Excitation and detection of standing gravitational waves

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A concrete scheme is considered of an experiment on the radiation and detection of gravitational waves under laboratory conditions. The scheme includes both a source and a receiver of radiation. The source of gravitational waves is an electromagnetic resonator in the form of a torus in which an alternating electromagnetic field is excited. Interference of the radiated gravitational waves leads to the picture of a standing cylindrical gravitational wave in the focal region of the radiator—near its symmetry axis. In the focal region there is placed a detector—an electromagnetic resonator in which a certain initial electromagnetic field is produced. As a result of the interaction with the gravitational wave an additional electromagnetic field appears in the detector which is capable of being observed. In the limiting case of a close positioning of the detector and the radiator at the boundary of the wave zone, a formula is derived relating the parameters of the radiator and the detector. The formula is obtained from the condition that the ratio of signal to noise in the detector be equal to unity. Preliminary estimates show that the construction of a system with the choice of parameters required for the realization of such an experiment presents a very difficult technical problem, but apparently soluble in principle.

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The negative result [1-4] of carefully repeated Weber's experiments forces one to suppose that in the universe there do not occur such grandiose and frequent processes which are accompanied by gravitational radiation accessible to already existing receivers. This forces one to embark on the lengthy and laborious path of increasing the sensitivity of receivers, and also to turn to a detailed study of the possibilities for the production and detection of gravitational waves under laboratory conditions.

The hope for the success of a laboratory experiment is associated with the possibility of a close positioning of the radiator and the detector, of an optimal choice of their shape, of the fullest possible utilization of the effect of resonance reception, etc. It is clear that if the size of the source exceeds the wavelength of the radiated wave, then for its effective operation one requires coherence of the whole volume of the radiator (so-called interference "focusing" of gravitational radiation). The concept of in-phase radiation propagated "from all sides" into a given region of space leads us to the picture of a standing wave. The standing gravitational wave is produced not because the wave is reflected from walls (the effect of reflection of a gravitational wave is negligibly small), but in virtue of the interference of waves freely propagating from different parts of the radiator. Placing the detector in the focal region of the radiator and seeking its resonance response to the produced alternating gravitational field we thereby utilize the geometry of the system in an optimum manner. Of course, there remains the problem of obtaining a sufficiently large amplitude of the gravitational wave and of an as complete as possible utilization of resonance (a detector with a very high Q is required).

Specifically, as a radiator (Sec. 1) we consider an electromagnetic (EM) resonator of toroidal shape (torus of rectangular cross section), in which a periodic EM field is produced. In the focal region of the radiator near its axis of symmetry the radiated gravitational waves have the nature of cylindrically-symmetric standing waves. Into the focal region an EM resonator is placed—a detector of cylindrical form in which a certain initial EM field is present (Sec. 2). Under the action of the gravitational wave the state of the EM field in the detector is altered and it is proposed to detect this change (Sec. 3). The possibilities and advantages of EM generators and detectors of gravitational waves have been discussed already [5,6]. Here we have given calculations of a concrete scheme including a radiator and a detector, with the gravitational waves being standing waves.

In order to obtain an idea of the size of the system capable of creating and recording gravitational waves we have considered the limiting case of a close positioning of the detector and the radiator—at the boundary of the wave zone. In this case the radiator and the detector themselves are also of the dimensions of the order of a wavelength of the gravitational wave. Utilizing formulas derived for the wave zone at the limit of their applicability we obtain a limitation interrelating the parameters of the system. This limitation can, for example, be satisfied by the following choice of parameters (Sec. 3): the characteristic intensity of the EM fields in the radiator of the detector $\sim 3 \times 10^5$ G, the Q of the resonator-detector $\sim 7 \times 10^{13}$, the wavelength of the gravitational wave $\sim 10^3$ cm, the accumulation time for the signal 4×10^5 sec, the total volume of the whole system $\sim 25 \times 10^9\ \mathrm{cm}^3$. Of course, such an experiment appears to be exceedingly difficult, but, nevertheless, its realization does not presuppose that "senselessly extremal" conditions must be satisfied, and from this point of view there remains a basis for optimism.

