Lagrangian of an intense electromagnetic field and quantum electrodynamics at short distances

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The next, nonlinear correction after the Heisenberg-Euler correction to the Lagrangian of a constant electromagnetic field is found and takes into account the change of the radiative interaction of the vacuum electrons by the external field. In perturbation theory with respect to the radiation field the Lagrangian of the field is described by a considerably smaller number of diagrams than is the photon polarization function, but contains, in the strong-field limit, the same information as the latter function contains in the limit of large values of the momentum squared. This makes the Lagrangian an object of interest not only in the electrodynamics of an intense field but also in electrodynamics at short distances. Not only is the charge of a real electron determined by the behavior of the Lagrangian of the field, but so too is the mass. The invariance under the renormalization group makes it possible to find an improved perturbation-theory series (and its applicability parameter) for the strong-field asymptotic form of the Lagrangian. This series diverges for an exponentially strong field, in which the lowest approximation to the Langrangian vanishes.

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1. INTRODUCTION

In 1936 Heisenberg and Euler[1] found a quantum correction to the Lagrangian of a constant electromagnetic field; the correction takes into account the polarization of the vacuum, i.e., the change induced by the external field in the motion of the vacuum electrons. With this correction included, the Lagrangian is equal to

$$\mathscr{L}=\frac{\varepsilon^2-\eta^2}{2}+\frac{1}{8\pi^2}\int_0^\infty\frac{ds}{s}e^{-im^2s}\left(\frac{e^2\eta\varepsilon}{\operatorname{tg} e\eta s\ \operatorname{th} e\varepsilon s}-\frac{1}{s^2}+\frac{e^2(\eta^2-\varepsilon^2)}{3}\right). \quad (1)$$

Here ϵ and η are the intensities of the electric and magnetic fields in the reference frame in which they are parallel. Other derivations of the polarization correction to $\mathscr{L}$ have been proposed by Weisskopf[2] and Schwinger[3]. The Lagrangian (1) has served as the basis for nonlinear quantum electrodynamics, in which the effects are sensitive to the intensity of the electromagnetic field and are optimal for fields F of the order of $F_0 = m^2c^3/e\hbar$ (in the proper frame of electrons interacting with a field).

Polarization of the vacuum is induced not only by the external electromagnetic field but also by the individual quanta of this field—the photons. The photon polarization function π_R, which describes this phenomenon, has been found (in different representations) by Serber[4], Uehling[5], Schwinger[6, 3] and Feynman[7]. It constitutes a basis for investigations of quantum electrodynamics at short distances[8-14]. The radiative corrections of fourth order[15-17] and (partially) sixth order[18] have been found for the function π_R.

It is interesting to compare the vacuum polarization by an intense field with the vacuum polarization by quanta of large momenta, thereby linking the electrodynamics of an intense field with the electrodynamics of short distances. For this it is natural to find the next radiative corrections to the Lagrangian of an intense field and, above all, the correction associated with the change induced by the external field in the radiative interaction of the vacuum electrons. This correction is found in the present paper; cf. formula (50). Its simplicity is explained by the compactness of the Green function G_0 that takes the interaction with the external field into account exactly. This also leads to the result that, unlike the corrections to π_R, the radiative corrections to the Lagrangian are described by the minimum possible number of topologically different diagrams (there are 1, 1, 3, ... diagrams for the corrections of order α, α^2, α^3, ..., in contrast to the 1, 2, 10, ... diagrams for π_R).

Comparison of two successive radiative corrections gives the applicability parameter for perturbation theory in the radiation field in electrodynamics. For electrons and photons interacting with an intense field, in the region of high particle energies or large fields this parameter has been found to be the quantity $\alpha\chi^{1/3}\ln\chi$ (cf.[19, 20]), where the invariant $\chi=\sqrt{(eF_{\mu\nu}p_\nu)^2}\,m^{-3}$ is proportional to the field and to the particle momentum and in the proper reference frame is equal to the field intensity in units of F_0. The fact that this parameter differs from the well-known parameter $\alpha\ln s$ of the electrodynamics of short distances has been one of the motivations for carrying out the present work. It is elucidated that in the electrodynamics of an intense field with no real particles the parameter of perturbation theory in the radiation field is $x=(\alpha/\pi)\ln(e\eta/m^2)$ or $(\alpha/\pi)\ln(e\epsilon/m^2)$, while the parameter of the improved perturbation theory is

$$\xi=\max\left[\frac{\alpha}{\pi},\frac{\alpha}{\pi}(1-x)^{-1}\ln(1-x)^{-1}\right].$$

The structure of the article and the principle results are as follows. In the next Section we give the derivation of the correction $\mathscr{L}^{(2)}$ to the Lagrangian that is due to the change of the radiative interaction of the vacuum electrons by a constant external field. In this calculation essential use is made of the Green function of an electron in a constant field, and of the diagonalization of functions of the field matrix $F_{\mu\nu}$. Section 3 is devoted to the renormalization of the external field and of the charge and mass of the electron. Like the charge renormalization, the mass renormalization is carried out directly in the framework of the calculation of the Lagrangian of the electromagnetic field (without a separate treatment of the mass operator or of the position of the pole of the electron Green function), using the general physical renormalization principle that requires that the radiative corrections to the observed charge and mass vanish when the field is switched off. Speaking more generally, from the Lagrangian of a boson field the mass spectrum of the fermions interacting with the bosons is established. For the renormalization of the charge (and of the field), the behavior in a weak field of the real part of the Lagrangian is important, while for the renormalization of the mass the behavior of its imaginary part is important (it is the latter which effects the coupling of the field with a channel of real particles). This is carried out explicitly in Sec. 4, where the imaginary part of the Lagrangian is calculated for weak and strong fields. Also in Sec. 4, it is observed that the asymptotic behavior of the Lagrangian for large fields coincides, with logarithmic accuracy, with the asymptotic behavior of the polarization function at large momenta. A deeper analogy between these quantities is traced in Sec. 5 by means of the renormalization group. This group makes it possible to find for the Lagrangian an improved perturbation-theory series in the radiation field, the region of applicability of which is determined by the smallness of the parameter ξ indicated above. It is instructive that the terms of this series become infinite at the point $x = 1$, at which, according to the zeroth approximation, the Lagrangian ought to vanish.

2. CALCULATION OF THE LAGRANGIAN

As is well known (cf.[21]), the motion of an electron in an external field, with radiative corrections taken into account, is described by the exact Green function $G(x,x')$, obeying the equation

$$(i\gamma\Pi+m)G(x,x')+\int d_4x'' M(x,x'')G(x'',x)=-i\delta(x-x'). \qquad (2)$$

The Green function $G(x,x')$ and mass function $M(x,x')$, which describes the self-energy effects, are conveniently considered as matrix elements of operators G and M in the coordinate representation: $G(x,x') = (x|G|x')$, $M(x,x') = (x|M|x)$. Then Eq. (2) is a matrix element of the operator equation

$$(i\gamma\Pi+m+M)G=-i, \qquad (3)$$

in which the kinetic-momentum operator $\Pi_\alpha = p_\alpha - eA_\alpha$ is characterized by the properties

$$[x_\alpha, \Pi_\beta]=i\delta_{\alpha\beta}, \quad [\Pi_\alpha, \Pi_\beta]=ieF_{\alpha\beta}, \qquad (4)$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the electromagnetic-field stress tensor.

It is also known that, on variation of the potential of the external field by an amount $\delta A_\mu(x)$, the action changes by an amount

$$\delta W=\int d^4x\,\delta A_\mu(x)\langle j_\mu(x)\rangle, \qquad (5)$$

where $\langle j_\mu(x)\rangle$ is the average value of the current operator, connected with the Green function by the relation

$$\langle j_\mu(x)\rangle=-ie\,\mathrm{tr}\,\gamma_\mu(x|G|x). \qquad (6)$$

Here the symbol tr denotes a diagonal sum over the spinor indices. The expression for δW acquires the compact form

$$\delta W=-ie\,\mathrm{Tr}\,\gamma\delta A G, \qquad (7)$$

if we use the matrix form of the function G, denote by δA_μ the operator with matrix elements

$$(x|\delta A_\mu|x')=\delta(x-x')\delta A_\mu(x) \qquad (8)$$

and introduce the symbol Tr for a complete diagonal summation both over the spinor indices and over the space-time coordinates; cf.[3, 21].

