Isomeric states of quantum fields

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(Submitted December 15, 1974)
Zh. Eksp. Teor. Fiz. **68**, 1975–1990 (June 1975)

It is shown that in theories with spontaneous symmetry breaking there exist particles of a peculiar type, which are blobs of classical fields with small quantum fluctuations. Such blobs exist only in the case when the state of the field at arbitrarily large distances from the blob cannot be continuously deformed into the usual vacuum. These isomeric states exhibit therefore rigorously conserved quantum numbers which will be called topological and which are not present in the original Lagrangian. The question of possible identification of isomers with observable particles is discussed.

PACS numbers: 03.70.

1. INTRODUCTION

The number of known elementary particles is presently so large that it becomes imperative to understand whether one can explain their existence in terms of quantum-field-theoretical Hamiltonians of more or less usual type, or whether qualitatively new ideas will be required for this. The solution of this problem is made difficult by the absence of reliable computational methods for strong coupling Hamiltonians. At the same time one customarily thinks that weak coupling Hamiltonians have obvious properties, described by perturbation theory and having no relation whatsoever to the hadronic world. Quite recently[1-4], it became clear that this is not quite so, and in many cases the states of the indicated Hamiltonians contain unusual particles, called "extremons"[4], which correspond to vibrational levels near nontrivial extremals of the potential energy of the quantized field. A remarkable peculiarity of the extremons is the fact that, in spite of the weak nonlinearity of the original Hamiltonian, they are subject to strong interactions.

In the present paper we study various types of extremons which appear in theories with internal symmetries. It turns out that the extremons have a topological meaning, related to the fact that the states of the field at arbitrarily large distances cannot be transformed into the vacuum by means of a continuous deformation.* As a result of this one must attribute to the extremons rigorously conserved quantum numbers, and even strong quantum fluctuations are not capable to destroy an extremon. For the same reason the qualitative results obtained below are not sensitive to the concrete choice of the Hamiltonian.

Yang-Mills fields are quite naturally included in the model under discussion and lead to new states. Some of these states exhibit a magnetic charge.

One attributes automatically new and exactly conserved quantum numbers to the isomers which will be designated as "topological quantum numbers." The conservation of these quantum numbers is related to the impossibility of continuously deforming the isomeric state of the field into the vacuum state.

The isomers obtained in the present paper have spherical or filament form. For the group SU(2) both these types of isomers are possible, but for SU(3) only filamentary objects have been found, the filaments exhibiting triality, which turns out to be a topological quantum number.

The paper is organized as follows. In Sec. 2 the general pattern of using classical solutions in quantum field theory is demonstrated on a simple one-dimensional model, the spectrum of isomers being calculated explicitly within this framework. In Sec. 3 we determine the spherical solutions of the isovector and isospinor Higgs fields with SU(2) symmetry. In Sec. 4 the objects of the preceding section are filled by the Yang-Mills field and form isomeric states. In Sec. 5 we consider filament-like isomers for the groups SU(2) and SU(3). The number of types of filaments is equal to the dimension of the center of the group. In Sec. 6 we discuss the topological meaning of the results and introduce the topological quantum numbers. In addition, we discuss the relation of the filament-like isomers to small-distance properties of the Hamiltonians.

2. CLASSICAL SOLUTIONS AND QUANTIZATION

In the present section we consider the problem of quantization of a field near some classical extremal. For more clarity we shall study a simple one-dimensional model field theory and shall use this example to demonstrate our methods. The extension of these methods to the general case will be trivial.

Let $\varphi(x)$ denote a scalar field in two-dimensional spacetime with the Hamiltonian

$$H=\int_{-\infty}^{\infty} dx\left\{\frac{\pi^2}{2}+\frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2-\frac{\mu^2}{2}\varphi^2+\frac{\lambda}{4}\varphi^4\right\},\quad \pi(x)=-i\delta/\delta\varphi(x),\quad \lambda\ll\mu^2. \tag{2.1}$$

In Eq. (2.1) the first term is the kinetic energy and the remaining terms are the potential energy.

One may interpret H as the Hamiltonian of an elastic linear string lying in a two-humped potential relief (here x labels the points of the string and $\varphi(x)$ is the displacement amplitude at x). In its ground state the string lies in one of the wells and undergoes small zero-point oscillations, i.e., $\bar{\varphi} = \pm\mu\lambda^{-1/2}$. However, this is not the only equilibrium point of the string. Other equilibrium configurations must be determined from the equation

$$d^2\varphi_c/dx^2+\mu^2\varphi_c-\lambda\varphi_c^3=0 \tag{2.2}$$

and the boundary conditions $\varphi_c^2(\pm\infty) = \mu\lambda^{-1/2}$, which guarantee the finiteness of the energy of the equilibrium state. The solution of (2.2) has the form

$$\varphi_c(x)=(\mu/\sqrt{\lambda})\,\mathrm{th}(\mu x/\sqrt{2}). \tag{2.3}$$

Thus, starting in the left-hand well, at some point the string is thrown over into the right-hand well. Owing to the joint action of the potential and elastic forces such a configuration is an equilibrium state. Let us now consider oscillations near the equilibrium:

$$\varphi(x)=\varphi_c(x)+\Phi(x),$$

$$H=\int dx\left\{\frac{1}{2}\left(\frac{d\varphi_c}{dx}\right)^2-\frac{\mu^2}{2}\varphi_c^2+\frac{\lambda}{4}\varphi_c^4\right\}+\int dx\left\{\frac{1}{2}\left(\frac{d\Phi}{dx}\right)^2+\frac{\pi^2}{2}+\left(\mu^2-\frac{3}{2}\mu^2\,\mathrm{ch}^{-2}\frac{\mu x}{\sqrt{2}}\right)\Phi^2-\mu\sqrt{\lambda}\,\mathrm{th}\frac{\mu x}{\sqrt{2}}\Phi^3+\frac{\lambda}{4}\Phi^4\right\}. \quad (2.4)$$