1. EXCITATION OF STANDING GRAVITATIONAL WAVES

The source of the gravitational field is an alternating EM field in a cavity bounded by two coaxial cylindrical surfaces of radii R_1 and R_2 and the planes $z = -L/2$ and $z = L/2$. Thus, the resonator consists of a toroid of rectangular cross section. The solution of the problem of EM oscillations in an ideal resonator of such a shape is known (cf., for example, [7]). For the sake of definiteness we choose an oscillation of the E type with fields independent of the coordinate φ. In terms of the so-called natural components the intensities of the electric and the magnetic fields satisfying the required boundary conditions are

$$E_r = f_r \sin\frac{m\pi}{L}\left(z+\frac{L}{2}\right)\sin\omega t, \quad E_z = f_z \cos\frac{m\pi}{L}\left(z+\frac{L}{2}\right)\sin\omega t, \quad (1)$$
$$H_\varphi = f_\varphi \cos\frac{m\pi}{L}\left(z+\frac{L}{2}\right)\cos\omega t,$$

where

$$f_r = \frac{m\pi}{L}[aJ_1(\varkappa r)+bN_1(\varkappa r)], \quad f_\varphi = \frac{kL}{m\pi}f_r,$$
$$f_z = \varkappa[aJ_0(\varkappa r)+bN_0(\varkappa r)];$$

J and N with indices denote Bessel and Neumann functions of corresponding orders. The characteristic frequencies $\omega = ck$ are obtained from the equation $k^2 = \kappa_n^2 + (m\pi/L)^2$, where m is an integer and κ_n are the roots of the equation

$$J_0(\varkappa R_1)N_0(\varkappa R_2) = J_0(\varkappa R_2)N_0(\varkappa R_1). \quad (2)$$

The constants a and b are related by the equations

$$aJ_0(\varkappa R_1)+bN_0(\varkappa R_1)=0, \quad aJ_0(\varkappa R_2)+bN_0(\varkappa R_2)=0,$$

of which only one is independent as a consequence of (2). In the final analysis a and b can be expressed in terms of the total energy stored in the resonator

$$\mathscr{E} = \frac{L}{16}\int_{R_1}^{R_2}(f_r^2+f_z^2+f_\varphi^2)r\,dr.$$

With the aid of (1) one can obtain the time-dependent part of the energy-momentum tensor of the EM field which is the source of the gravitational wave field.

We turn to the Einstein equations in the linear approximation [8,9]. Taking into account the gauge conditions

$$\psi_{\mu,\nu}^{\nu} = \left(h_\mu^\nu - \frac{1}{2}\delta_\mu^\nu h\right)_{,\nu} = 0 \quad (3)$$

the equations take on the form

$$\frac{1}{2}\Box\psi_{\mu\nu} = \frac{1}{2}\psi_{\mu\nu,\alpha}{}^{,\alpha} = \frac{8\pi G}{c^4}T_{\mu\nu}, \quad (4)$$

where $h_{\mu\nu}$ are corrections to the metric of a flat universe $\eta_{\mu\nu}$ in the Minkowski coordinates, $T_{\mu\nu}$ is the energy-momentum tensor of the source[1)]. Strictly speaking the source of the wave gravitational field is not only the energy-momentum tensor $T_{\mu\nu(e)}$ of the EM field in the resonator, but also the energy-momentum tensor $T_{\mu\nu(j)}$ of the charge carriers producing a current in the thin surface layer, and also the energy-momentum tensor $T_{\mu\nu(m)}$ of the elastic shell of the resonator. However, the contribution to the metric of the latter two terms is small. According to estimates in the case of a spherical resonator [5] the amplitudes of the wave fields produced by each of the tensors are interconnected by the relations (we do not write any indices on the components of the metric)

$$h(m)/h(e) \sim \lambda_s/\lambda_e \ll 1, \quad h(j)/h(e) \sim \lambda_L/\lambda_e \ll 1.$$

Here λ_e is the wavelength of the EM wave in the resonator, λ_s is the wavelength of the acoustic wave in the material of the resonator wall, λ_L is the "depth of penetration of the current" equal to $(mc^2/e^2n)^{1/2}$ where n is the number density for free electrons. We take henceforth $T_{\mu\nu}$ in (4) to mean $T_{\mu\nu(e)}$.