For the following it is convenient to introduce the Green function G_0 that does not take the radiative corrections into account, i.e., the Green function obeying Eq. (2) in which the mass function M is equal to zero. Since $-ie\gamma\delta A = \delta(i\gamma\Pi + m)$ and $i\gamma\Pi + m = -iG_0^{-1}$, we have

$$\delta W=-i\,\mathrm{Tr}\,\delta G_0^{-1}G. \qquad (9)$$

This expression for δW is convenient for representing it in the form of a series in the radiative corrections. Thus, if we write Eq. (3) in the form

$$G=G_0-iG_0MG \qquad (10)$$

and confine ourselves to the second-order corrections:

$$G\approx G_0-iG_0M^{(2)}G_0, \qquad (11)$$

then,

$$\delta W\approx-i\,\mathrm{Tr}\,\delta G_0^{-1}G_0-\mathrm{Tr}\,\delta G_0^{-1}G_0M^{(2)}G_0. \qquad (12)$$

Taking into account the identity $\delta G_0^{-1}G_0 + G_0^{-1}\delta G_0 = 0$, we can write the first term in (12) in the form[1)]

$$\delta W^{(1)}=i\,\mathrm{Tr}\,G_0^{-1}\delta G_0=i\delta\,\mathrm{Tr}\ln G_0, \qquad (13)$$

and the second term in the form

$$\delta W^{(2)}=\mathrm{Tr}\,\delta G_0M^{(2)}={}^1\!/_2\delta\,\mathrm{Tr}\,G_0M^{(2)}. \qquad (14)$$

Here we have used the fact that operators can by cyclically permuted under the trace Tr, and also the fact that $M^{(2)}$ depends linearly on G_0:

$$M^{(2)}=ie^2\int\gamma(\xi)G_0\gamma(\xi')D_0(\xi-\xi')d^4\xi d^4\xi' + ie^2\int\mathrm{Tr}[\gamma(\xi)G_0]\gamma(\xi')D_0(\xi-\xi')d^4\xi d^4\xi'. \qquad (15)$$

Here D_0 is the photon Green function without allowance for the radiative corrections and $\gamma(\xi)$ is the operator with matrix elements

$$(x|\gamma_\alpha(\xi)|x')=\gamma_\alpha\delta(\xi-x)\delta(x-x'), \qquad (16)$$

so that $-ie\mathrm{Tr}\,[\gamma_\alpha(\xi)G_0] = \langle j_\alpha(\xi)\rangle$.

Thus, to within an additive constant, the first-order nonlinear corrections to the Lagrangian of the Maxwell field is equal to

$$\mathscr{L}^{(1)}(x)=i\,\mathrm{tr}\,(x|\ln G_0|x) \qquad (17)$$

and corresponds to the diagram 1 in the figure, while the second-order nonlinear correction is equal to

$$\mathscr{L}^{(2)}(x)={}^1\!/_2\,\mathrm{tr}\,(x|M^{(2)}G_0|x) \qquad (18)$$

and is characterized by the diagrams 2a and 2b in the figure. The double line in this figure represents a virtual electron interacting with the external field, and the wavy line represents a virtual photon. In the case of a constant and uniform field $F_{\mu\nu}$, which is considered below, the diagram 2b makes no contribution to $\mathscr{L}^{(2)}$ since the average current $\langle j_\mu(x)\rangle = -ie\mathrm{tr}\gamma_\mu(x|G|x)$ induced in the vacuum by such a field is equal to zero. Thus, in this case the polarization correction $\mathscr{L}^{(2)}$ takes into account the interaction of the electron and positron of one virtual pair.

To calculate

$$\mathscr{L}^{(2)}=\frac{1}{2}ie^2\int d^4x'\,\mathrm{tr}[\gamma_\mu G_0(x,x')\gamma_\mu G_0(x',x)]D_0(x-x') \qquad (19)$$

we shall make use of the Green function of an electron in a constant field, found by Schwinger[3] (cf. also the paper by Fock[22]):

$$G_0(x,x')=\frac{-i\exp[i\eta(x,x')]}{(4\pi)^2}\int_0^\infty\frac{ds}{s^2}\left[m-\frac{i}{2}\gamma(\beta+eF)z\right]\times\exp\left[-im^2s-L(s)+\frac{iz\beta z}{4}+\frac{ie\sigma Fs}{2}\right]. \qquad (20)$$

Here,

$$\beta = eF \operatorname{cth} eFs, \quad L(s) = \frac{1}{2} \operatorname{tr} \ln[(eFs)^{-1} \operatorname{sh} eFs]$$

are a matrix function and scalar function of the matrix F, z = x − x′, and η is the off-diagonal phase of the Green function, equal to the line integral of the potential $eA_\mu(y)$ along the straight line joining the points x′ and x. Substituting into the expression (19) the functions (20) and the photon Green function in the proper-time representation:

$$D_0(z) = \frac{-i}{(4\pi)^2} \int_0^\infty \frac{dt}{t^2} \exp\left(\frac{iz^2}{4t}\right) \tag{21}$$

we obtain

$$\mathscr{L}^{(2)} = \frac{-e^2}{2(4\pi)^6} \iiint_0^\infty \frac{ds\,ds'\,dt}{(ss't)^2} \exp(-im^2(s+s') - L - L') \times \int d^4z \exp\left[\frac{i}{4} z \left(\beta + \beta' + \frac{1}{t}\right) z\right] \operatorname{tr} \Gamma, \tag{22}$$

where

$$\Gamma = \gamma_\mu \left[m - \frac{i}{2}\gamma(\beta + eF)z\right] \exp(ie\sigma Fs/2) \times \gamma_\mu \left[m + \frac{i}{2}\gamma(\beta' + eF)z\right] \exp(ie\sigma Fs'/2).$$

Here and below, primed quantities are obtained from the unprimed ones by the replacement s → s′.

If we represent

$$e^{ie\sigma Fs/2} = S + {}^1\!/_2 ie\sigma FT - i\gamma_5 P + {}^1\!/_2 e\gamma_5 \sigma F T^*, \tag{23}$$

where S and T are scalar, and P and T* pseudoscalar functions of the field F and of the proper time s, and denote the analogous functions in the expansion of $\exp[1/2ie\sigma F(s'-s)]$ by $\bar{S}$, $\bar{T}$, $\bar{P}$ and $\bar{T}^*$ (these are obtained from S, T, P, and T* by the replacement s → s′ − s), then tr Γ is easily calculated and acquires the form

$$\operatorname{tr} \Gamma = 2\{8m^2(SS' + PP') - [B'(\bar{S} + eF\bar{T} + eF^*\bar{T}^*)B]_{\beta\alpha} z_\alpha z_\beta\}, \tag{24}$$

where the matrix B = β + eF, the matrix $\tilde{B}'$ − eF, and $F^*_{\mu\nu} = 1/2i\epsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma}$ is the dual of the tensor $F_{\mu\nu}$. To calculate the integral over z in (22) we shall consider the quadruple Gaussian integral

$$I = \int d^4z \exp\left(-ipz + \frac{izAz}{4}\right) = \frac{i(4\pi)^2}{\sqrt{\det A}} \exp(-ipA^{-1}p), \tag{25}$$

in which A is a symmetric 4 × 4 matrix and p is a 4-vector. By differentiating with respect to p we obtain

$$\int d^4z\, z_\alpha z_\beta \exp\left(-ipz + \frac{izAz}{4}\right) = I[(2A^{-1}p)_\alpha (2A^{-1}p)_\beta + 2iA_{\alpha\beta}^{-1}]. \tag{26}$$

Now, putting p = 0, we obtain in our case

$$\int d^4z \operatorname{tr} \Gamma e^{izAz/4} = \frac{i(4\pi)^2 4}{\sqrt{\det A}} \left\{4m^2(SS' + PP') - i \operatorname{tr}[B'(\bar{S} + eF\bar{T} + eF^*\bar{T}^*)BA^{-1}]\right\}, \tag{27}$$

where the matrix $A = \beta + \beta' + t^{-1}$, and tr, as before, denotes diagonal summation.

Concrete expressions for all the functions appearing in formula (27) are conveniently obtained by means of the diagonal representation of the matrix $F_{\mu\nu}$. As is well-known (cf., e.g.,[23]), for an arbitrary constant field there exists a coordinate frame in which the magnetic and electric fields are parallel; their magnitudes η and ϵ in this frame are relativistic invariants of the field. Using the field matrix $F_{\mu\nu}$ in this frame and solving the characteristic equation $\det(F - \lambda) = 0$, we find that the eigenvalues of the field matrix are, to within phase factors, just the magnetic and electric fields in this frame:

$$\lambda_{1,2} = \pm i\eta, \quad \lambda_{3,4} = \pm\varepsilon. \tag{28}$$

By means of the unitary transformation

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & 1 \end{pmatrix} \tag{29}$$

the field matrix $F_{\mu\nu}$ can be brought to the diagonal form $F^d = U^+FU$ with the eigenvalues (28) on the diagonal. In the diagonal representation, matrix functions of the matrix F become diagonal matrices with the corresponding functions of the eigenvalues (28) on the diagonal.