For $\lambda \ll \mu^2$ one may omit the last two terms in (2.4). The quadratic Hamiltonian arising in this manner can be diagonalized. For this purpose we represent $\Phi(x)$ in the form

$$\Phi(x)=\sum_\lambda \xi_\lambda\psi_\lambda(x),$$

$$\frac{\delta}{\delta\Phi(x)}=\sum_\lambda\psi_\lambda(x)\frac{\partial}{\partial\xi_\lambda}=\sum_\lambda\psi_\lambda(x)\pi_\lambda, \quad (2.5)$$

where $\psi_\lambda(x)$ is a complete set of functions subject to the equation

$$\frac{d^2\psi_\lambda}{dx^2}+\left(m_\lambda^2-2\mu^2+3\mu^2\,\mathrm{ch}^{-2}\frac{\mu x}{\sqrt{2}}\right)\psi_\lambda(x)=0 \quad (2.6)$$

(here m_λ^2 are the eigenvalues of (2.6)). Substituting (2.6) into (2.4) we obtain (neglecting the nonlinear terms)

$$H=\int dx\left\{\frac{1}{2}\left(\frac{d\varphi_c}{dx}\right)^2-\frac{\mu^2}{2}\varphi_c^2+\frac{\lambda}{4}\varphi_c^4\right\}+\frac{1}{2}\sum_\lambda(\pi_\lambda^2+m_\lambda^2\xi_\lambda^2). \quad (2.7)$$

It is now easy to determine the masses of the particles corresponding to small oscillations of the string near the unusual equilibrium position. These masses are defined as the eigenvalues of the operator $H - E_0$, where E_0 is the energy of the genuine vacuum with $\bar{\varphi} = \pm\mu\lambda^{-1/2}$. This yields the mass spectrum

$$M=M_0+\left(\frac{1}{2}\sum m_k-\frac{1}{2}\sum_{vac} m_k\right)+\sum l_s m_s. \quad (2.8)$$

Here M_0 is the difference between the first term of (2.7) and its vacuum value

$$M_0={}^2/_3\sqrt{2}\mu^3/\lambda, \quad (2.9)$$

The second term gives the change in energy of the zero-point oscillations and l_s are integers.

Equation (2.8) is not related to the concrete model and yields a general prescription for calculating the masses of extremons in theories with small coupling constants. Corrections to this formula involve the first (and higher) powers of the coupling constant and can in principle be calculated.

An explicit expression for the second term in (2.8) can be obtained making use of a well known method from statistical mechanics[5], which expresses this term in terms of the scattering phaseshifts on the potential created by the extremon. We do not list here the explicit formulas, which are of no special interest, and turn our attention to the following important circumstance. It is easy to verify that in the one-dimensional model

$$\frac{1}{2}\left(\sum m_k-\sum_{vac} m_k\right)=-\frac{\mu\sqrt{2}}{8\pi}\ln\frac{\Lambda}{\mu}+\text{const},$$

where Λ is the ultraviolet cutoff. Thus, the first quantum correction diverges logarithmically. However, at the same time the first term in (2.8) given by Eq. (2.9) contains the bare mass $2^{1/2}\mu$ of the meson (the first excitation above the normal vacuum). The physical mass of the meson is given by

$$m^2=2\mu^2-(\lambda/4\pi)\ln(\Lambda/\mu). \quad (2.10)$$

Substituting (2.10) into (2.9) and thus renormalizing the mass we arrive at the conclusion that after this the quantum correction has a finite magnitude. Apparently this assertion is valid for any renormalizable theory: after appropriate renormalizations the mass of the extremon is finite.

The reasoning used in relation with Eq. (2.8) assumed that the nonlinear terms are small. One could have doubts about this assumption, since the problem always involves one or several oscillators of zero frequency. Their existence is related to translation invariance or to other invariance properties violated by the classical solution. Thus, with translations one associates a level with

$$\psi_0=d\varphi_c/dx,$$

with rotation one associates three levels with

$$\psi_{0,\alpha}=\left[\mathbf{x}\frac{d}{dx}\right]_\alpha\varphi_c$$

etc. In spite of the vanishing eigenfrequencies, the quantum corrections do not involve infrared divergences. Physically this is related to the fact that the indicated modes are motions of the extremon as a whole, and for sufficiently slow motions they cannot strongly affect the spectrum. A formal proof of this fact and a calculation of the moments of inertia of the extremon form the subject of another paper.

We have thus arrived at a simple quantization scheme for the classical solutions. We see that it is necessary first to find a stable stationary solution of the classical equations; secondly, to investigate the spectrum of small oscillations in the neighborhood of this solution and thirdly, to make use of Eq. (2.8). For small coupling constants this procedure yields the spectrum up to a dimensionless coupling constant.

3. THE YANG-MILLS FIELD WITH SU(2) SYMMETRY

In this section we investigate extremons in theories with broken SU(2) gauge symmetry.** Two types of symmetry breaking are possible, which have to be considered separately. In the first case the Higgs field is an isovector and in the second case it is an isospinor. In the theories of weak interactions these two cases correspond respectively to the Georgi-Glashow and Weinberg models.

We start with the first case. Following the program outlined in the preceding section we find for the Higgs field extremal boundary conditions which are topologically inequivalent to the usual ones. The Hamiltonian for the Higgs field has the form (the Yang-Mills field has been temporarily switched off)

$$H=\int\left\{\frac{1}{2}\pi_a^2+\frac{1}{2}(\partial\psi_a)^2+V\left(\sum\psi_a^2\right)\right\}d^3x, \quad (3.1)$$

where $a = 1, 2, 3$ is the isospin index. If one introduces the unit vector n^a, such that $\psi_a = un^a$, the potential energy in (3.1) takes the form

$$U=\int\left\{\frac{1}{2}(\partial u)^2+\frac{1}{2}u^2(\partial n^a)^2+V(u^2)\right\}d^3x. \quad (3.2)$$

Let us consider the contribution to (3.2) from large distances, where $u \to u_\infty$ and u_∞ is defined by the condition $V'(u_\infty) = 0$. We have

$$U\approx\frac{1}{2}u_\infty^2\sum_a\iint(\partial n^a)^2\,d^3x. \quad (3.3)$$

The minimum conditions for (3.3) yield all possible boundary conditions at infinity. We denote by θ and ϕ the polar and azimuthal angles of the vector n^a in isospace. The extremal equations are of the form

$$\nabla^2\theta = \tfrac{1}{2}\sin 2\theta(\nabla\phi)^2, \quad \nabla(\sin^2\theta\nabla\phi)=0. \qquad (3.4)$$

