}{a_1-a_2},\quad v_3=\frac{a_2b_3-a_3b_2}{a_2-a_3}, \tag{9}$$

while the quantity p occurring in Eqs. (7) and (8) is

$$p=q(a_1-a_2)^{1/2}(a_1-a_3)^{1/2}(a_2-a_3)^{1/2}[a_1b_2-a_2b_1+a_2b_3-a_3b_2+a_3b_1-a_1b_3]^{-1}.$$

Eliminating the quantities b_i from (9) we get

$$v_1\left(\frac{1}{a_3}-\frac{1}{a_1}\right)=v_2\left(\frac{1}{a_2}-\frac{1}{a_1}\right)+v_3\left(\frac{1}{a_3}-\frac{1}{a_2}\right); \tag{10}$$

this means that if $a_3>0$, the velocity of the wave 1 (i.e., the pumping wave in the case of the reduction (7)) lies between the velocities of the waves 2 and 3, i.e., either $v_2>v_1>v_3$, or $v_3>v_1>v_2$. If, however, $a_3<0$, $a_2>0$, the wave 3 has the intermediate velocity, while the wave 1 has the largest or the smallest: $v_1>v_3>v_2$ or $v_2>v_3>v_1$. The properties of the operator L, and hence the behavior of the solutions of the set (3), depend strongly on the sign of the quantity a_3.

We now give the necessary facts relating to the scattering problem for the operator L of (5). We consider the matrix equation

$$L\psi=\lambda\psi \tag{11}$$

with a real spectral parameter λ. Assuming that the functions $u_i(x)$ decrease sufficiently fast as $|x|\to\infty$ we determine two partial solutions of (11), $\psi^+(x, \lambda)$ and $\psi^-(x, \lambda)$, by their asymptotic behavior as $x\to\pm\infty$:

$$\psi^\pm(x,\lambda)\to e^{-i\lambda A^{-1}x},\quad x\to\pm\infty.$$

The scattering matrix of the operator L is defined as the ratio of these solutions:

$$\psi^+(x,\lambda)=\psi^-(x,\lambda)S(\lambda),$$

$S(\lambda)$ is unimodular:

$$\det S(\lambda)=1. \tag{12}$$

For the reduction (7) the scattering matrix is A-unitary:

$$S^{-1}(\lambda)=A^{-1}S^+(\lambda)A, \tag{13}$$

and for the reduction (8) it is GA-unitary:

$$S^{-1}(\lambda)=G^{-1}A^{-1}S^+(\lambda)AG,$$

where G is a diagonal matrix, $G_{11}=G_{33}=-G_{22}=1$. If $a_3>0$, the elements $S_{11}(\lambda)$ and $S_{33}(\lambda)$ of the scattering matrix allow an analytical continuation into the upper and lower half-planes of the spectral parameter λ, respectively. As $|\lambda|\to\infty$, S_{11}, $S_{33}\to 1$. If $a_3<0$, $a_2>0$, $S_{33}(\lambda)$ is analytical in the upper half-plane and $S_{22}(\lambda)$ in the lower one while the boundary values of these functions are also equal to unity.

If the functions $u_i(x)$ depend on the time, the scattering matrix also becomes a function of time. In that case, if the $u_i(x, t)$ change with time according to (3) or (4) one can easily find the function S(t) explicitly:

$$S_{ik}(\lambda,t)=S_{ik}(\lambda,0)\exp\left[i\left(\frac{b_k}{a_k}-\frac{b_i}{a_i}\right)\lambda t\right]. \tag{14}$$

Equation (14) solves, in fact, the problem of integrating the set (3) or (4); the Cauchy problem for either of these sets can thus be solved by the standard scheme of the inverse scattering method:

$$u_i(x,0)\to S_{ik}(\lambda,0)\to S_{ik}(\lambda,t)\to u_i(x,t).$$

Because of (14) the nontrivial stages in that scheme are the first and the last. One must in the first stage solve the direct scattering problem for the operator L and in the last one the inverse problem. Both these and the other problems reduce to considering linear equations, i.e., the set (11) or the set of equations of the inverse problem (see [20, 19]).

The inverse scattering method can, however, also be used for another problem: the construction of some exact solutions of the sets (3) or (4) using the given scattering matrix. The analysis of such solutions aids to formulate ideas about the solutions of the systems considered. An algorithm for constructing such solutions and the simplest of them are given in the next section.

The general scheme for applying the inverse scattering method is appreciably simplified if we are interested only in the connection between the asymptotic solutions of the set (3) as $t \to \pm\infty$, i.e., in the problem of the scattering of wave packets by one another. If we state it in that way, at $t \to -\infty$ the wave packets are spatially separated and, hence, do not interact. It is clear that the order in which these packets are positioned is determined by the ratio of their velocities v_i. To fix the ideas, let $v_1 > v_3 > v_2$. At $t \to -\infty$ the packet u_1 is at the left and then u_3 and u_2. The scattering matrix can then be written in the form $S = S_1^{(-)}S_3^{(-)}S_2^{(-)}$, where the $S_i^{(-)}$ are the "partial" scattering matrices, i.e., the scattering matrices of the operators $\hat{L}_i$ obtained from the operator L of (5) if we put in it all u bar u_i equal to zero. Each such scattering matrix is appreciably simpler than the complete one as it clearly contains only four nontrivial elements.

We can, further, assume that as $t \to +\infty$ the wave packets, having undergone a decay interaction, also turn out to be spatially separated. Their positions will then be the inverted one, and the scattering matrix can be written in the form $S = S_2^{(+)}S_3^{(+)}S_1^{(+)}$, where the $S_i^{(+)}$ are again partial scattering matrices. It follows from Eq. (14) that the two preceding relations, obtained in the limiting cases as $t \to \pm\infty$ are identically valid at all times. Hence follows a simple way to solve the problem of the connection of the asymptotic states of the wave packets as $t \to \pm\infty$. This problem can be solved using the scheme

$$u_i^{(-)}(x,t) \to S_i^{(-)}(\lambda,t) \to S(\lambda,0) \to S_i^{(+)}(\lambda,t) \to u_i^{(+)}(x,t).$$

A consistent application of this scheme enables us to obtain directly in terms of the initial partial scattering matrices $S_i^{(-)}(\lambda)$ explicit expressions for the integral intensities, produced as a result of the interaction of the packets $u_i^{(\pm)}(x, t)$. This is done in Sec. 3.

2. EXACT SOLUTIONS

In principle the inverse scattering method enables us to find an extremely rich set of exact solutions of the sets (3) and (4) which is everywhere dense. Such solutions are connected with scattering matrices with elements which, apart from an exponential factor, are rational functions of the spectral parameter λ. However, in the present paper we restrict ourselves to constructing solutions which are the analogues of the well-known N-soliton solutions of the KdV equation which correspond to "non-reflecting" scattering matrices.

We shall assume that $a_3 > 0$, i.e., when applying it to the set (3), we shall consider the case where the velocity of the pumping wave is the intermediate one. If the off-diagonal elements of the scattering matrix are identically equal to zero for real values of λ, the above described properties of the elements of the scattering matrix $S_{11}(\lambda)$ and $S_{33}(\lambda)$ enable us to determine these elements unambiguously through their zeroes:

$$S_{11}(\lambda)=\prod_{n=1}^{N_1}\frac{\lambda-\lambda_n}{\lambda-\lambda_n^*}\qquad S_{33}(\lambda)=\prod_{n=1}^{N_3}\frac{\lambda-\mu_n}{\lambda-\mu_n^*},$$

where $\operatorname{Im}\lambda_n > 0$ and $\operatorname{Im}\mu_n < 0$. For the sake of simplicity we shall assume that all zeroes are simple ones and we shall, moreover, assume that none of the zeroes of S_{11} and S_{33} are each other's complex conjugates.