As a result, we obtain

$$\sqrt{\det A} = [e\eta(\operatorname{ctg} e\eta s + \operatorname{ctg} e\eta s') + t^{-1}][e\varepsilon(\operatorname{cth} e\varepsilon s + \operatorname{cth} e\varepsilon s') + t^{-1}], \tag{30}$$

$$e^{-L} = \left[\det\left(\frac{\operatorname{sh} eFs}{eFs}\right)\right]^{-1/2} = \frac{e\eta s}{\sin e\eta s} \frac{e\varepsilon s}{\operatorname{sh} e\varepsilon s}, \tag{31}$$

where in the latter relation we have used the fact that the trace of the logarithm of a matrix is equal to the logarithm of its determinant—a property which is obvious in the diagonal representation.

To determine the invariant functions S, T, etc., appearing in the expansion (23) and formula (27), we note that in the aforementioned coordinate frame $1/2\sigma F = (\eta + ie\gamma_5\Sigma_3$ and, since the matrices γ_5 and Σ_3 have eigenvalues ±1, the eigenvalues of the matrix $1/2\sigma F$ will be $\pm(\eta + i\epsilon$'m $\pm(\eta - u\epsilon)$. Then,

$$S = \cos e\eta s \operatorname{ch} e\varepsilon s, \quad P = \sin e\eta s \operatorname{sh} e\varepsilon s,$$
$$T = \frac{\eta \sin e\eta s \operatorname{ch} e\varepsilon s + \varepsilon \cos e\eta s \operatorname{sh} e\varepsilon s}{e(\eta^2 + \varepsilon^2)}, \tag{32}$$
$$T^* = \frac{\varepsilon \sin e\eta s \operatorname{ch} e\varepsilon s - \eta \cos e\eta s \operatorname{sh} e\varepsilon s}{e(\eta^2 + \varepsilon^2)}.$$

Using these functions and the eigenvalues of the corresponding matrices, we find[2)]

$$\operatorname{tr}[B'(\bar{S} + eF\bar{T} + eF^*\bar{T}^*)BA^{-1}] = \frac{p}{a + t^{-1}} + \frac{q}{b + t^{-1}} \tag{33}$$

where

$$a = e\eta(\operatorname{ctg} e\eta s + \operatorname{ctg} e\eta s'), \quad b = e\varepsilon(\operatorname{cth} e\varepsilon s + \operatorname{cth} e\varepsilon s'),$$
$$p = \frac{2(e\eta)^2 \operatorname{ch} e\varepsilon(s' - s)}{\sin e\eta s \sin e\eta s'}, \quad q = \frac{2(e\varepsilon)^2 \cos e\eta(s' - s)}{\operatorname{sh} e\varepsilon s \operatorname{sh} e\varepsilon s'}. \tag{34}$$

Substituting the expressions (30)–(33) into formulas (27) and (22) and performing the integration over the photon proper time t, we obtain for $\mathscr{L}^{(2)}$ the following expression:

$$\mathscr{L}^{(2)} = \frac{-i\alpha}{32\pi^3} \iint_0^\infty \frac{ds\,ds' \exp[-im^2(s + s')](e^2\eta\varepsilon)^2}{\sin e\eta s \sin e\eta s' \operatorname{sh} e\varepsilon s \operatorname{sh} e\varepsilon s'} [4m^2(SS' + PP')I_0 - iI], \tag{35}$$

in which I_0 and I are functions arising as a result of the integration over t:

$$I_0 = \frac{1}{b - a} \ln \frac{b}{a}, \quad I = \frac{q - p}{(b - a)^2} \ln \frac{b}{a} - \frac{qb^{-1} - pa^{-1}}{b - a}. \tag{36}$$

The integration contours pass below the singularities on the real axes s, s′ (which is equivalent to a small damping[1] or to the rule $m^2 \to m^2 - i\delta$) and can be deformed with allowance for the singularities of the integrand.

1 2a 2b

3. RENORMALIZATION

The expression (35) obtained for $\mathscr{L}^{(2)}$ does not vanish when the field is switched off. This is connected with the fact that the initial expression (18) was defined to within an additive constant. We denote the integrand in (35) by f(s,s′) and expand it in the field

$$f(s,s')=f_0(s,s')+f_2(s,s')+\dots \quad (37)$$

In this expansion, the first term

$$f_0(s,s')=\frac{\exp[-im^2(s+s')]}{ss'(s+s')}\left(4m^2-\frac{2i}{s+s'}\right) \quad (38)$$

does not depend on the field, while the next term, quadratic in the field,

$$f_2(s,s')=\frac{e^2(\eta^2-\varepsilon^2)}{3}\frac{\exp[-im^2(s+s')]}{s+s'}\left[2m^2\left(1-\frac{2s}{s'}-\frac{2s'}{s}\right)-\frac{5i}{s+s'}\right] \quad (39)$$

is proportional to the Lagrangian of the Maxwell field:

$$\mathscr{L}^{(0)}=(\varepsilon^2-\eta^2)/2. \quad (40)$$

The subsequent terms of the expansion are proportional to the fourth, sixth, etc., powers of the field. Therefore, $\mathscr{L}^{(2)}$ can be represented in the following form:

$$\mathscr{L}^{(2)}=-\mathscr{L}^{(0)}\frac{\alpha^2}{4\pi^2}\left(\ln\frac{1}{i\gamma m^2s_0}-\frac{5}{3}+\frac{5}{3}\ln 2\right) -\frac{i\alpha}{32\pi^3}\iint\limits_0^\infty ds\,ds'(f-f_0-f_2), \quad (41)$$

if we omit the infinite constant–the integral of f_0 (which it is necessary to do if $L^{(2)}$ is to vanish in the absence of the field) and if we calculate the logarithmically divergent coefficient in the term proportional to $L^{(0)}$, i.e., the integral of f_2, by introducing an invariant cutoff with respect to the proper times s and s′ by means of equal lower limits s_0. In the expression (41), $\ln\gamma = 0.577\dots$ is the Euler constant.

The first term in (41) must be added to the Lagrangian of the Maxwell field. As a result, this function is multiplied by a logarithmically divergent factor, which should be included in the change of magnitude of all the fields and in the corresponding change of magnitude of the charge, i.e., in the renormalization of the field and charge. We shall discuss this somewhat later, and concentrate our attention now on the principal, integral term in (41), denoting its integrand by

$$K(s,s')=f(s,s')-f_0(s,s')-f_2(s,s'). \quad (42)$$

It is not difficult to see that this integral still diverges logarithmically as $s\to 0$, $s'\neq 0$ ($s'\to 0$, $s\neq 0$). Using the symmetry of the function K(s, s′) under the interchange $s\rightleftarrows s'$, we transform this integral as follows:

$$\frac{-i\alpha}{16\pi^3}\int\limits_0^\infty ds\int\limits_0^s ds'K(s,s')=\frac{-i\alpha}{16\pi^3}\int\limits_0^\infty ds\int\limits_0^s ds'\left[K(s,s')-\frac{K_0(s)}{s'}\right] -\frac{i\alpha}{16\pi^3}\int\limits_0^\infty ds\int\limits_{s_0}^s ds'\frac{K_0(s)}{s'}. \quad (43)$$

In the right-hand side we have subtracted from, and added to, the function K its limiting expression as $s'\to 0$:

$$\lim K(s,s')=K_0(s)/s',\quad s'\to 0;$$
$$K_0(s)=e^{-im^2s}\left(4m^2+i\frac{\partial}{\partial s}\right)\left(\frac{e^2\eta\varepsilon}{\operatorname{tg} e\eta s\,\operatorname{th} e\varepsilon s}-\frac{1}{s^2}+\frac{e^2(\eta^2-\varepsilon^2)}{3}\right). \quad (44)$$

In formula (43) the aforementioned logarithmic divergence has been separated out into the last term, which has been regularized by the introduction of the lower limit s_0 for the proper time s′. Thus, after integration over s′ this term acquires the form

$$-\frac{i\alpha}{16\pi^3}\ln\frac{1}{i\gamma m^2s_0}\int\limits_0^\infty ds K_0(s)-\frac{i\alpha}{16\pi^3}\int\limits_0^\infty ds K_0(s)\ln i\gamma m^2 s. \quad (45)$$