We shall search for the solution of (3.4) in the form $\theta = \theta(\vartheta)$ and $\phi = \phi(\varphi)$, where ϑ and φ are the corresponding angles in ordinary space. Then Eqs. (3.4) reduce to the form

$$\frac{d}{d\vartheta}\left(\sin\vartheta\frac{d\theta}{d\vartheta}\right)=\frac{1}{2}\sin 2\theta\frac{1}{\sin\vartheta}\left(\frac{d\phi}{d\varphi}\right)^2, \quad \frac{d^2\phi}{d\varphi^2}=0. \qquad (3.5)$$

Together with the continuity condition the second equation of (3.5) yields $\phi = \pm n\varphi$ (n is an integer). The first equation can be integrated by introducing the independent variable $\ln\tan(\vartheta/2)$. The regular and continuous solutions have the form

$$\theta=2\,\mathrm{arctg}\left\{\left[\mathrm{tg}\frac{\vartheta}{2}\right]^n\right\}, \quad \theta=2\,\mathrm{arctg}\left\{\left[\mathrm{ctg}\frac{\vartheta}{2}\right]^n\right\}. \qquad (3.6)$$

In order that these solutions be physically meaningful, we have to verify that they are stable, i.e., that they indeed realize a minimum of the functional (3.3). We have succeeded in showing this in the simplest case $n = 1$ (cf. Appendix); for $n > 1$ the problem remains open, although, in the opinion of the author, there is stability also in these cases. Therefore below we concentrate our attention on the case $n = 1$:

$$\phi=\varphi, \quad \theta=\vartheta, \qquad (3.7)$$

which we designate as the "hedgehog" (the "quills" of the hedgehog which determine the direction of the Higgs field are directed along the radius-vectors in ordinary space; solutions with $n \neq 1$ could be called twisted hedgehogs. We note that the number n is the "degree of the mapping" of a sphere onto a sphere, as defined in topology).

After imposing the boundary conditions (3.7) one can search for a minimum of the potential energy of the Higgs field by means of the trial function

$$\psi_a=x_a u(r)/r. \qquad (3.8)$$

Substituting (3.8) into the Euler equation yields

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right)+\left(\mu^2-\frac{2}{r^2}\right)u-\lambda u^3=0. \qquad (3.9)$$

In the derivation of (3.9) we have chosen, for the sake of concreteness,

$$V(u^2)=-\tfrac{1}{2}\mu^2u^2+\tfrac{1}{4}\lambda u^4. \qquad (3.10)$$

Equation (3.9) has a solution with the asymptotic behaviors:

$$u(r)\to\mathrm{const}\cdot r, \quad r\to 0;$$
$$u(r)\to\mu^2/\lambda, \quad r\to\infty.$$

Such a solution represents a bubble within which the Higgs field tends to zero. It is easy to understand qualitatively why this bubble is stable: the expansion of the bubble is prevented by the volume loss of energy related to the term $V(u^2)$ in (3.2) and the collapse is hindered by the term $\int u^2(\nabla n)^2 d^3x$, which yields in the energy difference a contribution of the order $-u_\infty^2 R$ (R is the size of the bubble). We see that the anomalous boundary conditions produce a negative "linear tension" which stabilizes the bubble.

The total energy of the hedgehog diverges linearly owing to the term $\int u^2(\nabla n)^2 d^3x$, and therefore an isolated hedgehog is not an extremon. However, in the sequel we shall construct extremon states, either by filling up the hedgehog with a Yang-Mills field or considering pairs of oppositely oriented hedgehogs.

Let us go over to a study of hedgehogs which appear in theories with an isospinor Higgs field ψ. It should be stated from the beginning that the hedgehogs which are obtained in this case are unstable. Nevertheless we list the solutions for the extremal equations (the extremum is now no longer a minimum) in the hope that by means of the introduction of some additional fields it will be possible to stabilize the hedgehog. We search for the solution at infinity in the form of a rotated standard solution

$$\psi=u_\infty\hat{R}(x)\begin{pmatrix}0\\1\end{pmatrix}, \quad x\to\infty, \qquad (3.11)$$

where $\hat{R}(x)$ is a matrix from the SU(2) group which must be determined from the condition that the boundary conditions (3.11) be extremal. This extremality must be ensured for the functional

$$U=\int\{\partial\psi^+\partial\psi+V(\psi,\psi^+)\}d^3x\approx u_\infty^2\int(\nabla R^+\nabla R)_{22}+\ldots$$
$$=\frac{1}{2}u_\infty^2\int\mathrm{Sp}(\nabla R^+\nabla R)+\ldots; \qquad (3.12)$$

the terms omitted in Eq. (3.12) give a finite contribution to the energy and are inessential for the formulation of the boundary conditions at infinity.

In order to determine the extremals of the functional (3.12) we parametrize the matrices R by means of Euler angles:

$$\hat{R}=\exp\left(i\frac{\sigma_z\gamma}{2}\right)\exp\left(i\frac{\sigma_x\beta}{2}\right)\exp\left(i\frac{\sigma_z\alpha}{2}\right). \qquad (3.13)$$

After relatively simple calculations we obtain

$$U=\frac{1}{4}u_\infty^2\int d^3x\{(\nabla\alpha)^2+(\nabla\beta)^2+(\nabla\gamma)^2+2\nabla\gamma\nabla\alpha\cos\beta\}. \qquad (3.14)$$

We shall look for solutions for which $\alpha = -\gamma$. In this case the functional (3.14) is equivalent to the functional (3.3) if one identifies $\beta/2 \leftrightarrow \theta$, $\alpha \leftrightarrow \phi$. Therefore we find a solution of the hedgehog type:

$$\alpha=\varphi, \quad \beta=2\vartheta. \qquad (3.15)$$

The corresponding boundary conditions for ψ have the form

$$\psi=u_\infty\begin{pmatrix}e^{-i\varphi}\sin\vartheta\\ \cos\vartheta\end{pmatrix}, \quad x\to\infty. \qquad (3.16)$$

If one now seeks a solution for the whole potential energy in the form

$$\psi=u(r)\begin{pmatrix}e^{-i\varphi}\sin\vartheta\\ \cos\vartheta\end{pmatrix}, \qquad (3.17)$$

one obtains for u the equation (3.9). As already mentioned, the solution (3.17) does not define a minimum of the energy. This fact, which is proved in the Appendix has an intuitive topological interpretation which will be discussed in Sec. 6.