The above determined partially solutions of the problem (11) $\psi^+(x, \lambda)$ and $\psi^-(\lambda)$ are then analytical in the whole of the complex λ-plane, except a finite number of points where the elements of these matrices have simple poles. One can, for instance, show (see[20, 19]) that

$$\psi_{i1}^+(x,\lambda)\exp(i\lambda x/a_1)=\delta_{i1}+\frac{a_1}{a_2}\sum_{n=1}^{N_1}\frac{G\alpha_n^*\psi_{i2}^+(x,\lambda_n^*)\exp(i\lambda_n^*x/a_1)}{\lambda-\lambda_n^*},\tag{15}$$

$$\psi_{i2}^+(x,\lambda)\exp(i\lambda x/a_2)=\delta_{i2}-\sum_{n=1}^{N_1}\frac{\alpha_n\psi_{i1}^+(x,\lambda_n)\exp(i\lambda_n x/a_2)}{\lambda-\lambda_n}-\sum_{n=1}^{N_3}\frac{\beta_n\psi_{i3}^+(x,\mu_n)\exp(i\mu_n x/a_2)}{\lambda-\mu_n},\tag{16}$$

$$\psi_{i3}^+(x,\lambda)\exp(i\lambda x/a_3)=\delta_{i3}+\frac{a_3}{a_2}\sum_{n=1}^{N_3}\frac{G\beta_n^*\psi_{i2}^+(x,\mu_n^*)\exp(i\mu_n^*x/a_3)}{\lambda-\mu_n^*}.\tag{17}$$

Here $\psi_{ij}^+(x, \lambda)$ are the elements of the matrix $\psi^+(x, \lambda)$, δ_{ij} is the Kronecker symbol,

$$\alpha_n=S_{12}(\lambda_n)/S_{11}'(\lambda_n),\quad \beta_n=S_{32}(\mu_n)/S_{33}'(\mu_n)\tag{18}$$

and G = 1 in the case of the decay instability (3), (7) or G = −1 for the explosive instability (4), (8). The quantities $S_{12}(\lambda_n)$ and $S_{32}(\mu_n)$ which occur in Eqs. (18) can in the case of finite functions $u_i(x)$ be considered to be the result of the analytical continuation of the corresponding elements of the scattering matrix on the real axis; when performing the subsequent calculations it will be sufficient if we know their time-dependence which follows from Eqs. (14) through analytically continuing the latter into the complex plane; $S_{12}(\lambda_n, t)$ and $S_{32}(\mu_n, t)$ are then arbitrary constants for t = 0.

We note now that Eqs. (15) to (17) enable us to determine the functions $\psi_{ij}^+(x, \lambda)$ uniquely. To do this it is sufficient to put $\lambda = \lambda_n$ in (15), $\lambda = \mu_n$ in (17), and in Eq. (16) let λ take on the values λ_n^* or μ_n^*, after which we obtain a closed set of linear equations for the functions $\psi_{i1}^+(x, \lambda)$, $\psi_{i2}^+(x, \lambda^*)$, $\psi_{i2}^+(x, \mu^*)$, $\psi_{i3}^+(x, \mu)$, and, solving them, we determine in that way the set of functions $\psi_{ij}^+(x, \lambda)$. It is clear, moreover, that to determine the functions $u_i(x, t)$ in the operator L of (5) it is sufficient to know only one of the solutions of (11).

One sees easily that

$$Q_{i2}(x)=\frac{1}{a_2}\sum_{n=1}^{N_1}\alpha_n\psi_{i1}^+(x,\lambda_n)\exp\left\{i\frac{\lambda_n}{a_2}x\right\}+\frac{1}{a_2}\sum_{n=1}^{N_3}\beta_n\psi_{i3}^+(x,\mu_n)\exp\left\{i\frac{\mu_n}{a_2}x\right\},$$

$$Q_{i3}(x)=-\frac{G}{a_2}\sum_{n=1}^{N_3}\beta_n^*\psi_{i2}^+(x,\mu_n^*)\exp\left\{i\frac{\mu_n^*}{a_3}x\right\},$$

where the $Q_{ij}(x)$ are the coefficient functions of the operator L of (5) which are connected with the solutions of the set (3) or (4), respectively, by Eqs. (7) or (8).

We have thus given a prescription to construct exact solutions of the sets (3) and (4). We consider the simplest of such solutions in the case of the decay instability.

Let $S_{11}(\lambda)$ have just a single zero at a point λ in the upper half-plane while $S_{33} \equiv 1$. Performing the appropriate calculations we find that such a scattering matrix corresponds to a "potential" Q of the form $Q_{13} = Q_{23} = 0$:

$$Q_{12}=\frac{\lambda-\lambda^*}{2\sqrt{a_1a_2}}\exp\left[i\frac{\lambda+\lambda^*}{2}\left(\frac{1}{a_2}-\frac{1}{a_1}\right)(x-v_2t)+i\varphi_2\right]$$
$$\times\,\mathrm{ch}^{-1}\left[i\frac{\lambda-\lambda^*}{2}\left(\frac{1}{a_2}-\frac{1}{a_1}\right)(x-v_2t-x_2^0)\right], \tag{19}$$

where

$$\varphi_2=\arg S_{12},\quad x_2^0=\frac{2}{i(\lambda-\lambda^*)}\left(\frac{1}{a_2}-\frac{1}{a_1}\right)^{-1}\ln\left(\frac{a_2}{a_1}\right)^{1/2}\frac{1}{|S_{12}|}.$$

Using (7) we call solution (19) a wave 2 soliton. Similarly, if $S_{11}\equiv 1$ and $S_{33}(\mu)=0$, we have $Q_{12}=Q_{13}=0$,

$$Q_{23}=\frac{\mu-\mu^*}{2\sqrt{a_2a_3}}\exp\left[i\frac{\mu+\mu^*}{2}\left(\frac{1}{a_3}-\frac{1}{a_2}\right)(x-v_3t)+i\varphi_3\right]$$
$$\times\,\mathrm{ch}^{-1}\left[i\frac{\mu-\mu^*}{2}\left(\frac{1}{a_3}-\frac{1}{a_2}\right)(x-v_3t-x_3^0)\right],$$
$$\varphi_3=-\arg S_{32}.$$
$$x_3^0=\frac{2}{i(\mu^*-\mu)}\left(\frac{1}{a_2}-\frac{1}{a_3}\right)^{-1}\ln\left(\frac{a_2}{a_3}\right)^{1/2}\frac{1}{|S_{32}|} \tag{20}$$

—a wave 3 soliton.

From the point of view of the set (3) these solutions are by themselves of no interest as they are trivial (the amplitude of only one of the packets is nonvanishing). We have, however, given the explicit form of these solutions as the picture of the scattering of such packets by one another looks very simple. The solution describing the collision of the secondary waves by one another can be obtained from the nonreflecting scattering matrix in which the elements $S_{11}(\lambda)$ and $S_{33}(\lambda)$ have up to a single zero. The above described procedure leads to the potential

$$Q_{13}=\frac{1}{a_2}\frac{(\lambda-\lambda^*)(\mu-\mu^*)}{\lambda-\mu^*}\frac{S_{12}S_{32}^*\exp[i\lambda\gamma_{21}(x-v_2t)+i\mu^*\gamma_{32}(x-v_3t)]}{\Delta(x,t)}, \tag{21}$$

$$Q_{12}=\frac{\lambda-\lambda^*}{a_2}\frac{S_{12}\exp\{i\lambda\gamma_{21}(x-v_2t)\}}{\Delta(x,t)}$$
$$\times\left\{1+\frac{a_3}{a_2}\frac{\mu-\lambda}{\mu^*-\lambda}|S_{32}|^2\exp[i(\mu^*-\mu)\gamma_{32}(x-v_3t)]\right\}, \tag{22}$$

$$Q_{23}=\frac{\mu-\mu^*}{a_2}\frac{S_{32}^*\exp\{i\mu^*\gamma_{32}(x-v_3t)\}}{\Delta(x,t)}$$
$$\times\left\{1+\frac{a_1}{a_2}\frac{\lambda-\mu}{\lambda^*-\mu}|S_{12}|^2\exp[i(\lambda-\lambda^*)\gamma_{21}(x-v_2t)]\right\}, \tag{23}$$

where $\gamma_{ik}=1/a_i-1/a_k$,

$$\Delta(x,t)=\left[1+\frac{a_1}{a_2}|S_{12}|^2e^{i(\lambda-\lambda^*)\gamma_{21}(x-v_2t)}\right]\left[1+\frac{a_3}{a_2}|S_{32}|^2e^{i(\mu^*-\mu)\gamma_{32}(x-v_3t)}\right]$$
$$+\frac{a_1a_3}{a_2^2}\frac{(\lambda-\lambda^*)(\mu-\mu^*)}{|\lambda-\mu^*|^2}|S_{12}|^2|S_{32}|^2$$
$$\times\exp[i(\lambda-\lambda^*)\gamma_{21}(x-v_2t)+i(\mu^*-\mu)\gamma_{32}(x-v_3t)]. \tag{24}$$