By means of integration by parts the integral of the function K_0 can be transformed to the expression

$$\frac{-i\alpha}{16\pi^3}\ln\frac{1}{i\gamma m^2s_0}\int\limits_0^{\infty} ds K_0(s)=\frac{3\alpha m^2}{2\pi}\ln\frac{1}{i\gamma m^2s_0}\cdot\frac{\partial\mathscr{L}_R^{(1)}(m^2)}{\partial m^2}, \quad (46)$$

which is proportional to the derivative of the Heisenberg-Euler Lagrangian function

$$\mathscr{L}_R^{(1)}=\frac{1}{8\pi^2}\int\limits_0^\infty\frac{ds\,e^{-im^2s}}{s}\left(\frac{e^2\eta\varepsilon}{\operatorname{tg} e\eta s\,\operatorname{th} e\varepsilon s}-\frac{1}{s^2}+\frac{e^2(\eta^2-\varepsilon^2)}{3}\right). \quad (47)$$

with respect to the mass squared. This proportionality is not accidental: it is necessary for the renormalization of the mass in the expression for $L_R^{(1)}$. The proportionality coefficient $(3\alpha m^2/2\pi)\ln(i\gamma m^2 s_0)^{-1}$ in formula (46) is, to within a constant added to the logarithm, the radiative correction to the square of the "bare" mass of the electron. This additive constant can be taken from the work of Schwinger [3], in which the electromagnetic part of the electron mass was found in the proper-time representation used in the present work; it turns out to be equal to 5/6:

$$\delta m^2=\frac{3\alpha m^2}{2\pi}\left(\ln\frac{1}{i\gamma m^2s_0}+\frac{5}{6}\right). \quad (48)$$

It is not necessary to take the additive constant 5/6 from Schwinger's work or, in general, from any independent treatment of the mass operator and of the position of the pole of the electron Green function; the constant can be derived in the framework of the present calculation of the Lagrangian of the electromagnetic field. Below we shall show that the principle of the renormalization of the electron mass, which makes it possible to find the exact expression (48) for the radiative correction to the electron mass and, consequently, the relation $m^2 = m_0^2 + \delta m^2$ between the real and bare electron masses, is intrinsic to the calculation of the Lagrangian of the electromagnetic field.

When we add the constant 5/6 to the logarithm in the first term of (45), we must subtract it from the logarithm in the second term of (45).

Thus, $\mathscr{L}^{(2)}$ is represented in the following form:

$$\mathscr{L}^{(2)}=-\mathscr{L}^{(0)}\frac{\alpha^2}{4\pi^2}\left(\ln\frac{1}{i\gamma m^2s_0}-\frac{5}{3}+\frac{5}{3}\ln 2\right)+\delta m^2\frac{\partial\mathscr{L}_R^{(1)}}{\partial m^2}+\mathscr{L}_R^{(2)}, \quad (49)$$

where

$$\mathscr{L}_R^{(2)}=\frac{-i\alpha}{16\pi^3}\int\limits_0^\infty ds\int\limits_0^s ds'\left[K(s,s')-\frac{K_0(s)}{s'}\right] -\frac{i\alpha}{16\pi^3}\int\limits_0^\infty ds K_0(s)\left(\ln i\gamma m^2 s-\frac{5}{6}\right), \quad (50)$$

and δm^2 is determined by formula (48) and is the electromagnetic part of the square of the electron mass, so that $m^2 = m_0^2 + \delta m^2$ is the square of the real-electron mass. At the same time, the first-order radiative correction to the Lagrangian has the form

$$\mathscr{L}^{(1)}=\mathscr{L}^{(0)}\frac{\alpha_0}{3\pi}\ln\frac{1}{i\gamma m_0^2s_0}+\mathscr{L}_R^{(1)}(m_0^2), \quad (51)$$

where $\mathscr{L}_R^{(1)}$ is determined by formula (47) and the unrenormalized fine-structure constant and electron mass are indicated by the subscript zero.

Thus, to within radiative corrections of second order in α, the nonlinear Lagrangian of a constant field is equal to

$$\mathscr{L}=\mathscr{L}^{(0)}+\mathscr{L}^{(1)}+\mathscr{L}^{(2)}=\mathscr{L}_R+\mathscr{L}_R^{(1)}(m^2)+\mathscr{L}_R^{(2)}(m^2), \quad (52)$$

in which the latter expression contains only the renormalized values of the field and of the electron charge and mass. In fact, the unrenormalized Lagrangian $\mathscr{L}^{(0)} = 1/2(\epsilon_0^2 - \eta_0^2)$ of the Maxwell field and the terms proportional to it in the expressions (51) and (49) are collected into the renormalized Lagrangian $\mathscr{L}_R^{(0)} = 1/2(\epsilon^2 - \eta^2)$:

$$\mathscr{L}_R^{(0)} = \mathscr{L}^{(0)} Z_3^{-1}, \qquad (53)$$
$$Z_3^{-1} = 1 + \frac{\alpha_0}{3\pi} \ln \frac{1}{i\gamma m_0^2 s_0} - \frac{\alpha_0^2}{4\pi^2}\left(\ln \frac{1}{i\gamma m_0^2 s_0} - \frac{5}{3} + \frac{5}{3}\ln 2\right),$$

thereby leading to renormalization of the field and charge:

$$\eta = \eta_0 Z_3^{-1/2}, \quad \varepsilon = \varepsilon_0 Z_3^{-1/2}, \quad e = e_0 Z_3^{1/2} \qquad (54)$$

(we have indicated the unrenormalized values of the field and charge by the subscript zero), and the term $\delta m^2 \partial \mathscr{L}_R^{(1)}/\delta m^2$ in (49) effects the renormalization of the electron mass:

$$\mathscr{L}_R^{(1)}(m_0^2) + \delta m^2 \frac{\partial \mathscr{L}_R^{(1)}(m_0^2)}{\partial m_0^2} = \mathscr{L}_R^{(1)}(m^2); \qquad (55)$$

the functions $\mathscr{L}_R^{(1)}(m^2)$ and $\mathscr{L}_R^{(2)}(m^2)$ are determined by formulas (47) and (50). The argument $e_0\eta_0 = e\eta$, $e_0\epsilon_0 = e\epsilon$ of these functions are invariant under the renormalization (54). The factor Z_3^{-1} can be expressed by means of (48) in terms of the renormalized electron mass:

$$Z_3^{-1} = 1 + \frac{\alpha_0}{3\pi} \ln \frac{1}{i\gamma m^2 s_0} + \frac{\alpha_0^2}{4\pi^2}\left(\ln \frac{1}{i\gamma m^2 s_0} + \frac{10}{3} - \frac{5}{3}\ln 2\right). \qquad (56)$$

The coefficients of the logarithms in this expression coincide with those known in the literature; cf., e.g.,[14].

4. ASYMPTOTIC PROPERTIES AND THE RENORMALIZATION PRINCIPLE

We shall find the limiting expressions for the Lagrangian function $\mathscr{L}_R^{(2)}$ for weak and strong fields, and consider its imaginary part in detail.

Expanding the functions $K(s, s')$ and $K_0(s)$ in the field (the lowest terms of these expansions, proportional to the fourth power of the field, are given in Appendix A) and integrating these expansions over the proper times s and s′, we obtain, according to formula (50),

$$\mathscr{L}_R^{(2)} = \frac{\alpha^3}{\pi m^4}\left[\frac{16}{81}(\eta^2 - \varepsilon^2)^2 + \frac{263}{162}(\eta\varepsilon)^2\right] + \ldots; \quad \frac{e\eta}{m^2}, \frac{e\varepsilon}{m^2} \ll 1. \qquad (57)$$

In the expression (57) the imaginary part of $\mathscr{L}_R^{(2)}$, which is exponentially small compared with the real part (cf. below) and cannot be represented by a series in powers of the field, is absent. In general, the function $\mathscr{L}_R^{(2)}$ has an essential singularity at the point $eF = 0$, so that the series (57) for the real part of $\mathscr{L}_R^{(2)}$ is asymptotic.