Thus, we have not found point extremons for the isospinor Higgs field.

4. SWITCHING-ON OF THE YANG-MILLS FIELDS

One of the methods of giving the hedgehog a physical meaning is filling up the above-mentioned bubble with a Yang-Mills field. It is easy to understand without calculations that the switching-on of these fields makes the mass of the hedgehog finite. Indeed, the infinite mass of the empty hedgehog appears on account of the large-distance divergence of the integral (3.3) which in turn is related to the fact that at arbitrary distances the direction of the Higgs field is not uniform in isospace. The introduction of the Yang-Mills field makes the Hamiltonian invariant with respect to non-uniform rotations in isospace, and therefore the indicated non-uniformity of directions stops contributing to the energy.

For a quantitative analysis of the problem we write the potential energy of the system of Higgs and Yang-Mills fields (as before we start with the case of isovector Higgs fields):

$$U=\int d^3x\{|D_i\psi^a|^2+{}^1/_4(F_{ik}^a)^2+V(|\psi|^2)\},$$
$$(D_i\psi)^a=\nabla_i\psi^a+g\varepsilon^{abc}A_i^b\psi^c,$$
$$F_{ik}^a=\partial_iA_k^a-\partial_kA_i^a+g\varepsilon^{abc}A_i^bA_k^c. \qquad (4.1)$$

Eq. (4) contains only the "magnetic" part of the energy of the Yang-Mills field, since in canonical quantization the "electric" part is expressed in terms of the generalized moments and must be considered as part of the kinetic energy (in the gauge $A_0^a = 0$).

Our problem will consist in determining the minima of the functional (4.1), which go over into the hedgehog solution discussed above as $g \to 0$. As before, we shall look for the field ψ^a in the form (3.8). In order to guess the form of the field A_i^a generated by the field ψ^a we calculate the current associated to the field ψ^a

$$J_i^a=\varepsilon^{abc}\psi^b\partial_i\psi^c=\varepsilon^{abc}\frac{u^2(r)}{r^2}x^b\partial_ix^c=\frac{u^2(r)}{r^2}\varepsilon_{iab}x_b. \qquad (4.2)$$

On account of (4.2) it is natural to search for the Yang-Mills field in the form

$$A_i^a=a(r)\varepsilon_{iab}x_b. \qquad (4.3)$$

The field strength has the form

$$F_{ik}^a=2\varepsilon_{ika}a(r)+\frac{a'}{r}[\varepsilon_{kas}x_sx_i-\varepsilon_{ias}x_sx_k]-ga^2\varepsilon_{kis}x_sx_a. \qquad (4.4)$$

Substituting these expressions into the Yang-Mills equation yields

$$a''+4r^{-1}a'-3ga^2-g^2r^2a^3-g^2u^2a=gu^2r^{-2},$$
$$u''+2r^{-1}u'-2r^{-2}u-4gau-2g^2a^2r^2u+\mu^2u-\lambda u^3=0.$$

The substitution

$$a(r)=b(r)-1/gr^2 \qquad (4.5)$$

leads to simpler equations:

$$b''+4r^{-1}b'+3r^{-2}b-g^2r^2b^3-g^2u^2b=0,$$
$$u''+2r^{-1}u'+(\mu^2-2g^2b^2)u-\lambda u^3=0. \qquad (4.6)$$

The asymptotic behaviors of the solutions of the system (4.6) are:

$$r\to 0:\ u(r)\propto r,\quad a(r)\propto \text{const};$$
$$r\to\infty:\ u(r)\to u_\infty,\quad a(r)\to -1/gr^2. \qquad (4.7)$$

The hedgehog described by Eqs. (4.6) is constructed in the following manner. At its center there is no Bose condensate and there are Yang-Mills fields. At a characteristic distance of the order of $1/\mu$ the condensate $u(r)$ reappears and practically takes on its asymptotic value. At another distance of the order $1/m_V = 1/gu_\infty$ the field $a(r)$ takes on its asymptotic value (4.7). A characteristic property of the hedgehog is the fact that up to lengths of the order $1/m_V$ it is completely analogous to the empty hedgehog described by Eqs. (4.9). In particular, it is not hard to show that the energy has the form

$$u=\frac{1}{2}u_\infty^2\int\limits_{|x|\lesssim 1/m_V}(\nabla n)^2d^3x\sim\frac{u_\infty^2}{m_V}\sim\frac{gu_\infty}{g^2}\sim\frac{m_V}{g^2}. \qquad (4.8)$$

We note that the Yang-Mills field extends to infinity in this case: it is easy to verify that the asymptotic potential (4.7) yields a nonvanishing field strength. This result is related to the fact that in the case of an isovector Higgs field only two out of the three Yang-Mills fields acquire a mass. The Bose condensate is impenetrable for the massive components, but easily lets the massless ones through. The presence of a nontrivial vector potential at infinity means, as will be shown in Appendix B, that the hedgehog is a magnetic monopole.

For the isospinor Higgs fields the solution of the extremum equations has only a methodological character, as already mentioned in Sec. 3. As before, the potential energy has the form (4.1), with the only difference that

$$D_i\psi=\partial_i\psi-{}^1/_2igA_i^a\tau_a\psi, \qquad (4.9)$$

where τ_a are the Pauli matrices.