One sees easily that the solution obtained is, asymptotically as $t\to\pm\infty$, a superposition of solitons of the secondary waves with amplitudes which as $t\to\infty$ are exactly the same as their values when $t\to-\infty$; the whole collision effect thus reduces to a shift in the phases and the coordinates of the centers of the solitons, expressions for which can easily be found from (21) to (24). If, for instance, $v_2>v_3$, we have

$$\Delta x_2^0\equiv x_2^0(+\infty)-x_2^0(-\infty)=\frac{2}{i(\lambda-\lambda^*)}\gamma_{21}^{-1}\ln\left|\frac{\lambda-\mu}{\lambda-\mu^*}\right|,$$
$$\Delta x_3^0=\frac{2}{i(\mu^*-\mu)}\gamma_{32}^{-1}\ln\left|\frac{\lambda-\mu^*}{\lambda-\mu}\right|.$$

We note that Δx_2^0 is always negative while $\Delta x_3^0>0$, and in the limit $|\lambda-\mu^*|\to 0$, $\Delta x_2^0\to-\infty$, $\Delta x_3^0\to\infty$. The cause for such a retardation of the solitons is obvious enough.

The solution (21) to (24) depends on four arbitrary complex constants. Of great interest, however, is the case when these constants are somehow interrelated. Let, for instance, $\lambda-\mu^*\to 0$, $S_{12}\to 0$, so that $S_{12}/(\lambda-\mu^*)\to\mathrm{const}=T$. Then

$$\Delta(x,t)=1+\frac{a_3}{a_2}|S_{32}|^2e^{i(\lambda-\lambda^*)\gamma_{32}(x-v_3t)}+\frac{a_1a_3}{a_2^2}|\lambda-\lambda^*|^2|S_{32}|^2|T|^2e^{i(\lambda-\lambda^*)\gamma_{31}(x-v_1t)},$$

$$Q_{13}=\frac{1}{a_2}|\lambda-\lambda^*|^2\frac{S_{32}^*Te^{i\lambda\gamma_{31}(x-v_1t)}}{\Delta(x,t)}. \tag{25}$$

If then $v_3>v_1$, we see easily from (21), (22), and (25) that as $t\to-\infty$ we have Q_{12}, $Q_{23}\to 0$, and

$$Q_{13}=\frac{|\lambda-\lambda^*|^2S_{32}^*Te^{i\lambda\gamma_{31}(x-v_1t)}a_2^{-1}}{1+a_1a_3a_2^{-2}|\lambda-\lambda^*|^2|S_{32}|^2|T|^2e^{i(\lambda-\lambda^*)\gamma_{31}(x-v_1t)}}.$$

The expression obtained for Q_{13} is completely analogous to (19) and (20); for that reason we shall call that kind of solution a pumping wave soliton. However, as $t\to\infty$ the solution again consists of secondary wave solitons; the solution found thus describes the total decay of the pumping soliton into secondary wave solitons. Completely analogously, taking in (21) to (24) the limits as $\lambda-\mu^*\to 0$, $S_{32}^*/(\lambda-\mu^*)\to\mathrm{const}$, we get a solution describing the fusion of secondary wave solitons into a pumping soliton, if $v_3>v_1>v_2$, and the decay of the pumping soliton when the velocity ratios are the opposite. If, however, we put in (21) to (24) $\lambda-\mu^*\to 0$, $S_{12}\to 0$, $S_{32}\to 0$, $S_{32}\to 0$, $S_{12}S_{32}^*/(\lambda-\mu^*)\to\mathrm{const}$, we get a "pure" pumping soliton.

We can similarly also consider the analogous situation when $S_{11}(\lambda)$ and $S_{33}(\lambda)$ have an arbitrary number of zeroes. It then turns out that the solutions obtained contain, asymptotically as $t\to\pm\infty$, only secondary waves; if, however, we take in the solutions obtained the limit as $\lambda_n\to\mu_k^*$ for some n and k, and putting the $S_{12}(\lambda_n)/(\lambda_n-\mu_k^*)\to\mathrm{const}$, the expressions obtained, which are also exact solutions of the set (3), will, as $t\to-\infty$, contain, in general, all three packets, including the pumping packet, which vanishes as $t\to+\infty$. The solutions corresponding to scattering matrices in which $S_{11}(\lambda)=S_{33}^*(\lambda^*)$ describe in the general case the spontaneous decay of the pumping packet into secondary wave packets or the inverse process of the total fusion of the latter into the pumping packet, or even both at the same time. If, on the other hand, some of the λ_n are "close" to some μ_k^*, in the scattering of the secondary wave packets there arises a situation when after a sufficiently long time there exists a pumping packet which is spatially separated from the main bulk of the secondary waves which have amplitudes which are small in the region where the pumping wave is localized. The period during which such an "intermediate asymptotic behavior" exists can be very long; it depends logarithmically on $|\lambda_n-\mu_k^*|$.

We now consider the case of the explosive instability. We must then put $G=-1$ in Eqs. (15) to (17). It is clear that we can use the formulae which we have already obtained, replacing in them $S_{12}^*(\lambda)$ and $S_{32}^*(\mu_n)$, respectively, by $-S_{12}^*(\lambda_n)$ and $-S_{32}^*(\mu_n)$. Amongst the solutions occurring then only the solution obtained from (25) is regular as $t\to-\infty$. It takes the form

$$Q_{13}=\frac{\lambda-\lambda^*}{a_3}\frac{S_{13}e^{i\lambda\gamma_{31}(x-v_1t)}}{\Delta(x,t)},\quad Q_{23}=\frac{\lambda-\lambda^*}{a_2}\frac{S_{32}^*e^{i\lambda\gamma_{32}(x-v_3t)}}{\Delta(x,t)},$$
$$Q_{12}=-\frac{\lambda-\lambda^*}{a_2}\frac{S_{13}S_{32}}{\Delta(x,t)}$$
$$\times\exp\left[i\frac{\lambda+\lambda^*}{2}\gamma_{21}(x-v_2t)+i\frac{\lambda-\lambda^*}{2}(\gamma_{31}(x-v_1t)+\gamma_{32}(x-v_3t))\right],$$
$$\Delta(x,t)=1+\frac{a_1}{a_3}|S_{13}|^2e^{i(\lambda-\lambda^*)\gamma_{31}(x-v_1t)}-\frac{a_3}{a_2}|S_{32}|^2e^{i(\lambda-\lambda^*)\gamma_{32}(x-v_3t)}. \tag{26}$$

Here S_{32} and S_{13} are arbitrary constants. One sees easily that $\Delta(x, t)$, starting at some time $t \geq t^*$, vanishes for several values of x. Simple calculations give

$$t^*=\frac{i}{(\lambda-\lambda^*)(v_3-v_1)}\left\{\gamma_{32}^{-1}\ln\frac{a_1-a_3}{a_1-a_2}\frac{a_2^2a_3^{-2}}{|S_{32}|^2}-\gamma_{31}^{-1}\ln\frac{a_2-a_3}{a_1-a_2}\frac{1}{|S_{13}|^2}\right\},$$

i.e., at $t = t^*$ there occurs in the solution (26) a singularity at the point

$$x^*=\frac{i}{(\lambda-\lambda^*)(v_3-v_1)}\left\{v_1\gamma_{32}^{-1}\ln\frac{a_1-a_3}{a_1-a_2}\frac{a_2^2a_3^{-2}}{|S_{32}|^2}-v_3\gamma_{31}^{-1}\ln\frac{a_2-a_3}{a_1-a_2}\frac{1}{|S_{13}|^2}\right\}.$$

The amplitudes of the packets then behave as $(x - x^*)^{-2}$. The solution obtained describes the spontaneous explosion of a soliton of the wave which has the intermediate velocity.