In the case of a strong magnetic field ($e\eta/m^2 \gg 1$, $e\epsilon/m^2 \lesssim 1$) the calculations are conveniently carried out by first rotating the contours of integration over the proper times: $s, s' \to -is, -is'$. We then obtain, with logarithmic accuracy,

$$\mathscr{L}_R^{(2)} \approx \frac{\alpha^2\eta^2}{8\pi^2}\left(\ln \frac{e\eta}{\gamma\pi m^2} + a_2\right), \quad \frac{e\eta}{m^2} \gg 1, \quad \frac{e\varepsilon}{m^2} \lesssim 1. \qquad (58)$$

The uncalculated constant a_2, additive to the logarithm, is real. The case of a strong electric field ($e\epsilon/m^2 \gg 1$, $e\eta/m^2 \lesssim 1$) can be obtained from (58) by the transformation $\eta \to -i\epsilon$:

$$\mathscr{L}_R^{(2)} \approx -\frac{\alpha^2\varepsilon^2}{8\pi^2}\left(\ln \frac{e\varepsilon}{\gamma\pi m^2} - \frac{i\pi}{2} + a_2\right), \quad \frac{e\varepsilon}{m^2} \gg 1, \quad \frac{e\eta}{m^2} \lesssim 1. \qquad (59)$$

In this way the correct asymptotic expression for the imaginary part of $\mathscr{L}_R^{(2)}$ is obtained. (With regard to the transformation $\eta \rightleftarrows -i\epsilon$ and the imaginary part of the Lagrangian, see below.) It is useful to compare formulas (57) and (58) with the corresponding formulas for $\mathscr{L}_R^{(1)}$:

$$\mathscr{L}_R^{(1)} = \begin{cases} \dfrac{2\alpha^2}{45m^4}[(\eta^2 - \varepsilon^2)^2 + 7(\eta\varepsilon)^2], & \dfrac{e\eta}{m^2}, \dfrac{e\varepsilon}{m^2} \ll 1, \\[2ex] \dfrac{\alpha\eta^2}{6\pi}\left(\ln \dfrac{e\eta}{\gamma\pi m^2} + \dfrac{6}{\pi^2}\zeta'(2)\right), & \dfrac{e\eta}{m^2} \gg 1, \quad \dfrac{e\varepsilon}{m^2} \lesssim 1 \end{cases}. \qquad (60)$$

Here $6\pi^{-2}\zeta'(2) = -0.5699610\ldots$; $\zeta(x)$ is the Riemann zeta-function. As can be seen from the asymptotic formulas obtained, the function $\mathscr{L}_R^{(2)} \sim \alpha\mathscr{L}_R^{(1)}$, irrespective of the magnitude of the field. In Sec. 5 it will be shown that this is an exception, and the ratio of the subsequent radiative corrections will be of order $(\alpha/\pi)\ln(eF/m^2)$ for large fields.

A remarkable property of the functions $\mathscr{L}_R^{(1)}$ and $\mathscr{L}_R^{(2)}$ is the fact that for large magnetic (electric) fields the ratios

$$\frac{\mathscr{L}_R^{(1)}}{\mathscr{L}_R^{(0)}} \approx \frac{\alpha}{\pi}\left(-\frac{1}{3}\ln \frac{e\eta}{\gamma\pi m^2} - \frac{2}{\pi^2}\zeta'(2)\right),$$
$$\frac{\mathscr{L}_R^{(2)}}{\mathscr{L}_R^{(0)}} \approx \left(\frac{\alpha}{\pi}\right)^2\left(-\frac{1}{4}\ln \frac{e\eta}{\gamma\pi m^2} + a_{20}\right), \qquad (61)$$

behave, with logarithmic accuracy, in the same way as the polarization functions of second and fourth order in e at large values of the squares of the space-like (time-like) momenta[3)]

$$\pi_R^{(2)} \approx \frac{\alpha}{\pi}\left(-\frac{1}{3}\ln \frac{k^2}{m^2} + \frac{5}{9}\right), \quad \pi_R^{(4)} \approx \left(\frac{\alpha}{\pi}\right)^2\left(-\frac{1}{4}\ln \frac{k^2}{m^2} + \frac{5}{24} - \zeta(3)\right). \qquad (62)$$

By the nature of the problem, the Lagrangian and the polarization function describe the same phenomenon and are determined by the effective value of the operator Π^2 ($\Pi_\alpha = p_\alpha - eA_\alpha$) responsible for the interaction of the vacuum electrons with the quanta or with the field. At large momenta of the quanta, or high fields, the average value of Π^2 becomes of the order of $k^2 \gg m^2$ for the quanta and of the order of $(eFx)^2 \sim eF \gg m^2$ for the field, since in this case the Lagrangian is formed over distances $x \sim (eF)^{-1/2}$ that are short compared with the Compton wavelength.

Therefore, the Lagrangian of the constant field also correctly describes the polarization corrections for varying fields, if the fields are sufficiently intense. For example, the well-known Uehling-Serber correction to the Coulomb field at short distances:

$$\varepsilon = \frac{q}{4\pi r^2}\left[1 + \frac{\alpha}{3\pi}\left(\ln \frac{1}{(\gamma m r)^2} - \frac{5}{3}\right)\right], \quad r \ll m^{-1}, \qquad (63)$$

calculated with the aid of the polarization function π_R, can also be found, with logarithmic accuracy, with the aid of the Lagrangian $\mathscr{L} = \epsilon^2/2 - (\alpha\epsilon^2/6\pi)\ln(e\epsilon/\gamma\pi m^2)$ of a constant electric field, if we solve for ϵ the expression for the induction

$$\frac{q}{4\pi r^2} = \frac{\partial \mathscr{L}}{\partial \varepsilon} \approx \varepsilon\left(1 - \frac{\alpha}{3\pi}\ln \frac{e\varepsilon}{\gamma\pi m^2}\right), \qquad (64)$$

which is equal to $q/4\pi r^2$ since its divergence is equal to zero.

We consider now the imaginary part of the Lagrangian function $\mathscr{L}^{(2)}$. We recall, in this connection, that $e^{i\mathscr{L}VT}$ is the amplitude and $|e^{i\mathscr{L}VT}|^2 = \exp(-2\,\mathrm{Im}\mathscr{L}VT)$

the probability that the field in volume V remains the field for a time T. Therefore, $2\,\mathrm{Im}\,\mathscr{L}$ is the probability of real creation of pairs and photons by the field, per unit time and unit volume. Here $2\,\mathrm{Im}\,\mathscr{L}_R^{(1)}$ is the probability of creation of a pair, and $2\,\mathrm{Im}\,\mathscr{L}_R^{(2)}$ is the sum of the probability of formation of a pair and a photon and the radiative correction to the probability of formation of a pair.

The imaginary part of the Lagrangian of a constant field is nonzero only if $\epsilon \neq 0$—only the electric field can do work and lead to real production of particles. Therefore, for simplicity, we shall confine ourselves to the case when $\eta = 0$ and $\epsilon \neq 0$. It is convenient to rotate the proper-time integration contours to lie along the negative imaginary half-axis: $s \to -is$, $s' \to -is'$. Then the expression (50) for $\mathscr{L}_R^{(2)}$ becomes

$$\mathscr{L}_R^{(2)}=\frac{i\alpha}{16\pi^3}\int_0^\infty ds\int_0^s ds'\left[K(-is,-is')-\frac{iK_0(-is)}{s'}\right]-\frac{\alpha}{16\pi^3}\int_0^\infty ds\,K_0(-is)\left(\ln\gamma m^2 s-\frac{5}{6}\right). \tag{65}$$

Here,

$$K(-is,-is')=\frac{\exp[-m^2(s+s')](e\varepsilon)^2}{ss'\sin e\varepsilon s\sin e\varepsilon s'}[4m^2\cos e\varepsilon s\cos e\varepsilon s' I_0-iI]+\frac{i\exp[-m^2(s+s')]}{ss'(s+s')^2}\left[4m^2(s+s')+2+\frac{(e\varepsilon)^2}{3}ss'\left(2m^2(s+s')\left(1-\frac{2s}{s'}-\frac{2s'}{s}\right)+5\right)\right], \tag{66}$$

$$K_0(-is)=e^{-m^2 s}\left(4m^2-\frac{\partial}{\partial s}\right)\left(\frac{-e\varepsilon}{s\,\mathrm{tg}\,e\varepsilon s}+\frac{1}{s^2}-\frac{(e\varepsilon)^2}{3}\right), \tag{67}$$

where I_0 and I are the functions (36) in which, now,

$$a=i\frac{s+s'}{ss'},\qquad b=\frac{ie\varepsilon\sin e\varepsilon(s+s')}{\sin e\varepsilon s\sin e\varepsilon s'},\qquad p=-\frac{2}{ss'}\cos e\varepsilon(s'-s),\qquad q=\frac{-2(e\varepsilon)^2}{\sin e\varepsilon s\sin e\varepsilon s'}. \tag{68}$$

It can be seen from (65) that $\mathrm{Im}\,\mathscr{L}_R^{(2)}$ arises on account of taking an upper path around the poles of the integrand (the contribution from the poles) and on account of the imaginary part of the logarithms $\ln(b/a)$ in the regions of their branch-cuts where $b/a \le 0$ (the contribution of the branch-cuts).