We shall look for the solutions of the extremal equations in the form (4.3) for A_i^a and (3.17) for ψ. In order to get the equations it is necessary to calculate the quantity

$$D^2\psi=\nabla^2\psi-ig(\tau_aA_i^a)\nabla_i\psi-{}^1/_4g^2(A_i^a)^2\psi. \qquad (4.10)$$

We have

$$(A_i^a)^2=2r^2a^2(r),\quad A_i=\tau_aA_i^a=ra(r)\left[\tau\times\frac{\mathbf{r}}{r}\right]. \qquad (4.11)$$

From the second equation of (4.11) we determine the potentials A_i in spherical coordinates:

$$A_r=0,\quad A_\vartheta=ra(r)(-i)\begin{pmatrix}0 & e^{-i\varphi}\\ -e^{i\varphi} & 0\end{pmatrix},$$
$$A_\varphi=ra(r)\sin\vartheta\begin{pmatrix}1 & -\mathrm{ctg}\,\vartheta e^{-i\varphi}\\ -\mathrm{ctg}\,\vartheta e^{i\varphi} & -1\end{pmatrix}. \qquad (4.12)$$

Making use of Eqs. (3.17), (4.10) and (4.12) we obtain the equation for $u(r)$:

$$u''+\frac{2}{r}u'+\left\{\mu^2-\frac{g^2r^2}{2}\left[a(r)+\frac{2}{gr^2}\right]^2\right\}u-\lambda u^3=0. \qquad (4.13)$$

In order to find the equation for $a(r)$ it is necessary to calculate the current associated to the field ψ:

$$J_i^a={}^1/_2ig\psi^+\tau^a\overleftrightarrow{\nabla}_i\psi+{}^1/_2g^2\psi^+\psi A_i^a. \qquad (4.14)$$

We must substitute

$$\psi=u(r)R\begin{pmatrix}0\\1\end{pmatrix} \qquad (4.15)$$

into Eq. (4.14), where R is defined by (4.13) and (4.15). This yields

$$J_i^a={}^1/_2igu^2(r)\{R^+\tau^a\nabla_iR-\nabla_iR^+\tau^aR\}_{22}+{}^1/_2g^2u^2(r)a(r)\varepsilon_{ias}x_s. \qquad (4.16)$$

After rather long calculations which make use of the explicit form of the matrix R, we obtain

$$J_i^a={}^1/_2g^2u^2(r)(a(r)+2/gr^2)\varepsilon_{ias}x_s. \qquad (4.17)$$

One can understand Eq. (4.17) also without calculations, since the field $a(r) = -2/gr^2$ is in its entirety related to the gauge transformation (the corresponding field strength F_{ij}^a vanishes). It is easy to check that the gauge transformation is given by R^+. Therefore for $a(r) = -2/gr^2$ the covariant derivatives of the field $R\binom{0}{1}$ must vanish. This fact is reflected in Eq. (4.17).

Making use of Eq. (4.17) and remembering Eq. (4.4) we find the equation for $a(r)$:

$$a''+(4/r)a'-3ga^2-g^2r^2a^3={}^1/_2 g^2u^2[a(r)+2/gr^2]. \quad (4.18)$$

Equations (4.13) and (4.18) simplify somewhat if one introduces the field

$$\begin{gathered} b(r)=a(r)+2/gr^2, \\ b''+4b'/r+3gb^2-g^2r^2b^3-{}^1/_2 g^2u^2b=0, \\ u''+2u'/r+[\mu^2-{}^1/_2 g^2r^2b^2(r)]u-\lambda u^3=0. \end{gathered} \quad (4.19)$$

Without dwelling on the obvious similarity of the objects described by Eqs. (4.19) and (4.6), we will underline their differences. In the case under consideration now the physical Yang-Mills field (i.e. F^a_{ik}) is concentrated within a region of size m_V^{-1} and decays exponentially outside this region. At the same time, in the case of an isovector hedgehog there is a "magnetic" field at infinity which decays according to (4.3) and (4.7) as $1/r^2$. Thus, the isovector hedgehog is a magnetic monopole[1)]. This property will be discussed in more detail in the Appendix.

For the quantities F^a_{ik} one can obtain the estimates:

$$F_{ik}{}^a \infty a(r) \infty 1/gr^2 \infty m_V{}^2/g.$$

Comparing these two estimates we find that the contribution of the Yang-Mills field to the energy is of the order m_V/g^2. Similarly, we convince ourselves that the Higgs field is of the same order. We note that these estimates are correct if $\mu > m_V$ (a situation analogous to vortices in type-II superconductors). In the other case the hedgehog has size $1/\mu$ and in place of m_V in this equation there appears μ.

5. FILAMENT-LIKE ISOMERS

Until now we have studied isomeric states representing small balls inside which the vacuum is different from the normal vacuum. In the present section we shall study filament-like (string-like) objects of the type of quantum vortices. For the case of abelian groups this problem has already been considered by Nielsen and Olesen in[1]. We shall therefore study only non-abelian groups.

It should be noted that infinite filaments like those studied below are devoid of physical meaning owing to their infinite mass. However, they can be used as building material for the creation of extremons (e.g., by means of the formation of rotating rings). The problem of construction of extremons from filaments will be studied elsewhere; here we shall concern ourselves only with the properties of the building material.

As before we start with the boundary conditions for the group SU(2). For this it is necessary to study the minima of the functional (3.14). It is convenient to rewrite this functional, replacing the Euler angles by the unit vector n which characterizes the rotation axis, and by the rotation angle χ. Thus the SU(2) matrices are

$$R=\exp\left\{i\frac{\chi}{2}(\sigma\cdot\mathbf{n})\right\}, \quad (5.1)$$

where $0 < \chi < 2\pi$. Substituting (5.1) into (3.12), we obtain

$$U=\frac{1}{2}u_\infty{}^2\int \mathrm{Sp}\,\nabla R^+\,\nabla R\,d^3x=\frac{1}{2}u_\infty{}^2\int\left\{\frac{1}{4}(\nabla\chi)^2+(\nabla\mathbf{n})^2\sin^2\frac{\chi}{2}\right\}d^3x. \quad (5.2)$$

The expression (5.2) has an intuitive interpretation: the group SU(2) is a sphere in four-dimensional space and (5.2) is the length element of an arc on this sphere. We have to find the minimum of the functional (5.2) searching for the solution in an axially-symmetric form (the filament is directed along the z-axis).