In the general case given initial conditions lead as the result of the development of the explosive instability to the formation of a singularity if the elements $S_{11}(\lambda)$ and $S_{33}(\lambda)$ have zeroes. If, for instance, initially the amplitudes of the waves with the largest and smallest velocities are small compared to the amplitude of the wave with the intermediate velocity the condition for the occurrence of zeroes in $S_{11}(\lambda)$ and $S_{33}(\lambda)$ is of the form

$$q|u_1|L_1/\sqrt{|v_1-v_2||v_1-v_3|}\geqslant c, \tag{27}$$

where u_1 and L_1 are the characteristic amplitude and length of the packet of wave 1, while the number $c \sim 1$; the exact value of c depends on the shape of packet 1 and can not be evaluated for a packet of arbitrary shape. In accordance with what has been said (27) is the criterion for the formation of a singularity from the initial conditions of the type indicated.

3. SCATTERING OF WAVE PACKETS

In the present section we consider the problem of the connection between the asymptotic solutions of the set (3), i.e., the problem of the resonance scattering of wave packets. We obtain explicit expressions for the integral intensities of the wave packets arising as $t \to \infty$, directly in terms referring to $t \to -\infty$. First, however, we must study the integrals of motion of the set (3). To do this we write $\psi^+(x, \lambda)$ in the form

$$\psi_{ij}^+(x,\lambda)=\delta_{ij}\exp\left(-i\frac{\lambda}{a_j}x+\int_x^\infty\chi_j(s,\lambda)\,ds\right)$$
$$+(1-\delta_{ij})A_{ij}(x,\lambda)\exp\left(-i\frac{\lambda}{a_j}x+\int_x^\infty\chi_j(s,\lambda)\,ds\right) \tag{28}$$

and, eliminating $\chi_j(x, \lambda)$, we get for $A_{ij}(x, \lambda)$

$$ia_ia_j\frac{\partial A_{ij}}{\partial x}-a_iA_{ij}\sum_k(a_j-a_k)Q_{jk}A_{kj}+a_j(a_i-a_j)Q_{ij}$$
$$+a_j\sum_{k\neq j}(a_i-a_k)Q_{ik}A_{kj}=\lambda(a_j-a_i)A_{ij}.$$

From this equation there follows a recurrence relation for the coefficients of the asymptotic expansion of $A_{ij}(x, \lambda)$ in powers of $1/\lambda$. Putting

$$A_{ij}(x,\lambda)=\sum_{n=1}^\infty\frac{A_{ij}^{(n)}(x)}{\lambda^n},$$

we find

$$A_{ij}^{(n+1)}(x)=\frac{1}{a_j-a_i}\left\{ia_ia_j\frac{\partial A_{ij}^{(n)}}{\partial x}+a_j\sum_{k\neq j}(a_i-a_k)Q_{ik}A_{kj}^{(n)}\right.$$
$$\left.-a_i\sum_{n_1+n_2=n}A_{ij}^{(n_1)}\sum_k(a_j-a_k)Q_{jk}A_{kj}^{(n_2)}\right\},\quad n=1,2,\ldots,$$
$$A_{ij}^{(1)}(x)=-a_jQ_{ij}(x). \tag{29}$$

Using now the expression for $\chi_j(x, \lambda)$ in terms of the $A_{ij}(x, \lambda)$:

$$ia_j\chi_j(x,\lambda)=\sum_{k\neq j}(a_j-a_k)Q_{jk}(x)A_{kj}(x,\lambda),$$

we get an expansion for the quantity $\chi_j(x, \lambda)$:

$$\chi_j(x,\lambda)=\sum_{n=1}^\infty\chi_j^{(n)}(x)/\lambda^n,$$
$$\chi_j^{(n)}(x)=-\frac{i}{a_j}\sum_k(a_j-a_k)Q_{jk}(x)A_{kj}^{(n)}(x).$$

We note further that we get directly from (28) the relation

$$\int_{-\infty}^\infty\chi_j(s,\lambda)\,ds=\ln S_{jj}(\lambda), \tag{30}$$

whence by virtue of the conservation of $S_{jj}(\lambda)$ it follows that the quantities

$$I_j^{(n)}=i\int_{-\infty}^\infty\chi_j^{(n)}(x)\,dx$$

are integrals of the set (3). Equations (29) give us the possibility to evaluate all $I_j^{(n)}$. As applied to the set (3) the $I_j^{(1)}$ give the well-known Manley-Rowe relations. We give also for the set (3) the second integrals:

$$I_1^{(2)}=p^2\int_{-\infty}^\infty\left\{i\frac{a_1a_2}{a_1-a_2}u_2\frac{\partial u_2^*}{\partial x}+i\frac{a_1a_3}{a_1-a_3}u_1\frac{\partial u_1^*}{\partial x}\right.$$
$$\left.+pa_1\left[\frac{a_2-a_3}{(a_1-a_2)(a_1-a_3)}\right]^{1/2}(u_1^*u_2u_3+u_1u_2^*u_3^*)\right\}dx.$$
$$I_2^{(2)}=p^2\int_{-\infty}^\infty\left\{i\frac{a_1a_2}{a_1-a_2}u_2^*\frac{\partial u_2}{\partial x}+\frac{ia_2a_3}{a_2-a_3}u_3\frac{\partial u_3^*}{\partial x}\right.$$
$$\left.-pa_2\left[\frac{a_1-a_3}{(a_1-a_2)(a_2-a_3)}\right]^{1/2}(u_1^*u_2u_3+u_1u_2^*u_3)\right\}dx.$$

The Hamiltonian of the set (3) can be expressed in terms of these integrals:

$$H=p^{-2}\left(\frac{a_1-a_3}{a_1a_3}v_1I_1^{(2)}+\frac{a_2-a_3}{a_2a_3}v_3I_2^{(2)}\right),$$

as can the momentum P:

$$P=p^{-2}\left(\frac{a_1-a_3}{a_1a_3}I_1^{(2)}+\frac{a_2-a_3}{a_2a_3}I_2^{(2)}\right).$$

There is no simple physical interpretation for the other integrals $I_j^{(n)}$ $(n \geq 3)$.

We now turn directly to the problem of the scattering of the packets. We first of all consider the case where the velocity of the pumping wave is the intermediate one. To fix the ideas we shall assume that $v_2 < v_1 < v_3$. It is clear that as $t \to \pm\infty$ any solution decays into non-interacting packets. As $t \to -\infty$ the packets are then positioned along the x-axis in the order 3, 1, 2. As $t \to +\infty$ the sequence of the packets is the reverse. The transition matrix (i.e., the scattering matrix) can be written in the form

$$S(t)=S_3^{(-)}S_1^{(-)}S_2^{(-)},\quad t\to-\infty,$$
$$S(t)=S_2^{(+)}S_1^{(+)}S_3^{(+)},\quad t\to\infty, \tag{31}$$

where the $S_i^{(\pm)}$ are the partial transition matrices, described below. It is clear that the asymptotic behavior (31) is by virtue of (14) valid at any time. Referring both representations (31) to the time t = 0 we get

$$S_2^{(+)}S_1^{(+)}S_3^{(+)}=S_3^{(-)}S_1^{(-)}S_2^{(-)}. \tag{32}$$

Relation (32) and the analytical properties of the diagonal elements of the S_i enable us, as we shall show below, to determine unambiguously the $S_i^{(+)}$ for given $S_i^{(-)}$. Re-establishing then the amplitudes of the packets $u_i^{(+)}(x, t)$ in terms of the $S_i^{(+)}$ we completely determine the characteristics of the wave packets which occur as a result of the scattering.