We seek the space components of the metric in the form of retarded solutions of Eqs. (4):

$$\psi_{ik} = -\frac{4G}{c^4}\int\frac{[T_{ik}]}{R}dV, \quad (5)$$

where T_{ik} is taken at the retarded time instant $[t] = t - R/c$, R is the distance between the point of observation and the element of integration.

We seek the components $\psi_{0\alpha}$ in terms of the known ψ_{ik} from the gauge conditions (3). The $\psi_{0\alpha}$ obtained in this manner satisfy Eqs. (4) (and the homogeneous wave equations $\Box\psi_{0\alpha} = 0$ outside the source), but they do not necessarily coincide with the retarded solutions of the corresponding Eqs. (4). Such an agreement is automatically attained only under the condition of sufficient smoothness of the functions $T_{\mu\nu}$.

The gauge conditions do not fix the coordinate system completely, they remain unchanged under the small transformations $\bar{x}^\alpha = x^\alpha + \xi^\alpha$ which satisfy the equations $\Box\xi^\alpha = 0$. We use these transformations in order to make the components $h_{0\alpha}$ vanish outside the source, i.e., to introduce a synchronous system of coordinates. Such a choice of the coordinate system will be convenient for the formulation of boundary conditions in the resonator-detector. Simultaneously with the vanishing of the components $h_{0\alpha}$ h also vanishes: $h \equiv h_{\mu\nu}\eta^{\mu\nu} = 0$.

We denote the coordinates of the point of observation by r, φ, z, and the coordinates of the element of integration by r', φ', z'. Then we have

$$R = [r^2+r'^2-2rr'\cos(\varphi-\varphi')+(z-z')^2]^{1/2}.$$

The determination of the gravitational field everywhere outside the resonator-radiator under arbitrary assumptions concerning the ratios of its dimensions presents a complicated problem. We restrict ourselves to the determination of the gravitational field in the focal region, i.e., in the neighborhood of the center of symmetry—the point with the coordinates r = 0 and z = 0. In order to do this we assume that $|r| \ll R_1$ and $|z| \ll R_1$. Moreover, we assume that the height of the resonator is small compared to its radius, $L \ll R_1$. (The calculations carried out below are valid for $L \ll \sqrt{R_1/k}$ and $|z| \ll \sqrt{R_1/k}$, and they remain valid in order of magnitude as L and |z| are increased right up to $\sqrt{R_1/k}$.) We can write approximately

$$R \approx r' - r\cos(\varphi-\varphi'). \quad (6)$$

We write out the time dependent terms in T_{ik}:

$$T_{xx} = \frac{1}{32\pi}e^{i2\omega t}\left\{-f_z^2+(f_r^2+f_\varphi^2)\cos 2\varphi + [-f_z^2-(f_\varphi^2-f_r^2)\cos 2\varphi]\cos\frac{2m\pi}{L}\left(z+\frac{L}{2}\right)\right\},$$
$$T_{yy} = \frac{1}{32\pi}e^{i2\omega t}\left\{-f_z^2-(f_r^2+f_\varphi^2)\cos 2\varphi + [-f_z^2-(f_\varphi^2-f_r^2)\cos 2\varphi]\cos\frac{2m\pi}{L}\left(z+\frac{L}{2}\right)\right\},$$
$$T_{zz} = \frac{1}{32\pi}e^{i2\omega t}\left\{(f_\varphi^2-f_r^2+f_z^2)+(f_\varphi^2+f_r^2+f_z^2)\cos\frac{2m\pi}{L}\left(z+\frac{L}{2}\right)\right\},$$
$$T_{xy} = \frac{1}{32\pi}e^{i2\omega t}\left\{(f_r^2+f_\varphi^2)+(f_\varphi^2-f_r^2)\cos\frac{2m\pi}{L}\left(z+\frac{L}{2}\right)\right\}$$
$$T_{xz} = \frac{1}{16\pi}e^{i2\omega t}f_r f_z\cos\varphi\sin\frac{2m\pi}{L}\left(z+\frac{L}{2}\right), \quad T_{yz} = T_{xz}\,\mathrm{tg}\,\varphi.$$