The problem of boundary conditions for $\chi(\beta)$ and $n(\beta)$ is extremely important, where β is the azimuthal angle. In going around the filament, i.e., when β changes from 0 to 2π, we must obtain the original transformation of the factor-group $O(3) = SU(2)/Z_2$, since the physical symmetry group is O(3). This means that two variants are possible

1) $\chi(2\pi)=\chi(0), \quad \mathbf{n}(2\pi)=\mathbf{n}(0),$
2) $\chi(2\pi)=2\pi-\chi(0), \quad \mathbf{n}(2\pi)=-\mathbf{n}(0).$

The possibility of boundary conditions of the second type is related to the fact that the matrix $R(2\pi - \chi, -n)$, although differing in sign from the matrix $R(\chi, n)$, represents the same rotation. There are no minima satisfying conditions of the first type, since the energy can always be diminished by means of a continuous deformation with $\chi = 0$. However, the boundary conditions of the second type are topologically inequivalent to the trivial ones and allow the existence of minima.

The simplest minimum of this type is obtained if one chooses $\chi = \pi$, and picks n from the minimum condition for the functional:

$$\int\left(\frac{d\mathbf{n}}{d\beta}\right)^2 d\beta=\min, \quad \mathbf{n}(2\pi)=-\mathbf{n}(0). \quad (5.3)$$

This problem has been studied before in the theory of liquid crystals[7]. It was shown that its general stable solution represents a uniform rotation of the vector n in an arbitrary plane, such that in going around the filament n describes a half-rotation. In the special case when the plane is chosen perpendicular to the filament axis the motion of n is described by the equations

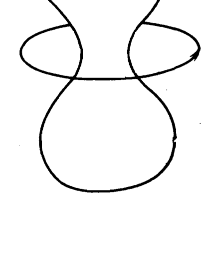
$$\mathbf{n}=(\cos\beta/2,\ \sin\beta/2,\ 0), \quad \chi=\pi. \quad (5.4)$$

Thus, in the case of the group SU(2) there exist two types of boundary conditions. At the same time in the abelian case of the group O(2) the number of types is infinite, since in going around the filament the phase of the field can change by $2\pi m$, where m is an arbitrary integer.

We have shown that for the group SU(2) there exists only one type of boundary conditions leading to filament isomers. However, for this type of conditions there exists a continuous infinity of solutions for $\chi(\beta)$ and $n(\beta)$ leading to the same energy. Indeed, the plane in the four-dimensional space in which the 4-vector

$$u=\left(\cos\frac{\chi}{2},\ \mathbf{n}\sin\frac{\chi}{2}\right),$$

rotates can be arbitrarily situated. In addition, the direction of rotation of the vector n as one goes around the circle, can take on two values. These results should be interpreted in the following manner. The wave functional of the ground state of the isomer does not depend on the orientation of the rotation plane. Therefore all positions of this plane are equally likely.



In the general case the number of different boundary conditions is equal to the connectivity of the group under consideration or, what amounts to the same, the dimension of the center of the universal covering group. In the case of SU(3) there are, thus, three types of boundary conditions. We stress the fact that these conclusions refer only to filament-like objects. The filaments for the group SU(3) are characterized by the fact that in going around the filament the SU(3) matrices are multiplied either by $\exp[2\pi i/3]$ or by $\exp[4\pi i/3] = \exp[-2\pi i/3]$. This shows that a fusion can take place either between three filaments of one type or between a pair of filaments of different types. If one imagines that inside the filaments there are fermions and interprets the fermion captured by the filament as a quark, then normal states of such a system can be formed either out of three quarks, or out of a quark and antiquark (cf. the figure).

We shall not write down the equations for the gauge field which fills up the filament, since the physical consequences of this are the same as for the cases considered before. The field strength is concentrated inside the filament and decays exponentially from the exterior over a length of order $1/m_V$. The mass per unit length of an isolated filament becomes finite and has the order of magnitude

$$\rho\sim u_\infty^2\log\frac{\mu}{m_V}\sim\frac{m_V^2}{g^2}\log\frac{\mu}{m_V}. \tag{5.5}$$

The logarithm appeared on account of the contribution of the term $(\nabla\cdot n)^2$ in the region $1/\mu < x < 1/m_V$.

6. ISOMERIC STATES AND TOPOLOGY

In this section we give a qualitative analysis of the properties and origin of the isomer states. This analysis is necessary for understanding the degree of generality of the results obtained above.

We start with the simplest case, the one-dimensional "inflection" of Sec. 2. What prevents the disappearance of this inflection? It is obvious that only the unusual boundary conditions: $\varphi(\pm\infty) = \pm\mu\lambda^{-1/2}$ are responsible for this. If we had $\varphi(\pm\infty) = \pm\mu\lambda^{-1/2}$, then it would be energetically advantageous to remove any deviation of $\varphi(x)$ from a constant. This reasoning shows that for the existence of the extremon it is necessary that a rigorous symmetry group exist (in order that the states with $+\mu\lambda^{-1/2}$ and $-\mu\lambda^{-1/2}$ have the same energy) and that a condensate be present (otherwise the only possible boundary condition is $\varphi(\pm\infty) = 0$).

The boundary conditions must be such that it should be impossible to transform them into the trivial boundary condition by means of a continuous deformation. Thus, if in the one-dimensional model considered above the field $\varphi(x)$ is complex, then, although the formal solution (2.3) continues to exist, it turns out to be unstable. Indeed, by means of a continuous rotation of the phase of the field $\varphi(x)$ one can then reduce the boundary conditions to the trivial ones. Formally one can convince oneself of the instability in the following manner. Owing to the independence of the energy of the constant phase the variations of the field

$$\delta\varphi(x)=i\varepsilon\varphi_c(x)$$

do not change the energy. Therefore the corresponding Schrödinger equation must have a zero eigenvalue (compare with the reasoning in Sec. 2 relative to $\delta\varphi = \epsilon d\varphi_c/dx$). The corresponding eigenfunction is $\varphi_c(x)$ and consequently it has a zero. Therefore there exists one level with a negative eigenvalue which signifies the instability of the extremon (in this case in place of a minimum there is a saddle point).

Let us consider from this point of view the extremons of the hedgehog type. The hedgehogs in the isovector Higgs field can be considered as maps of the unit sphere in three-space onto the unit sphere in isospace. Such a mapping is given by the equations (3.6). Trivial boundary conditions correspond to $n = 0$ in that equation. The number n is the "degree of the map" as defined in topology[8]. It is known that mappings of different degrees are not homotopic, i.e., they cannot be continuously deformed into one another. This fact explains the stability of the hedgehog.