First of all we study in detail the properties of the partial transition matrices. Putting in the operator L of (5) all $Q_{ij} = 0$ except $Q_{\alpha\beta}$, $Q_{\beta\alpha}$ $(= -Q^*_{\alpha\beta})$ we get instead of (11)

$$
\begin{aligned}
& i a_\alpha \frac{\partial \psi_\alpha}{\partial x} + (a_\alpha - a_\beta) Q_{\alpha\beta} \psi_\beta = \lambda \psi_\alpha, \\
& i a_\beta \frac{\partial \psi_\beta}{\partial x} + (a_\alpha - a_\beta) Q_{\alpha\beta}^* \psi_\alpha = \lambda \psi_\beta, \\
& i a_i \frac{\partial \psi_i}{\partial x} = \lambda \psi_i, \quad i \neq \alpha, \beta.
\end{aligned} \tag{33}
$$

The set (33) is a particular case of (11). It is clear that $S_{ik} = \delta_{ik}$, if $i, k \neq \alpha, \beta$. There are thus in this case only four nontrivial elements in the transition matrix: $S_{\alpha\alpha}$, $S_{\alpha\beta}$, $S_{\beta\alpha}$, and $S_{\beta\beta}$. We put $\alpha < \beta$. In that case, as $a_3 > 0$, the quantities $S_{\alpha\alpha}(\lambda)$ are analytical in the upper λ-half-plane, and the $S_{\beta\beta}$ in the lower one. By virtue of (12) and (13) we then have

$$
S_{\beta\beta}(\lambda) = S_{\alpha\alpha}^*(\lambda^*), \quad S_{\beta\alpha} = -\frac{a_\alpha}{a_\beta} S_{\alpha\beta}^*. \tag{34}
$$

It will be more convenient to use in what follows instead of the transition matrix S the matrix s with the elements

$$
s_{ik} = |a_i/a_k|^{1/2} S_{ik}, \tag{35}
$$

which is clearly unitary and unimodular. In terms of s (34) can be rewritten as

$$
s_{\beta\beta}(\lambda) = s_{\alpha\alpha}^*(\lambda^*), \quad s_{\beta\alpha} = -s_{\alpha\beta}^*.
$$

(We note that the diagonal elements of the matrices S and s are the same.)

We now introduce the following notation:

$$
s_1^\pm = \begin{pmatrix} \sigma_1^\pm & 0 & \eta_1^\pm \\ 0 & 1 & 0 \\ -\eta_1^{\pm *} & 0 & \sigma_1^{\pm *} \end{pmatrix}, \quad
s_2^\pm = \begin{pmatrix} \sigma_2^\pm & \eta_2^\pm & 0 \\ -\eta_2^{\pm *} & \sigma_2^{\pm *} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad
s_3^\pm = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma_3^\pm & \eta_3^\pm \\ 0 & -\eta_3^{\pm *} & \sigma_3^{\pm *} \end{pmatrix},
$$

where the $s_i^\pm$ are matrices connected through Eq. (35) with the partial transition matrices $S^{(\pm)}$. The $\sigma_i^\pm(\lambda)$ are then analytical in the upper λ-half-plane and

$$
|\sigma_i(\lambda)|^2 + |\eta_i(\lambda)|^2 = 1. \tag{36}
$$

We shall in what follows consider only the scattering of finite wave packets. In that case all elements of the transition matrix are analytical in the whole λ-plane.

Equation (32) which for the $s^\pm$ matrices has, as before, the form

$$
s_2^+ s_1^+ s_3^+ = s_3^- s_1^- s_2^- = s(0),
$$

enables us to find at once $|\sigma_i^+(\lambda)|$ and $\eta_i^+(\lambda)$ for given $s_i^-(\lambda)$. Using (36) we have

$$
\begin{gathered}
|\sigma_1^+|^2 = 1 - |s_{31}|^2, \quad |\sigma_2^+|^2 = \frac{|s_{11}|^2}{1 - |s_{31}|^2}, \quad |\sigma_3^+|^2 = \frac{|s_{33}|^2}{1 - |s_{31}|^2} \\
\eta_1^+ = -s_{31}^*, \quad \eta_2^+ = -\frac{s_{21}^*}{\sigma_1^{+*}}, \quad \eta_3^+ = -\frac{s_{32}^*}{\sigma_1^+}.
\end{gathered} \tag{37}
$$

Expressing the elements of the matrix s(0) in terms of σ_i^- and η_i^- we get

$$
\begin{aligned}
& s_{31} = -\sigma_2^- \sigma_3^{-*} \eta_1^{-*} + \eta_2^{-*} \eta_3^{-*}, \quad s_{11} = \sigma_1^- \sigma_2^-, \quad s_{33} = \sigma_1^{-*} \sigma_3^{-*}, \\
& s_{21} = -\sigma_3^- \eta_2^{-*} - \sigma_2^- \eta_1^{-*} \eta_3^-, \quad s_{32} = -\sigma_2^{-*} \eta_3^{-*} - \sigma_3^{-*} \eta_1^{-*} \eta_2^-.
\end{aligned} \tag{38}
$$

It is clear from (37) that for a complete determination of the s_i^+ matrices we must still know the arguments of the σ_i^+. In view of the analytical properties of σ_i it is sufficient for this to determine the position of the zeroes of σ_i^+ for Im $\lambda > 0$. Equating the matrix elements $(s_2^+ s_1^+ s_3^+)_{11}$, $(s_2^+ s_1^+ s_3^+)_{33}$ and $(s_3^- s_1^- s_2^-)_{11}$, $(s_3^- s_1^- s_2^-)_{33}$ we find

$$
\sigma_1^+ \sigma_2^+ = \sigma_1^- \sigma_2^-, \quad \sigma_1^+ \sigma_3^+ = \sigma_1^- \sigma_3^-. \tag{39}
$$

From this it follows at once that if $\sigma_2^-(\lambda) = 0$, $\sigma_{1,3}^- \neq 0$, also $\sigma_2^+(\lambda) = 0$, $\sigma_{1,3}^+(\lambda) \neq 0$ and if $\sigma_3^-(\lambda) = 0$, $\sigma_{1,2}^-(\lambda) \neq 0$, then $\sigma_3^+(\lambda) = 0$, $\sigma_{1,2}^+ \neq 0$.

We restrict here the considerations to the situation when $S_{11}(\lambda)$ and $S_{33}(\lambda)$ have only simple zeroes, i.e., the case when the zeroes of σ_1^- and σ_2^- or of σ_1^- and σ_3^- are not the same. The general case can be obtained by taking the limit. Bearing this restriction in mind we see that if $\sigma_1^-(\lambda) = 0$, it follows from (39) that either $\sigma_1^+(\lambda) = 0$, or $\sigma_2^+(\lambda) = \sigma_3^+(\lambda) = 0$. Equating the elements $(s_2^+ s_1^+ s_3^+)_{21}$ and $(s_3^- s_1^- s_2^-)_{21}$ $((s_2^+ s_1^+ s_3^+)_{21} = -\sigma_1^+ \eta_2^{+*} = -\sigma_3^- \eta_2^{-*} - \sigma_2^- \eta_1^{-*} \eta_3$ $= (s_3^- s_1^- s_2^-)_{21})$ we see that if $(s_3^- s_1^- s_2^-)_{21} \neq 0$, $\sigma_1^+(\lambda) \neq 0$, $\sigma_2^+(\lambda) = \sigma_3^+(\lambda) = 0$. Similarly, if $\sigma_1^-(\lambda) \neq 0$, $\sigma_2^-(\lambda) = \sigma_3^-(\lambda) = 0$, it follows then from the equation

$$
(s_2^+ s_1^+ s_3^+)_{12} = \sigma_3^+ \eta_2^+ - \sigma_2^+ \eta_1^+ \eta_3^{+*} = \sigma_1^- \eta_2^- = (s_3^- s_1^- s_2^-)_{12}
$$

since $\eta_2^-(\lambda)$ and in view of (39), that $\sigma_1^+(\lambda) = 0$, $\sigma_{2,3}^+(\lambda) \neq 0$. If, however, $\sigma_1^-(\lambda) = 0$ and $(\sigma_3^- \eta_2^{-*} + \sigma_2^- \eta_1^{-*} \eta_3^-)(\lambda) = 0$, then also $\sigma_1^+(\lambda) = 0$, $\sigma_{2,3}^+(\lambda) \neq 0$. We thus see that the position of the zeroes of $\sigma_i^+(\lambda)$ is uniquely determined by the partial scattering matrices s_i^-. We found that the zeroes of $\sigma_1^-(\lambda)$ either produce zeroes in $\sigma_2^+(\lambda)$, $\sigma_3^+(\lambda)$ (lying at the same points as for $\sigma_1^-(\lambda)$), or remain with $\sigma_1^+(\lambda)$. The latter occurs only when $(\sigma_3^- \eta_2^{-*} + \sigma_2^- \eta_1^{-*} \eta_3^-)(\lambda) = 0$. It is clear, however, that any change in the amplitudes of the colliding packets, however small, shifts in general the zero of the last expression from the point λ to a point close to it. However, we must then already relate this zero in σ_1^- to the zeroes of $\sigma_{2,3}^+$ and not to σ_1^+. In the general case the vanishing of the combination mentioned here has zero probability. We shall therefore not consider such a situation just now. (We shall, however, describe below the behavior of the solution in that case.) The zeroes of $\sigma_1^+(\lambda)$ arise therefore only when the zeroes of σ_2^- and σ_3^- coincide. We denote the zeroes of $\sigma_i^-(\lambda)$ by $\lambda_i^{(n)}$, and we then number the coincident zeroes of σ_2^- and σ_3^- (let their total number be M) by the numbers from 1 to M and we shall assume that $\lambda_2^{(n)} = \lambda_3^{(n)}$, $n \leq M$. The zeroes of $\sigma_1^+(\lambda)$ are then $\lambda_2^{(n)}$, $n = 1, \ldots, M$. The zeroes of $\sigma_2^+(\lambda)$ lie at the points $\lambda_3^{(n)}$, $n = M + 1, \ldots, N_3$; $\lambda_1^{(n)}$, $n = 1, \ldots, N_1$. To determine $\arg \sigma_i^+(\lambda)$ we consider the function