Substituting

$$[t] = t - \frac{r'}{c} + \frac{r}{c}\cos(\varphi-\varphi'),$$

replacing R by r' in the denominator of the integrand in (5) and carrying out the integration over φ' from 0 to 2π and over z' from $-L/2$ to $L/2$, we obtain the components ψ_{ik}:

$$\psi_{xx} = F[\alpha_1 J_0(2kr)+\alpha_2\cos 2\varphi J_2(2kr)],$$
$$\psi_{yy} = F[\alpha_1 J_0(2kr)-\alpha_2\cos 2\varphi J_2(2kr)],$$
$$\psi_{zz} = -F\alpha_3 J_0(2kr), \quad \psi_{xy} = F\alpha_2\sin 2\varphi J_2(2kr), \quad \psi_{xz} = \psi_{yz} = 0,$$

where

$$F=\frac{GL}{4c^4}e^{i2\omega t},\quad \alpha_1=\int\limits_{R_1}^{R_2} f_z^2 e^{-i2kr'}\,dr',$$

$$\alpha_2=\int\limits_{R_1}^{R_2}(f_r^2+f_\varphi^2)\,e^{-i2kr'}\,dr',$$

$$\alpha_3=\int\limits_{R_1}^{R_2}(f_\varphi^2-f_r^2+f_z^2)\,e^{-i2kr'}\,dr'.$$

From (3) we obtain

$$\psi_{0x}=-Fi(\alpha_2-\alpha_1)\cos\varphi J_1(2kr),\qquad \psi_{0y}=\mathrm{tg}\,\varphi\,\psi_{0x},\qquad \psi_{0z}=0,$$
$$\psi_{00}=-F(\alpha_2-\alpha_1)J_0(2kr).$$

From this we obtain

$$\psi_{\mu\nu}\eta^{\mu\nu}=F(\alpha_3-\alpha_1-\alpha_2)J_0(2kr).$$

The small transformations $\bar{x}^\alpha = x^\alpha + \xi^\alpha$ which eliminate $h_{0\alpha}$ and which do not violate the conditions (3) have the form

$$\xi_x=F\frac{1}{8k}(\alpha_1-3\alpha_2+\alpha_3)\cos\varphi J_1(2kr),\qquad \xi_y=\mathrm{tg}\,\varphi\,\xi_x,\qquad \xi_z=0,$$
$$\xi_0=-F\frac{1}{8k}i(3\alpha_1-\alpha_2-\alpha_3)J_0(2kr).$$

After carrying out these transformations and introducing the notation

$$\Omega=2\omega,\quad K=2k,\quad A=-\frac{GL}{8c^4}|\alpha_1+\alpha_2+\alpha_3|,$$

where $\alpha_1 + \alpha_2 + \alpha_3 = |\alpha_1 + \alpha_2 + \alpha_3|e^{i\psi}$, we finally obtain

$$\begin{aligned}
h_{xx}&=-{}^1/_2A\cos(\Omega t+\psi)\,[J_0(Kr)+\cos 2\varphi J_2(Kr)],\\
h_{yy}&=-{}^1/_2A\cos(\Omega t+\psi)\,[J_0(Kr)-\cos 2\varphi J_2(Kr)],\\
h_{zz}&=A\cos(\Omega t+\psi)J_0(Kr),\\
h_{xy}&=-{}^1/_2A\cos(\Omega t+\psi)\sin 2\varphi J_2(Kr),\qquad h_{xz}=h_{yz}=0.
\end{aligned}\tag{7}$$

The nonvanishing relative corrections to the metric in cylindrical coordinates are expressed in the following manner:

$$h_{rr}=-A\cos(\Omega t+\psi)\frac{J_1(Kr)}{Kr},\qquad h_{\varphi\varphi}=-A\cos(\Omega t+\psi)\frac{dJ_1(Kr)}{d(Kr)}$$
$$h_{zz}=A\cos(\Omega t+\psi)J_0(Kr).$$

As should have been expected, the gravitational field in the focal plane has the form of a standing cylindrical wave. The dependence of the field on z is practically absent as long as the z-coordinate of the point of observation satisfies the inequality $|z| \ll \sqrt{R_1/k}$.