In the isospinor case one associates to each point of the unit sphere in three-space a transformation in SU(2). However, the transformation which results is not in general continuous with respect to the group SU(2), since to near-lying points on the sphere there might correspond SU(2) matrices differing in sign (the sign of an isospinor is chosen by convention, if isospin is conserved). Therefore the mapping obtained in this manner is continuous if one can consider it as a mapping of the sphere onto the factor group $O(3)\sim SU(2)/Z_2$. It is known from topology that any mapping of the two-dimensional sphere onto the group O(3) can be continuously deformed into the trivial mapping[8]. Therefore, it is not surprising that the solution we have found for the isospinor hedgehog is unstable.

Indeed, in the general case the hedgehogs are mappings of the unit sphere onto Higgs fields at large distances. These fields either determine uniquely a transformation from the symmetry group (as was the case in the isospinor situation), or determine it up to a subgroup (in the isovector case—the last Euler rotation). In each concrete case a classification of such mappings can be carried out by means of methods from topology (determination of the homotopy groups).

The same reasoning applies to filamentlike isomers, only in this case one does not map a sphere but a circle. It is easy to understand why in this case there are several classes of inequivalent mappings, the number of classes being equal to the connectivity of the group. This follows directly from the definition of connectivity. Therefore, for the group O(2) we have an infinite number of different vortices (characterized by integers), for O(3) there is only one class of nontrivial mappings, and consequently filaments of one type only.

For the group SU(3) a filament is characterized by its triality, which can take on two values, so that in going around the filament the SU(3) matrices are multiplied either by $\exp(2\pi i/3)$ or by $\exp(4\pi i/3)$.

What was said requires an important remark. What in fact do we have in mind when we talk about the conventionality of the sign of an isospinor? In order to clarify this let us consider the discrete analog of the Hamiltonian (3.12):

$$H=l^D\sum_{r,\alpha}\left\{\frac{1}{l^2}(\psi^{+}_{r+a_\alpha}-\psi_r)(\psi^{+}_{r+a_\alpha}-\psi_r)+V(\psi_r^{+}\psi_r)\right\}. \tag{6.1}$$

Here l is the size of the cell of the cubic lattice we have introduced, $a_{1,2,3,4}$ are the basis vectors of this lattice. After substituting $\psi_r = R_r\varphi$, where φ is some standard spinor and R_r is a matrix from SU(2), the Hamiltonian takes the form

$$H=l^D\sum_{\mathbf{r},\alpha}\left\{\left(\varphi^+,\ \frac{1}{l^2}(R^+_{\mathbf{r}+\mathbf{a}_\alpha}-R_\mathbf{r})^2\varphi\right)+V(\varphi^2)\right\}$$

$$=l^D\sum_{\mathbf{r},\alpha}2\left(\varphi^+,\ \frac{1}{l^2}(1-R_\mathbf{r}R^+_{\mathbf{r}+\mathbf{a}_\alpha})\varphi\right)+V(\varphi^2). \qquad (6.2)$$

Let us now assume that two neighboring matrices R_r differ in sign: $R_{r+a}\approx -R_r$. It is clear that in (6.2) such a situation leads to a tremendous energy, which becomes infinite as $l\to 0$. Therefore the boundary conditions (5.3) are inadmissible for the Hamiltonian (6.2) and there are no isomeric filaments.

However, the transition from the continuous Hamiltonian (3.12) to the lattice Hamiltonian is not unique. Consider in place of (6.2) the expressions

$$\dot{H}=l^D\sum_{\mathbf{r},\alpha}\frac{1}{l^2}\{\varphi^+[4-(R_\mathbf{r}R^+_{\mathbf{r}+\mathbf{a}_\alpha}+R_{\mathbf{r}+\mathbf{a}_\alpha}R_\mathbf{r}^+)^2]\varphi+V(\varphi^+\varphi)\}. \qquad (6.3)$$

In the continuum case

$$R_{\mathbf{r}+\mathbf{a}_\alpha}\approx R_\mathbf{r}+l\,\partial R/\partial r_\alpha+{}^1/_2 l^2\,\partial^2R/\partial r_\alpha{}^2. \qquad (6.4)$$

Substitution of (6.4) into (6.3) and an integration by parts taking into account the condition $R^+R = 1$ lead to the correct Hamiltonian (3.12). But in the lattice Hamiltonian (6.3) the boundary condition $R_{r+a_\alpha}\approx -R_r$ is quite acceptable!

Thus, the problem of the possibility of nonabelian filament isomers unexpectedly turns out to be related with the structure of the theory at ultrasmall distances. We can only assert that there exists a method for extending the theory into this region for which nonabelian filament isomers must appear.

The last question which we would like to discuss in relation to the isomer states is the automatic appearance of new conservation laws. Let us consider the one-dimensional model and show that the parity of the number of existing extremons is conserved. Indeed, let two normal particles collide. Can a single extremon be produced? It is clear that an infinite time will be required for this, since one must transfer the field in an infinite region of space from the state with $\bar{\varphi} = -\mu\lambda^{-1/2}$ into the state with $\bar{\varphi} = +\mu\lambda^{-1/2}$. This barrier penetration has probability $\sim e^{-\infty} = 0$. At the same time nothing prevents the formation of a pair of extremons, since for this the field has to jump the barrier only in a finite region of space.

Generalizing this result one may say that the degree of the map of the space onto the internal symmetry group is a conserved quantity. In particular, the magnetic charge of the hedgehog is conserved. We will call such quantum numbers topological.

One can obtain a formal proof of all this by representing the transition amplitude in the form

$$A\infty\int D\varphi(x,t)\,e^{iS},$$

where S is the classical action. The integration is to be carried out over those $\varphi(x, t)$ such that φ describes the hedgehog as $t\to -\infty$ and φ describes its decay products as $t\to +\infty$. However, on account of the topological inequivalence of $\varphi_{in}(x, t)$ and $\varphi_{out}(x, t)$ there is no path $\varphi(x, t)$ continuous in time joining these two states. Therefore for any $\varphi(x, t)$ the action is $S = \infty$.