$$
\ln \left(\sigma_i^+(\lambda) \prod \frac{\lambda - \lambda_i^{+(n)*}}{\lambda - \lambda_i^{+(n)}} \right)
$$

where the $\lambda_i^{+(n)}$ are all the zeroes of $\sigma_i^+(\lambda)$. It is clear that the function considered is analytical in the upper half-plane and tends to zero as $|\lambda| \to \infty$, Im $\lambda \geq 0$. In view of this we have

$$
\arg \sigma_i^+(\lambda) = \frac{1}{i} \sum \ln \frac{\lambda - \lambda_i^{+(n)}}{\lambda - \lambda_i^{+(n)*}} - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln |\sigma_i^+(\lambda')|}{\lambda' - \lambda} d\lambda'.
$$

We have thus completely determined the partial transition matrices s_i^+. Using the inverse method equations (see [10, 20]) we can now in principle completely determine the asymptotic behavior of the solution as $t \to \infty$. However, we can find an answer to the most interesting physical problems also without solving the inverse method equations. We find directly from (30), by calculating the first term in the expansion of $\ln \sigma_i^\pm(\lambda)$ in powers of $1/\lambda$:

$$\int_{-\infty}^{\infty}|u_i^{\pm}(x,t)|^2dx=p^{-2}\left(2\sum\operatorname{Im}\lambda_i^{\pm(n)}+\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln\frac{1}{|\sigma_i^{\pm}(\lambda)|^2}d\lambda\right),\quad(40)$$

where the $u_i^{\pm}(x, t)$ are the asymptotic expressions of the $u_i(x, t)$ as $t\to\pm\infty$, and the $\lambda_i^{\pm(n)}$ the zeroes of the $\sigma_i^{\pm}(\lambda)$. Equations (40) together with (37) give expressions for the integral intensities of the packets which occur as $t\to\infty$.

We consider the most interesting special cases. When a strong ($qL_1u_1/v\gg 1$) pumping packet collides with small secondary wave packets ($qL_{2,3}u_{2,3}/v\ll 1$) for which the $\sigma_{2,3}^-(\lambda)$ have no zeroes, we can neglect in (40) the integral terms. As a result we get

$$\int_{-\infty}^{\infty}|u_1^+(x,t)|^2dx=\frac{1}{2\pi p^2}\int_{-\infty}^{\infty}\ln(1-|\sigma_2^{-*}\sigma_3^{-*}\eta_1^{-*}-\eta_2^{-*}\eta_3^{-*}|^2)^{-1}d\lambda.$$

$$\int_{-\infty}^{\infty}|u_2^+(x,t)|^2dx\approx\int_{-\infty}^{\infty}|u_3^+(x,t)|^2dx=\frac{2}{p^2}\sum_{n=1}^{N_1}\operatorname{Im}\lambda_1^{(n)}.$$

The initial intensity of the pumping wave is (see (40))

$$\int_{-\infty}^{\infty}|u_1^-(x,t)|^2dx\approx\frac{2}{p^2}\sum_{n=1}^{N_1}\operatorname{Im}\lambda_1^{(n)}.$$

We see thus that in the case considered the initial intensity of the pumping wave decays practically completely.

In general the partial problems of (33) for the intensities of the packets simplify considerably. Indeed, putting in (33)

$$\psi_{\alpha,\beta}=a_{\alpha,\beta}^{-1/2}\exp\left(-i\frac{a_\alpha+a_\beta}{2a_\alpha a_\beta}\lambda x\right)\varphi_{\alpha,\beta}$$

and neglecting terms $\sim\partial Q_{\alpha\beta}/\partial x$ we get for φ_α:

$$-\frac{\partial^2\varphi_\alpha}{\partial x^2}-V_{\alpha\beta}(x)\varphi_\alpha=\xi^2\varphi_\alpha,\quad(41)$$

where

$$V_{\alpha\beta}=\frac{(a_\alpha-a_\beta)^2}{a_\alpha a_\beta}|Q_{\alpha\beta}|^2,\quad\xi=\frac{\lambda}{2}\left(\frac{1}{a_\alpha}-\frac{1}{a_\beta}\right).$$

The potential $V_{\alpha\beta}(x)$ in the one-dimensional Schrödinger equation (41) satisfies then in the case of strong packets the quasi-classical condition $V_{\alpha\beta}^{1/2}L_{\alpha\beta}\gg 1$ and the quantities $\eta_i(\lambda)$ turn out to be exponentially small in the parameter $V^{1/2}L$. From the point of view of the inverse scattering method such potentials are completely determined by the zeroes of the $\sigma_i(\lambda)$, i.e., the $\lambda_i^{(n)}$ and the corresponding quantities $\eta_i(\lambda_i^{(n)})$. The zeroes of the σ_i lie then at the points $\lambda_n=i\zeta_na_\alpha a_\beta/|a_\alpha-a_\beta|$ where the $-\zeta_n^2$ are the eigenvalues of (41) which are determined by the quasi-classical quantization rule:

$$\oint\sqrt{V_{\alpha\beta}(x)-\zeta_n^2}\,dx=2\pi(n+1/2).$$

Their total number $N\sim V^{1/2}L$ is large. Therefore, in the general case of the collision of large packets the intensities of the waves which occur are described by Eqs. (40) in which we must neglect the contributions from the continuous spectrum.

We note also that in the case of a collision of strong secondary wave packets there occurs, as a rule, no pumping wave in the asymptotic behavior as $t\to\infty$, as the coincidence of the zeroes of $\sigma_2^-(\lambda)$ and $\sigma_3^-(\lambda)$ is a very improbable event. By a special "preparation" of the packets 2 and 3 one can, however, achieve the coincidence of the zeroes of σ_2^- and σ_3^-. If, in particular,

$$u_2^-(x)/\sqrt{|v_2-v_1|}=u_3^-(x)/\sqrt{|v_3-v_1|},$$

one sees easily that the partial transition matrices s_2^- and s_3^- are exactly the same. If the packets u_2^- and u_3^- satisfy the quasi-classical condition, there then occurs a practically complete fusion of the secondary waves into a pumping wave. However, as the quasi-classical condition is very sensitive to small changes in $V_{\alpha\beta}(x)$, the appearance of a strong pumping wave occurs clearly only in an intermediate stage; asymptotically as $t\to+\infty$ it decays. A similar situation occurs if some of the zeroes of $\sigma_2^-(\lambda)$ and $\sigma_3^-(\lambda)$ are very close to one another. In that case we can state that the scattering of the secondary wave packets takes place in stages. As a result of the interaction in the first stage there appear three packets with distances between them which can be large. The intensity of the pumping wave is then

$$\int|u_1(x)|^2dx=\frac{2}{p^2}\sum\operatorname{Im}\lambda_2^{(n)}$$

where the summation is over the zeroes of $\sigma_2^-(\lambda)$ which are close to zeroes of $\sigma_3^-(\lambda)$ while the amplitudes of the packets 2 and 3 are small, but nevertheless non-zero, in the region where the pumping packet is localized. The time during which such an intermediate state exists may be very long. Finally, in the last stage the pumping packet decays practically completely. Simple estimates show that the lifetime of the intermediate state depends on the distance $|\Delta\lambda|$ between the close zeroes of σ_2^- and σ_3^- as $\ln(1/|\Delta\lambda|)$.