In the case of a weak electric field ($e\epsilon/m^2 \equiv \beta \ll 1$) we can confine ourselves to the contribution of the pole and branch-cut nearest to the origin, since these contributions are proportional to $\exp(-\pi/\beta)$ while the contributions of the singularities further from the origin are proportional to $\exp(-k\pi/\beta)$, $k = 2, 3, \ldots$. Then the imaginary part of the second term in (55) is equal to

$$\mathrm{Im}\left[-\frac{\alpha}{16\pi^3}\int_0^\infty ds\,K_0(-is)\left(\ln\gamma m^2 s-\frac{5}{6}\right)\right]=-\frac{\alpha(e\varepsilon)^2e^{-\pi/\beta}}{16\pi^4}\left[\frac{3\pi}{\beta}\left(\ln\frac{\gamma\pi}{\beta}-\frac{5}{6}\right)+1\right]+\ldots, \tag{69}$$

where, in the right-hand side, only the half-residue at the nearest pole $e\epsilon s = \pi$ is given, and the small (for $\beta \ll 1$) contribution of the other poles is denoted by the dots. To calculate the imaginary part of the first term in (65) it is convenient to change from s, s' to the dimensionless variables x, ξ in accordance with the formulas

$$e\varepsilon s=(1-\xi)x,\qquad e\varepsilon s'=\xi x,\qquad \int_0^\infty ds\int_0^s ds'=(e\varepsilon)^{-2}\int_0^{1/2}d\xi\int_0^\infty dx\,x. \tag{70}$$

Then,

$$\mathrm{Im}\left\{\frac{i\alpha}{16\pi^3}\int_0^\infty ds\int_0^s ds'\left[K(-is,-is')-\frac{iK_0(-is)}{s'}\right]\right\}=-\frac{\alpha m^2 e\varepsilon}{16\pi^3}e^{-\pi/\beta}\int_0^{1/2}d\xi g(\xi,\beta)+\ldots, \tag{71}$$

where

$$g(\xi,\beta)=\int_\pi^{\pi/(1-\xi)}dx\,f(\xi,x)e^{-(x-\pi)/\beta}-\frac{2\pi\beta}{\sin^2\xi\pi}-\frac{3}{\xi(1-\xi)} \tag{72}$$

$$f(\xi,x)=\frac{4\pi\,\mathrm{ctg}\,\xi x\,\mathrm{ctg}(1-\xi)x}{b/a-1}+\frac{2\pi\beta}{(b/a-1)^2}\frac{\xi(1-\xi)x\,\mathrm{cosec}\,\xi x\,\mathrm{cosec}(1-\xi)x-x^{-1}\cos(1-2\xi)x}{\sin\xi x\sin(1-\xi)x}. \tag{73}$$

The first, integral term of the function $g(\xi)$ is the contribution from the nearest branch cut $\pi \le x \le \pi(1-\xi)^{-1}$ of the logarithm, where $\mathrm{Im}[\ln(b/a)] = -\pi$. The second term of the function $g(\xi)$ is the contribution of the nearest pole, at the point $x = \pi$, which is possessed by the term $qb^{-1} = 2ie\epsilon/\sin x$ appearing in I (cf. (36) and (68)). Finally, the last term of the function $g(\xi)$ is the contribution of the nearest pole $e\epsilon s = \pi$ of the function $-iK_0(-is)/s'$. The dots in (71) denote the contribution of the singularities further from the origin; for $\beta \ll 1$ this contribution is exponentially small compared with the one cited. The calculation of the integral of the function $g(\xi)$ over ξ is carried out in Appendix B and leads to the following result for (71):

$$\frac{\alpha(e\varepsilon)^2}{16\pi^4}e^{-\pi/\beta}\left[\frac{3\pi}{\beta}\left(\ln\frac{\gamma\pi}{\beta}-\frac{5}{6}\right)+1+2\pi^2+\ldots\right], \tag{74}$$

in which ... denotes terms of order β.

Thus, the sum of the expressions (74) and (69), which constitutes the imaginary part of $\mathscr{L}_R^{(2)}$, is equal to

$$\mathrm{Im}\,\mathscr{L}_R^{(2)}=\frac{\alpha(e\varepsilon)^2}{8\pi^2}e^{-\pi/\beta}(1+\ldots),\qquad \beta\ll 1. \tag{75}$$

It is also useful to give the sum $\mathrm{Im}\,\mathscr{L}_R = \mathrm{Im}(\mathscr{L}_R^{(1)}+\mathscr{L}_R^{(2)})$, in which the above expression (75) plays the role of the radiative correction:

$$\mathrm{Im}\,\mathscr{L}_R=\frac{(e\varepsilon)^2}{8\pi^3}e^{-\pi/\beta}(1+\pi\alpha+\ldots),\qquad \beta\ll 1. \tag{76}$$

We call attention to an important circumstance: the leading terms $(3\pi/\beta)[\ln(\gamma\pi/\beta) - 5/6]$ of the expressions (74) and (69) have cancelled each other in the sum (75). This cancellation arises from the correct choice of the constant 5/6 added to the logarithm in the expression (48) for δm^2, i.e., it arises from the correct definition of the electromagnetic mass of the electron. If another constant b had been taken in place of 5/6 in formula (48), it would have appeared in the form of the constant $-b$ added to the logarithm in the second term of the expressions (50) and (65) for $\mathscr{L}_R^{(2)}$ and in formula (69) for the imaginary part of this term (as we have seen, the origin of this second term is intimately connected with the renormalization of the electron mass). At the same time, the first, principal term of the expressions (50) and (65) for $\mathscr{L}_R^{(2)}$ and its imaginary part (74) would not have changed. Thus, for $\mathrm{Im}\,\mathscr{L}_R^{(2)}$ we would then have obtained

$$\frac{\alpha(e\varepsilon)^2}{16\pi^4}e^{-\pi/\beta}\left[\frac{3\pi}{\beta}\left(b-\frac{5}{6}\right)+2\pi^2+\ldots\right], \tag{77}$$

and $\mathrm{Im}\,\mathscr{L}_R$ would have had the form

$$\frac{(e\varepsilon)^2}{8\pi^3}e^{-\pi/\beta}\left[1+\frac{3\alpha m^2}{2e\varepsilon}\left(b-\frac{5}{6}\right)+\pi\alpha+\ldots\right]=\frac{(e\varepsilon)^2}{8\pi^3}\exp\left\{-\frac{\pi}{e\varepsilon}\left[m^2-\frac{3\alpha m^2}{2\pi}\left(b-\frac{5}{6}\right)-\alpha e\varepsilon\right]\right\}. \tag{78}$$

In the latter form the radiative correction has been transferred into the exponent. It can be seen from this expression that if $b \neq 5/6$, then, even in the limit of an arbitrarily weak field, a finite radiative correction $-(3\alpha m^2/2\pi)(b-5/6)$ is added to the parameter m^2. Therefore, m cannot be the observable mass of a free electron, since, according to the physical principle of renormalization, all radiative corrections are already included in the observed free-electron mass. Thus, m has the meaning of the electron mass only if $b = 5/6$.

Nevertheless, in a formal renormalization we can choose $b \neq 6$, but in this case $\delta m^2 = (3\alpha m^2/2\pi)[\ln(i\gamma m^2 s_0)^{-1} + b]$ is not the electromagnetic part of the square of the electron mass, and $m^2 = m_0^2 + \delta m^2$ is not the square of the electron mass. In this case, the square of the electron mass will be the quantity (cf. (78))

$$m^2 - \frac{3\alpha m^2}{2\pi}\left(b - \frac{5}{6}\right) = m_0^2 + \frac{3\alpha m_0^2}{2\pi}\left(\ln\frac{1}{i\gamma m_0^2 s_0} + \frac{5}{6}\right), \quad (79)$$

which, of course, coincides with m^2 when $b = 5/6$. The arbitrariness in the quantity b, which leads to arbitrariness in the magnitude and meaning of the parameter m, lies at the basis of the renormalization group.

Thus, knowing the imaginary part of the Lagrangian of a weak electromagnetic field and using the physical principle of renormalization, we can find the electromagnetic mass of the electron and relate its observed mass to the bare mass. This is explained by the fact that the imaginary part of the Lagrangian connects the field channel with the real-particle channel. Analogously, the real part of the Lagrangian of a weak field and the renormalization principle (the weak field should be a Maxwell field, i.e., the radiative corrections to the field and to its Lagrangian should vanish with the field) enable us to establish the relation of the real field and charge to the bare field and charge (cf. (54)).