It is tempting to assume that all conservation laws existing in nature have a topological character, but at the present time I do not know how to realize this possibility.

It should be noted that in principle there could also exist nontopological extremons. The geometric picture may be such as pictured in the Figure. As is apparent from the figure, the line going around the handle of the dumbbell has minimal length, but moving it upward, one can contract it into a point. We do not know whether there exist classical solutions of this type. If they would then, as L. B. Okun' has remarked, they would be metastable states.

I am profoundly grateful to V. L. Berezinskiĭ, A. I. Vaĭnshteĭn, Ya. B. Zel'dovich, L. B. Okun', L. D. Faddeev and A. S. Shvarts for important discussions which have clarified many questions which were unclear to the author. In addition, it is my pleasure to thank Ya. G. Sinaĭ for intuitive explanations of the basic concepts of topology.

APPENDIX A

STABILITY OF THE HEDGEHOGS

We show that the extremal (3.7) of the functional (3.3) is a minimum. Consider the first and second variation of the functional:

$$F=\int\{(\nabla\mathbf{n})^2+\rho(x)\mathbf{n}^2\}d\Omega, \qquad (A.1)$$

where $\rho(x)$ is a Lagrange multiplier. The condition $\delta F = 0$ implies

$$\nabla^2\mathbf{n}=\rho(x)\,\mathbf{n}. \qquad (A.2)$$

Substituting n in the form (3.7) we find that $\rho = -2$. We must now investigate the eigenvalues of the second variation:

$$\delta^2F=\int\{(\nabla\delta\mathbf{n})^2-2(\delta\mathbf{n})^2\}d\Omega, \qquad (A.3)$$

$$\nabla^2\delta n+2\delta n=-\lambda\delta n \qquad (A.4)$$

with the condition that $n\cdot\delta n = 0$. It is obvious that the stability condition is $\lambda\geq 0$. From (A.4) we find $\lambda = l(l + 1) - 2$, where $l\geq 1$. The latter restriction appears on account of the fact that for $l = 0$ the variation δn = const and consequently $n\cdot\delta n\neq 0$. The zero eigenvalue with $l = 1$ corresponds to the possibility of rotation of the hedgehog as a whole.

Let us now prove the instability of the solution for the isospinor Higgs field. In this case the expression (3.14) can be rewritten according to (5.2):

$$F=\sum_{\alpha=1}^{4}\int d\Omega(\nabla\nu_\alpha)^2, \qquad (A.5)$$

where ν_a is a unit vector in four-space. It is easy to verify that the Lagrange multiplier corresponding to the solution (3.15) is again -2. This leads to the equation (A.4) for $\delta\nu_\mu$. However, now $l = 0$ is admissible, since the solution (3.15) corresponds to

$$\nu_i=x_i/r,\quad \nu_4=0$$

and selecting $\delta\nu_i = 0$, $\delta\nu_4$ = const we satisfy the condition $\sum\nu_\mu\delta\nu_\mu = 0$.

Thus there is an eigenvalue $\lambda = -1$, which means instability.

APPENDIX B

We show that if one identifies the massless component of the vector field with the photon the hedgehog exhibits a magnetic charge. We go over to a gauge where the Higgs field at infinity is directed along the third axis. For this it is necessary to carry out a rotation in isospace characterized by the Euler angles:

$$\alpha=\varphi, \quad \beta=\theta, \quad \gamma=\gamma(r, \theta, \varphi) \tag{B.1}$$

(the last Euler angle is of course arbitrary, it is a rotation of the vector around itself). We consider the quantity

$$A_\mu^3=(\psi \mathbf{A}_\mu)/|\psi|. \tag{B.2}$$

As can be seen from (3.8) and (4.3), before the rotation $A_\mu = 0$. After the rotation A_μ changes, since the term $\mathrm{Tr}\,\tau R^+ \partial R/\partial x$ is added to A_μ. Thus after the rotation

$$A_\mu^3=\mathrm{Sp}\,\tau_3 R^+\partial R/\partial x_\mu. \tag{B.3}$$

Making use of (B.3) and the well-known expression of R in terms of the Euler angles, we obtain

$$A_\mu^3=\nabla_\mu\gamma+\cos\beta\nabla_\mu\alpha. \tag{B.4}$$

Going over to spherical coordinates, we obtain from (B.4) and (B.1)

$$A_\varphi=\left(\frac{\partial\gamma}{\partial\varphi}+\cos\theta\right)\frac{1}{r\sin\theta}, \quad A_\theta=\frac{1}{r}\frac{\partial\gamma}{\partial\theta}, \quad A_r=\frac{\partial\gamma}{\partial r}. \tag{B.5}$$

The magnetic field corresponding to the potential (B.5) is

$$H=-1/r^2. \tag{B.6}$$

The Dirac potential is obtained from (B.5) if one takes $\gamma = -\varphi$. Then the Dirac string is directed oppositely to the z axis.

It is remarkable that (B.6) is valid even at small distances. But, it is clear that the magnetic field acquires a physical meaning only far from the hedgehog, where the test particle is not subject to the action of the charged components of the Yang-Mills field. In addition, these components act inside the filament and the potential (B.5) is valid only far from it.

[1)]This fact has been brought to my attention by L. B. Okun'. Somewhat later we have received the preprint [6] of 'tHooft containing the same conclusion. Cf. also [4].

TRANSLATOR'S NOTES

*Similar considerations have been advanced independently by Arafune, Freund and Goebel, J. Math. Phys. 16, 433 (1975); Proc. of Intern. Symposium on Math. Methods in Physics, Kyoto, Jan. 1975, p. 240. Springer Lect. Notes in Physics, 1975.

**Some of the problems related to gauge fields and the associated Higgs fields are due to the fact that one has overlooked the different geometric nature of these fields: gauge potentials are the connection one-form and the gauge field the curvature two-form of a connection in a bundle associated to a principal fibration which has the gauge group as its structure group. The fields themselves are sections of this associated bundle and the choice of a gauge corresponds to a particular trivialization of the principal fibration. Cf., e.g., M. E. Mayer, Proceedings of the Symposium on Differential-Geometric Methods in Theoretical Physics, Bonn, July 1975 (to be published).

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Translated by Meinhard E. Mayer
212