A similar effect of the formation of an "intermediate asymptotic behavior" also occurs in the case where a pumping packet collides with secondary wave packets, if the quantity $\epsilon(\lambda)=\sigma_3^-(\lambda)\eta_2^{-*}(\lambda^*)+\sigma_2^-(\lambda)\eta_1^{-*}(\lambda^*)\eta_3^-(\lambda)$ is small at the zero of $\sigma_1^-(\lambda)$. We shall show below that, if $\epsilon=0$, a zero of $\sigma_1^-(\lambda)$ does not lead to zeroes of $\sigma_2^+(\lambda)$ or $\sigma_3^+(\lambda)$, but to a zero of $\sigma_1^+(\lambda)$ which, according to (40), means the conservation of the corresponding contribution to the pumping intensity as $t\to\infty$. One understands easily, however, that the case of a small ϵ means the occurrence of an intermediate state of the kind just described. The lifetime of such a state depends logarithmically on $|\epsilon|$.

Equations (40) and (37) show also that when "small" packets, for which the $\sigma_i^-(\lambda)$ do not have zeroes, scatter, the integral intensities of the waves are of the same order as $t\to\pm\infty$. For instance, when small secondary wave packets collide, a pumping wave is produced with intensity

$$\int_{-\infty}^{\infty}|u_1^+(x,t)|^2dx=\frac{1}{2\pi p^2}\int_{-\infty}^{\infty}\ln\frac{1}{1-|\eta_2^-\eta_3^-|^2}d\lambda.$$

We emphasize, however, that the absence of zeroes of the $\sigma_i^-(\lambda)$ does not at all mean that the $u_i(x)$ are small. The amplitude of the packet (and its integral intensity) can then be arbitrarily large, provided the spectral width of the packet is sufficiently large. However, in this case a strong (but in k-space sufficiently wide) pumping packet when colliding with small (in the sense of having a small amplitude) secondary wave packets does practically not decay and even suffers little change in its shape. Only such a situation can be considered in the approximation of a given pumping field. Indeed, one can see from (37) that the partial transition matrix $s_1^+(\lambda)$ is close to $s_1^-(\lambda)$ provided $\sigma_1^-(\lambda)$ has no zeroes and $|\eta_2^-(\lambda)\eta_3^-(\lambda)|\ll|\eta_1^-(\lambda)$. This is, strictly speaking, also the criterion for the applicability for the approximation of a given pumping field in the case when the pumping velocity is neither the largest nor the smallest.

We summarize what has been said. A sufficiently strong pumping wave which satisfies the quasi-classical condition decays completely when it collides with any secondary wave packets. The interaction process can then be accompanied by the appearance of "intermediate asymptotic behavior" and the residual pumping intensity decreases when its amplitude in the "input" increases.

We studied the scattering of packets in the case when the pumping velocity is the intermediate one. In most physical situations the group velocity of the pumping wave is larger (in absolute magnitude) than the velocities of the secondary waves which appear when it decays. It is clear from Eq. (10) that this is possible only when $a_3 < 0$ (if $a_1 > a_2 > 0$). To fix the ideas we put $v_1 > v_3 > v_2$. For such velocity ratios we must write instead of (32)

$$S_2^{(+)}S_3^{(+)}S_1^{(+)}=S_1^{(-)}S_3^{(-)}S_2^{(-)}=S(0). \tag{42}$$

As above, we can easily find $S_i^{(+)}$ for given $S_i^{(-)}$ after which we can without difficulty get explicit expressions for the integral intensities of the packets which are formed. However, we must then bear in mind that as $a_3 < 0$, the regions of analyticity of the diagonal elements of the $S_i^{(\pm)}$ are, in general, not the same as in the case considered above.

We introduce the following notation for the elements of the partial transition matrices:

$$s_1^{\pm}=\begin{pmatrix}\sigma_1^{\pm *} & 0 & \eta_1^{\pm *}\\ 0 & 1 & 0\\ \eta_1^{\pm} & 0 & \sigma_1^{\pm}\end{pmatrix},\quad s_2^{\pm}=\begin{pmatrix}\sigma_2^{\pm} & \eta_2^{\pm} & 0\\ -\eta_2^{\pm *} & \sigma_2^{\pm *} & 0\\ 0 & 0 & 1\end{pmatrix},\quad s_3^{\pm}=\begin{pmatrix}1 & 0 & 0\\ 0 & \sigma_3^{\pm *} & \eta_3^{\pm *}\\ 0 & \eta_3^{\pm} & \sigma_3^{\pm}\end{pmatrix}$$

(the $s_i^{\pm}$ are connected with the $S_i^{(\pm)}$ by Eqs. (35)). The functions $\sigma_i^{\pm}(\lambda)$ defined in this way are analytical in the upper λ-half-plane while, in view of the unimodularity of the $s_i^{\pm}$, $\sigma_1^{\pm}$ and $\sigma_3^{\pm}$ do not have zeroes there. However, one sees easily from (42) that the zeroes of σ_2^{+} are the same as the zeroes of $\sigma_2^{-}(\lambda)$.

Performing the appropriate calculations we find

$$|\sigma_1^+|^2=|\sigma_1^-|^2|\sigma_3^-|^2F,\quad |\sigma_2^+|^2=|\sigma_2^-|^2|\sigma_3^-|^2F,\quad |\sigma_3^+|^2=F^{-1},$$
$$F^{-1}=1+|\sigma_1^-\sigma_2^{-*}\eta_3^-+\eta_1^-\eta_2^-|^2. \tag{43}$$

Equation (42) enables us to evaluate also all other elements of the partial transition matrices; we do not give the expressions for them.

The integral intensities of the packets which are formed can be expressed directly in terms of the $\sigma_i^{\pm}(\lambda)$. For i = 2 we have, as before, (40) and for i = 1, 3 we have

$$\int_{-\infty}^{\infty}|u_i^{\pm}(x,t)|^2\,dx=\frac{1}{2\pi p^2}\int_{-\infty}^{\infty}\ln|\sigma_i^{\pm}(\lambda)|^2\,d\lambda. \tag{44}$$

We consider concrete situations.

First of all we note that if in the input there was a strong packet of wave 2, the $\eta_2^{-}(\lambda)$ are exponentially small quantities and as $t\to-\infty$

$$\int_{-\infty}^{\infty}|u_2^-(x,t)|^2\,dx=\frac{2}{p^2}\sum_n \operatorname{Im}\lambda_2^{(n)}.$$

As $\sigma_2^{+}(\lambda)$ inherits the zeroes of $\sigma_2^{-}(\lambda)$ this (large) contribution from the discrete part of the spectrum is conserved also in the intensity of the second wave as $t\to\infty$. One sees easily from (43) and (44) that if then the pumping amplitude were small, even when a strong packet $u_3^{-}(x)$ were present no pumping wave is produced as $t\to\infty$.

A small change in the pumping intensity (and in its shape) occurs also when the pumping wave collides with small secondary wave packets. The expressions for the amplitudes of these waves can then be obtained in the approximation of a given pumping field.