We shall consider certain analytical properties of the Lagrangian. Under the transformation $\eta \rightleftarrows -i\epsilon$ the integrands in (47) and (50) are transformed into themselves, and their singularities do not intersect the integration contours. Therefore,

$$\mathscr{L}_R(\eta, \varepsilon) = \mathscr{L}_R(-i\varepsilon, i\eta). \quad (80)$$

If the magnetic field is large and the electric field is small or comparable with m^2/e, the asymptotic form of the Lagrangian is logarithmic in $e\eta/m^2$ and does not depend on $e\epsilon/m^2$ (cf. (58), (60)):

$$\mathscr{L}(\eta, \varepsilon) \approx \mathscr{L}_{ac}(\eta, 0), \quad \frac{e\eta}{m^2} \gg 1, \quad \frac{e\varepsilon}{m^2} \lesssim 1. \quad (81)$$

The weak, logarithmic singularity of $\mathscr{L}$ in $e\eta/m^2$ at infinity allows us to state that the asymptotic form of the analytically-continued function $\mathscr{L}(-i\eta, i\epsilon)$ will be the analytic continuation $\mathscr{L}_{ac}(-i\eta, 0)$ of the asymptotic form $\mathscr{L}_{ac}(\eta, 0)$:

$$\mathscr{L}(-i\eta, i\varepsilon) \approx \mathscr{L}_{ac}(-i\eta, 0), \quad \frac{e\eta}{m^2} \gg 1, \quad \frac{e\varepsilon}{m^2} \lesssim 1. \quad (82)$$

But it then follows from the symmetry (80) and formula (82) that for a strong electric and a not-so-strong magnetic field

$$\mathscr{L}(\eta, \varepsilon) = \mathscr{L}(-i\varepsilon, i\eta) \approx \mathscr{L}_{ac}(-i\varepsilon, 0), \quad \frac{e\varepsilon}{m^2} \gg 1, \quad \frac{e\eta}{m^2} \lesssim 1. \quad (83)$$

This formula was used in obtaining the asymptotic form (59).

5. LAGRANGIAN OF AN INTENSE FIELD FROM THE VIEWPOINT OF THE RENORMALIZATION GROUP

The relation (52) for the Lagrangian expresses its invariance with respect to renormalization of the field, charge and mass:

$$\mathscr{L}(e_0F_0, \alpha_0, m_0^2, im_0^2 s_0) = \mathscr{L}_R(eF, \alpha, m^2). \quad (84)$$

If in the left-hand side of (84) we carry out the mass renormalization and use the invariance of the product $e_0F_0 = eF$, this relation takes the form

$$\mathscr{L}(eF, \alpha_0, m^2, im^2 s_0) = \mathscr{L}_R(eF, \alpha, m^2), \quad (85)$$

with $\alpha = \alpha_0 Z_3(im^2 s_0, \alpha_0)$; cf. (54) and (56). By dividing both sides of the equality (85) by the renormalization-invariant function $\alpha\mathscr{L}_R^{(0)} \equiv -(eF)^2/16\pi$ and denoting the ratios $\mathscr{L}/\mathscr{L}^{(0)}$ and $\mathscr{L}_R/\mathscr{L}_R^{(0)}$ by l and l_R, we obtain in place of (85) an invariance relation for the dimensionless functions:

$$\alpha_0^{-1} l\left(\frac{eF}{m^2}, \alpha_0, im^2 s_0\right) = \alpha^{-1} l_R\left(\frac{eF}{m^2}, \alpha\right). \quad (86)$$

The function $\alpha^{-1} l_R$ may be called the invariant square of the inverse charge.

We shall assume that for $eF/m^2 \to \infty$ the function l has an asymptotic form that does not depend on m^2: $\lim l = l_\infty(ieFs_0, \alpha_0)$. More precisely, we assume that when one of the field parameters tends to infinity, e.g., $e\eta/m^2 \to \infty$, the asymptotic form of l depends neither on m^2 nor on the second parameter $e\epsilon$. This assumption is fulfilled for the function $l = 1 + l^{(1)} + l^{(2)}$ in the approximation found (cf. (49) and (51)). Then, substituting into l_∞ the quantity $is_0 = m^{-2}\varphi(\alpha, \alpha_0)$ found from the relation $\alpha = \alpha_0 Z_3(im^2 s_0, \alpha_0)$, we obtain, according to (86),

$$\alpha_0^{-1} l_\infty\left(\frac{e\eta}{m^2}\varphi(\alpha, \alpha_0), \alpha_0\right) = \alpha^{-1} l_{R\infty}\left(\frac{e\eta}{m^2}, \alpha\right). \quad (87)$$

Inasmuch as the right-hand side does not depend on α_0, the left-hand side should also not depend on α_0. Consequently, the left-hand side has the form $\Phi[e\epsilon m^{-2}\varphi(\alpha)]$, i.e., is a function of one variable. This means that the function

$$\alpha^{-1} l_{R\infty}\left(\frac{e\eta}{m^2}, \alpha\right) = \Phi\left[\frac{e\eta}{m^2}\varphi(\alpha)\right]$$

satisfies the Callan-Symanzik equation[24, 25]

$$\left(m^2\frac{\partial}{\partial m^2} + \beta(\alpha)\alpha\frac{\partial}{\partial\alpha}\right)\frac{1}{\alpha} l_{R\infty}\left(\frac{e\eta}{m^2}, \alpha\right) = 0, \quad \beta(\alpha) = \frac{\varphi(\alpha)}{\alpha\varphi'(\alpha)} = \frac{m^2\, dZ_3}{Z_3\, dm^2}. \quad (88)$$

The derivative $dZ_3 dm^2$ is calculated at fixed s_0 and α_0. According to perturbation theory, the function $l_{R\infty}$ can be expanded in the following series in powers of α:

$$l_{R\infty} = 1 + \frac{\alpha}{\pi}(a_{10} + a_{11}z) + \sum_{n=2}^{\infty}\left(\frac{\alpha}{\pi}\right)^n \sum_{k=0}^{n-1} a_{nk} z^k, \quad z = \ln\frac{e\eta}{\gamma\pi m^2}. \quad (89)$$

Its coefficients a_{nk} are, in accordance with (88), related to each other and to the coefficients of the power series for $\beta(\alpha)$[4)]

$$\beta(\alpha) = \sum_{k=1}^{\infty}\beta_k\left(\frac{\alpha}{\pi}\right)^k = \frac{1}{3}\left(\frac{\alpha}{\pi}\right) + \frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2 - \frac{121}{288}\left(\frac{\alpha}{\pi}\right)^3 + \ldots \quad (90)$$

by the relations

$$a_{00} = 1, \quad a_{n1} = \sum_{i=0}^{n-1}(i-1)a_{i0}\beta_{n-i},$$
$$k a_{nk} = \sum_{i=k}^{n-1}(i-1)a_{ik-1}\beta_{n-i}, \quad k \geq 2. \quad (91)$$

It follows from these relations that in terms of order $\alpha^n (n \geq 2)$ the exponent of the highest power of the logarithm will be $n-1$. The coefficients of the highest powers of the logarithms in (89) are determined entirely by the coefficients β_1 and β_2:

$$a_{11}=-\beta_1, \quad a_{nn-1}=-\beta_2\beta_1^{n-2}/(n-1), \tag{92}$$

while the coefficients $a_{n,n-r}$ $(2 \leq r \leq n-1)$ of the other logarithms are determined by the coefficients β_1, β_2, $\ldots$, β_{r+1} and the constants a_{20}, a_{30}, $\ldots$, a_{r0}. The latter constants, together with a_{10}, reflect the structure of the function Φ and its arbitrariness–the arbitrariness of the renormalization group: the function $l_{R\infty}$ is characterized by fully determined values of its constants a_{n0} if $e=\sqrt{4\pi\alpha}$ and m are the charge and mass of a real electron; other values of a_{n0} will correspond to a different meaning of the parameters e and m. On the other hand, only by the constants a_{n0} can the asymptotic form $l_{R\infty}$ of the Lagrangian function at large fields be distinguished from the asymptotic form $\pi_{R\infty}$ of the polarization function at large momenta.

Using (92) and formulas not written out here for the coefficients $a_{n,n-r}$ of the logarithms of lower rank r $(2 \leq r \leq n-2)$, we can find the contribution to the sum (89) from the logarithms of the first, second, etc., ranks $(r=1, 2, \ldots)$ in all orders in α:

$$\left(\frac{\alpha}{\pi}\right)^r L_r=\sum_{n=r+1}^{\infty}\left(\frac{\alpha}{\pi}\right)^n a_{n,n-r}z^{n-r},$$

$$L_1=\frac{\beta_2}{\beta_1}\ln(1-x), \quad x=\frac{\alpha}{\pi}\beta_1 z,$$

$$L_2=\left(\frac{\beta_2}{\beta_1}\right)^2\frac{\ln(1-x)+x}{1-x}+\left(a_{20}-\frac{\beta_3}{\beta_1}\right)\frac{x}{1-x}, \tag{93}$$

$$L_3=-\left(\frac{\beta_2}{\beta_1}\right)^3\frac{\ln^2(1-x)-x^2}{2(1-x)^2}+\frac{\beta_2\beta_3}{\beta_1^2}\frac{\ln(1-x)+x(1-x)}{(1-x)^2}-\frac{\beta_2 a_{20}}{\beta_1}\frac{\ln(1-x)}{(1-x)^2}$$
$$+\left(a_{30}-\frac{\beta_4}{2\beta_1}\right)\frac{x(2-x)}{(1-x)^2}, \ldots$$

and so the improved series for $l_{R\infty}$ is of the form

$$l_{R\infty}=1-x+\sum_{n=1}^{\infty}\left(\frac{\alpha}{\pi}\right)^n(L_n(x)+a_{n0}), \quad x=\frac{\alpha}{\pi}\beta_1\ln\frac{e\eta}{\gamma\pi m^2}. \tag{94}$$

In this series, L_1 and $a_{10}=-2\pi^{-2}\zeta'(2)$ are completely determined, and to determine L_2 we need only know the constant a_{20}, since β_3 is known from the study of the polarization function π_R [5)]; cf. (90).

We call attention to the singularities of the functions $L_n(x)$ at the point $x=1$, which prevent us from using the series (94) near this point, at which the zeroth approximation for $l_{R\infty}$ vanishes. Unlike the initial perturbation-theory series, which is applicable for $x \ll 1$, the improved series (94) is applicable in a wider range of x, including values of x that are ~ 1 but not too close to unity, viz., for

$$\alpha/\pi, \ (\alpha/\pi)(1-x)^{-1}\ln(1-x)^{-1} \ll 1. \tag{95}$$

The larger of the quantities on the left is the applicability parameter of the improved perturbation theory with respect to the radiation field in the electrodynamics of an intense field with no particles.

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APPENDIX A

We give here the lowest (proportional to the fourth power of the field) terms of the expansions of the functions $K(s, s')$ and $K_0(s)$, which are necessary for the calculation of the asymptotic form of the Lagrangian of a weak field:

$$K(s,s')=-\frac{\exp[-im^2(s+s')]}{ss'(s+s')^2}\left\{\frac{2e^4(\eta^4+\varepsilon^4)}{135}[m^2(s+s')(6(s+s')^4\right.$$
$$-12ss'(s+s')^2-49(ss')^2)+i(3(s+s')^4-4ss'(s+s')^2+32(ss')^2)]$$
$$+\frac{2e^4(\eta\varepsilon)^2}{27}[2m^2(s+s')(3(s+s')^4-15ss'(s+s')^2-8(ss')^2)$$
$$\left.+i(3(s+s')^4-19ss'(s+s')^2+11(ss')^2)]\right\}+\ldots, \tag{A.1}$$

$$K_0(s)=-\frac{2}{45}[e^4(\eta^4+\varepsilon^4)+5e^4(\eta\varepsilon)^2]se^{-im^2s}(2m^2s+i)+\ldots. \tag{A.2}$$

APPENDIX B

We shall calculate the integral

$$J(\beta)=\int_0^{1/2}d\xi g(\xi,\beta), \quad \beta=\frac{e\varepsilon}{m^2}\ll 1. \tag{B.1}$$

The function $g(\xi, \beta)$ is given in the main text; cf. (72), (73). We represent J in the following form:

$$J(\beta)=\int_0^{1/2\beta}dz\,G(z)+\int_0^{1/2\beta}dz\left[\beta g(\beta z,\beta)-G(z)+\frac{3\beta}{1-\beta z}\right.$$
$$\left.-2\pi\beta^2\left(\frac{1}{\sin^2\xi\pi}-\frac{1}{(\xi\pi)^2}-2\right)\right]-3\ln 2-2\beta\left(\pi-\frac{2}{\pi}\right), \tag{B.2}$$

where $\xi=\beta z$, and choose as the function $G(z)$ the limit

$$\lim\beta g(\beta z,\beta)=G(z)=\left(\frac{4}{\pi z^2}+\frac{2}{\pi^2 z^3}\right)(1-e^{-\pi z})-\frac{2}{\pi z^2}-\frac{3}{z}, \quad \beta\to 0. \tag{B.3}$$

Then, to within terms of order β, the first integral in (B.2) is equal to

$$\int_0^{1/2\beta}dz\,G(z)=-3\ln\frac{\gamma\pi}{2\beta}+\frac{5}{2}-\frac{4\beta}{\pi}+\ldots. \tag{B.4}$$

To within terms $\sim\beta$ in the region $z \lesssim 1$ and terms $\sim\beta^2$ in the region $z \sim (2\beta)^{-1}$, the integrand in the second integral of (B.2) is equal to

$$\beta g(\beta z,\beta)-G(z)+\frac{3\beta}{1-\beta z}-2\pi\beta^2\left(\frac{1}{\sin^2\xi\pi}-\frac{1}{(\xi\pi)^2}-2\right)=\beta G_1(\pi z)+\ldots, \tag{B.5}$$

where

$$G_1(x)=e^{-x}\left(4+\frac{6}{x}+\frac{8}{x^2}+\frac{4}{x^3}\right)-\frac{4}{x^2}-\frac{4}{x^3}.$$

Substituting the function $\beta G_1(\pi z)$ for the integrand in the second integral and replacing the upper limit by infinity, we obtain for the integral the value $-\beta/\pi$.

Thus,

$$\int_0^{1/2}d\xi\, g(\xi,\beta)=-3\left(\ln\frac{\gamma\pi}{\beta}-\frac{5}{6}\right)-(1+2\pi^2)\frac{\beta}{\pi}+\ldots. \tag{B.6}$$

[1)]By tradition, the superscript on the action function and on the Lagrangian denote the order of the correction in α, while the superscript on M denotes the order of the correction in the charge e.

[2)]If the tensor F has eigenvalues $\pm i\eta$, $\pm\epsilon$, its dual tensor F^* has eigenvalues $\pm i\epsilon$ and $\mp\eta$, respectively.

[3)]The polarization function π_R is defined by the relation $D=D_0(1+\pi_R)^{-1}$, where D and D_0 are the photon propagators in the vacuum with and without allowance for the radiative corrections. The structure of the exact polarization function at large momenta is described in e.g., [14].

4)At present the values of the first three coefficients β_k, given in (90), are known (cf. [14]). The values of two of them, β_1 and β_2, also follow from the results of the present work, in accordance with formulas (88), (56) or (61), (92). We note that our function β is smaller than the analogous function in [13,14] by a factor of two.

5)In the analogous series for $1 + \pi_{R\infty}$, in which $x = (\alpha/\pi)\beta_1 \ln(k^2/m^2)$, the coefficients L_1 and L_2 are completely determined, since here $a_{10} = 5/9$, $a_{20} = (5/24) - \zeta(3)$ (cf. [14]), and to determine L_3 we need only β_4 and a_{30}.

1W. Heisenberg and H. Euler, Z. Phys. **98**, 714 (1936).
2V. Weisskopf, K. Danske Vidensk. Selsk. Mat.-fys. Medd. **14**, No. 6 (1936).
3J. Schwinger, Phys. Rev. **82**, 664 (1951).
4R. Serber, Phys. Rev. **48**, 49 (1935).
5E. A. Uehling, Phys. Rev. **48**, 55 (1935).
6J. Schwinger, Phys. Rev. **75**, 651 (1949).
7R. P. Feynman, Phys. Rev. **76**, 769 (1949).
8L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR **95**, 773, 1177; **96**, 261 (1954) [English translations in "Collected Works of L. D. Landau", Gordon and Breach, N.Y., 1965].
9M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).
10K. Johnson, M. Baker and R. Willey, Phys. Rev. **136**, B1111 (1964).
11K. Johnson, R. Willey and M. Baker, Phys. Rev. **163**, 1699 (1967).
12M. Baker and K. Johnson, Phys. Rev. **183**, 1292 (1969); **D3**, 2516, 2541 (1971).
13S. L. Adler, Phys. Rev. **D5**, 3021 (1972).
14E. de Rafael and J. L. Rosner, Ann. Phys. (N.Y.) **82**, 369 (1974).
15R. Jost and J. M. Luttinger, Helv. Phys. Acta **23**, 201 (1950).
16M. Baranger, F. J. Dyson and E. E. Salpeter, Phys. Rev. **88**, 680 (1952).
17G. Källen and A. Sabry, K. Danske Vidensk. Selsk. Mat.-fys. Medd. **29**, No. 17 (1955).
18J. Rosner, Phys. Rev. Lett. **17**, 1190 (1966); Ann. Phys. (NY) **44**, 11 (1967).
19V. I. Ritus, Ann. Phys. (NY) **69**, 555 (1972).
20V. I. Ritus, Nucl. Phys. **B44**, 236 (1972).
21J. Schwinger, Proc. Nat. Acad. Sci. **37**, 452, 455 (1951).
22V. Fock, Phys. Z. Sowjet. **12**, 404 (1937).
23L. D. Landau and E. M. Lifshitz, Teoriya polya (Field Theory), Nauka M (1973) [English translation of an earlier edition (The Classical Theory of Fields) published by Pergamon Press, Oxford, 1962].
24C. G. Callan, Jr., Phys. Rev. **D2**, 1541 (1970).
25K. Symanzik, Commun. Math. Phys. **18**, 227 (1970); **23**, 49 (1971).

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