A nontrivial dynamics occurs when strong packets of the pumping wave and of wave 3 collide.[1)] Assuming that the conditions $qu_1L_1/v \gg 1$, $qu_3L_3/v \gg 1$ are satisfied, we are again led from problem (33) to (41); however, now the potential $V_{\alpha\beta}(x)$ in Eq. (41) is positive. Using the well-known formulae of the quasi-classical approximation we easily find expressions for $\sigma_1^{-}(\lambda)$ and $\sigma_3^{-}(\lambda)$. Substituting then the answers obtained into (43) and (44) we see that if

$$\max|u_3^-(x)|^2>\frac{a_1(a_2-a_3)}{a_2(a_1-a_3)}\max|u_1^-(x)|^2, \tag{45}$$

the integral $\int|u_1^{+}(x,t)|^2dx$ is exponentially small, i.e., the pumping wave decays almost completely. (45) is thus the condition for the annihilation of the pumping wave when strong packets of waves 1 and 3 collide. Using Eq. (10) to eliminate the quantities a_i from (45) we find that the condition for the total decay of the pumping wave is

$$\max|u_3^-(x)|^2>\frac{|v_1-v_2|}{|v_3-v_2|}\max|u_1^-(x)|^2.$$

If, however, this condition is not satisfied, we have

$$\int_{-\infty}^{\infty}|u_1^+(x,t)|^2\,dx=\int dx\left\{1-\frac{2}{\pi}\arcsin\left[\gamma\frac{\max|u_3^-(x)|}{|u_1^-(x)|}\right]\right\}$$
$$-\frac{2}{\pi}\gamma\max|u_3^-(x)|\int\left[|u_1^-(x)|^2-\gamma^2\max|u_3^-(x)|^2\right]^{1/2}dx,\quad \gamma=\left|\frac{v_3-v_2}{v_2-v_1}\right|^{1/2}. \tag{46}$$

The integration in (46) is over the region where

$$|u_1^-(x)|>\gamma\max|u_3^-(x)|.$$

The expression given here is valid provided one maximum of $|u_1^{-}(x)|$ is larger than $\max\gamma|u_3^{-}(x)|$. The intensities of the secondary waves which are formed as a result of the collision are obtained from (46) and the Manley-Rowe relations.

We note the following facts. It follows from (46) that the integral intensities of the waves which occur are independent of the shape of the packet of wave 3 and are determined solely by the maximum value of its amplitude. The result of the interaction is also independent of the magnitude of the interaction constant q.

We note, finally, that (48) does not contain the phases of the colliding packets which is the result of assuming the functions $Q_{\alpha\beta}(x)$ to be smooth, as we did when deriving Eq. (41); for this it is necessary that the phases of the packets vary sufficiently slowly with x.

APPENDIX (JULY 10, 1975)

THE EQUATIONS OF THE INVERSE PROBLEM ($a_3 > 0$)

We give here the equations of the inverse problem for the systems considered; through lack of space we omit their derivation which can be found in [19, 20].

When studying the inverse problem a fundamental role is played by the analytical properties of the solutions of the direct spectral problem. One can show that the functions $\psi_{i1}^{+}(x,\lambda)$ and $\zeta_{i2}^{(+)}(x,\lambda)$, where by definition

$$\zeta_{i2}^{(+)}(x,\lambda)=\psi_{i2}^{+}(x,\lambda)-\frac{S_{12}(\lambda)}{S_{11}(\lambda)}\psi_{i1}^{+}(x,\lambda)$$

(for real λ) allow an analytical continuation into the upper λ-half-plane and the functions $\psi_{i3}^{+}(x, \lambda)$ and

$$\zeta_{i2}^{(-)}(x,\lambda)=\psi_{i2}^{+}(x,\lambda)-\frac{S_{32}(\lambda)}{S_{33}(\lambda)}\psi_{i3}^{+}(x,\lambda)$$

(for real λ) one into the lower one; however, the functions $\zeta_{i2}^{(\pm)}(x, \lambda)$ have then simple poles, respectively, at the zeroes of $S_{11}(\lambda)$ and $S_{33}(\lambda)$ if the latter zeroes are simple ones (as we shall assume in what follows; we shall also assume that none of the zeroes of $S_{11}(\lambda)$ and $S_{33}^{*}(\lambda^{*})$ are coincident). The functions enumerated here can be written in the form (cf. (15) to (18))

$$\psi_{i1}^{+}(x,\lambda)\exp\left(i\frac{\lambda}{a_1}x\right)=\delta_{i1}+\frac{a_1}{a_2}\sum_{n=1}^{N_1}\frac{G\alpha_n^{*}\zeta_{i2}^{-}(x,\lambda_n^{*})\exp(i\lambda_n^{*}x/a_1)}{\lambda-\lambda_n^{*}}$$

$$+\frac{ia_1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{G}{a_2}\frac{S_{12}^{*}(\lambda')}{S_{11}^{*}(\lambda')}\psi_{i2}^{+}(x,\lambda')+\frac{1}{a_3}\frac{S_{13}^{*}(\lambda')}{S_{11}^{*}(\lambda')}\right.$$

$$\left.\times\psi_{i3}^{+}(x,\lambda')\right\}\exp\left(i\frac{\lambda'}{a_1}x\right)\frac{d\lambda'}{\lambda'-\lambda},\qquad(\mathrm{A.1})$$

$$\psi_{i3}^{+}(x,\lambda)\exp\left(i\frac{\lambda}{a_3}x\right)=\delta_{i3}+\frac{a_3}{a_2}\sum_{n=1}^{N_3}\frac{G\beta_n^{*}\zeta_{i2}^{(+)}(x,\mu_n^{*})\exp(i\mu_n^{*}x/a_3)}{\lambda-\mu_n^{*}}\qquad(\mathrm{A.2})$$

$$+\frac{ia_3}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{1}{a_1}\frac{S_{31}^{*}}{S_{33}^{*}}\psi_{i1}^{+}(x,\lambda')+\frac{G}{a_2}\frac{S_{32}^{*}}{S_{33}^{*}}\psi_{i2}^{+}(x,\lambda')\right\}\frac{\exp(i\lambda'x/a_3)\,d\lambda'}{\lambda'-\lambda},$$

$$\exp\left(i\frac{\lambda}{a_2}x\right)\zeta_{i2}^{(+)}(x,\lambda)=\delta_{i2}-\sum_{n=1}^{N_1}\frac{\alpha_n\psi_{i1}^{+}(x,\lambda_n)\exp(i\lambda_n x/a_2)}{\lambda-\lambda_n}$$

$$-\sum_{n=1}^{N_3}\frac{\beta_n\psi_{i3}^{+}(x,\mu_n)\exp(i\mu_n x/a_2)}{\lambda-\mu_n}+\frac{i}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{S_{12}}{S_{11}}\psi_{i1}^{+}(x,\lambda')\right.$$

$$\left.-\frac{S_{32}(\lambda')}{S_{33}(\lambda')}\psi_{i3}^{+}(x,\lambda')\right\}\exp\left(i\frac{\lambda'}{a_2}x\right)\frac{d\lambda'}{\lambda'-\lambda}.\qquad(\mathrm{A.3})$$

These expressions allow us in an obvious way to obtain a closed set of equations for the functions $\psi_{ij}^{+}(x, \lambda)$, Im λ = 0 and the quantities $\psi_{i1}^{+}(x, \lambda_n)$, $\psi_{i3}^{+}(x, \mu_n)$, $\zeta_{i2}^{(+)}(x, \mu_n^{*})$, and $\zeta_{i2}^{(-)}(x, \lambda^{*})$ which is just the set of equations of the inverse problem. The potential $Q_{ij}(x)$ is determined by the solutions of the latter through the formula

$$Q_{i2}(x)=\frac{i}{2\pi a_2}\int_{-\infty}^{\infty}\left[\frac{S_{12}}{S_{11}}\psi_{i1}^{+}-\frac{S_{32}}{S_{33}}\psi_{i3}^{+}\right]\exp\left(i\frac{\lambda}{a_2}x\right)d\lambda$$

$$+\frac{i}{a_2}\sum_{n=1}^{N_1}\alpha_n\psi_{i1}^{+}(x,\lambda_n)\exp\left(i\frac{\lambda_n}{a_2}x\right)+\frac{1}{a_2}\sum_{n=1}^{N_3}\beta_n\psi_{i3}^{+}(x,\mu_n)\exp\left(i\frac{\mu_n}{a_2}x\right),$$

$$Q_{i3}(x)=-\frac{i}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{a_1}\frac{S_{31}^{*}}{S_{33}^{*}}\psi_{i1}^{+}(x,\lambda)+\frac{G}{a_2}\frac{S_{32}^{*}}{S_{33}^{*}}\psi_{i2}^{+}\right]\exp\left(i\frac{\lambda}{a_3}x\right)d\lambda$$

$$-\frac{G}{a_2}\sum_{n=1}^{N_3}\beta_n^{*}\zeta_{i2}^{(+)}(x,\mu_n^{*})\exp\left(i\frac{\mu_n^{*}}{a_3}x\right).$$

[1)]We note that the numbers of the secondary waves are fixed by their velocities. The wave 3 has the intermediate velocity, and the wave 2 the largest or smallest. If we needed to make the ratio of the velocities the opposite one we should put $a_2 < 0$ in the operator L.

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