It remains for us now to find the amplitude A of the wave and to formulate the conditions for the optimal choice of the dimensions of the resonator-radiator in order that A should take on the greatest possible values. First of all we note that A is proportional to L. Such a dependence is a consequence of the assumption $L \ll \sqrt{R_1/k}$. As L increases the amplitude A will grow until L becomes comparable to $\sqrt{R_1/k}$, and this corresponds to the difference in the displacement from the edges of the system of the order of the wavelength of the gravitational wave. A further increase in L does not lead to an increase in A, and therefore the upper limit on L is $L \sim \sqrt{R_1/k}$. Further, from the expression

$$\alpha_1+\alpha_2+\alpha_3=2\int\limits_{R_1}^{R_2}(f_\varphi^2+f_z^2)\,e^{-i2kr'}\,dr'\tag{8}$$

it can be seen that the thickness of the resonator $\Delta R = R_2 - R_1$ also should not be chosen to be too great. The choice $k\Delta R \sim 1$ is optimal, since as ΔR is increased from zero the value of the integral increases only until ΔR becomes of order $1/k$. As ΔR is increased further the integral can not take on larger values.

In order that at least one wavelength of the gravitational wave should be contained within the focal region we must have $kr \sim 1$, and since, in accordance with our assumption, $|r| \ll R_1$, we have $kR_1 \gg 1$. From this and from the condition $k\Delta R \sim 1$ we obtain $\Delta R/R_1 \ll 1$. Taking this inequality into account it follows from (2) that $\kappa R_1 \gg 1$ and, consequently, that $\kappa R_2 \gg 1$. Then, utilizing the asymptotic expressions for the Bessel functions for large values of the argument we find from (2) that $\kappa_n = n\pi/(R_2 - R_1)$, where n is an integer. Evaluating the integral (8) by utilizing the value of κ_n obtained above and expressing the constants a and b in terms of the total energy $\mathscr{E}$, we obtain finally

$$A=-2\frac{G}{c^4}\frac{\mathscr{E}}{R_1}\left(2+\frac{\varkappa_n^2}{k^2}\right)\frac{\sin(k\Delta R)}{k\Delta R},\quad \psi=-2kR_1-k\Delta R.$$

Since the total energy $\mathscr{E} = \epsilon V$, where ϵ is the average energy density in the resonator while $V \approx 2\Delta R R_1 L$ is its volume, then the dependence of A on R_1 with all the other parameters kept constant disappears. This means that coaxially situated resonators with the same values of ϵ, ΔR, L, k, κ produce identical amplitudes of the gravitational field. A diminution of the amplitude due to the greater separation from the resonator ($\sim 1/R_1$) is completely compensated by an increase in its radius and its volume ($\sim R_1$). For a properly chosen phase of the EM oscillations (taking retardation into account) the contributions of all the coaxially situated resonators to the amplitude of the gravitational wave are summed.

It follows from the equation $k^2 = \kappa_n^2 + (m\pi/L)^2$, that the ratio $(\kappa_n/k)^2$ can not exceed unity. Equality of κ_n and k is attained for m = 0, and this corresponds to purely radial cylindrically-symmetric oscillations of the EM waves, which do not produce gravitational radiation. Indeed

$$\sin(k\Delta R)=\sin\pi[n^2+m^2(\Delta R/L)^2]^{1/2},$$

and this quantity (together with other degenerate cases) vanishes in the special case m = 0. From the same equation it can also be seen that since it is advantageous to choose L considerably larger than ΔR, then the number m characterizing the periodicity of the EM field with respect to the coordinate z must be sufficiently great: $(m/n)(\Delta R/L) \sim 1$, in order that there should not appear in the expression for A additional small factors.

For purposes of illustration we calculate A on the assumption that

$$k\Delta R={}^3/_2\pi.\tag{9}$$

This case is realized only in the case of the minimally possible value n = 1, and this leads to $\kappa/k = 2/3$ and to the relationship $m\Delta R/L = \frac{1}{2}\sqrt{5}$. Then for A we obtain

$$A\approx 1.05\frac{G}{c^4}\frac{\mathscr{E}}{R_1},\tag{10}$$

and this can also be rewritten in the form $A \approx 0.52 r_g/R_1$, where r_g is the gravitational radius for the mass of the EM of the field in the resonator.

In order of magnitude the value of the amplitude A can be obtained directly from (5). Since T_{ik} coincides with an accuracy up to numerical coefficients with the energy density ϵ, then for the characteristic amplitude of the metric h (we do not write out the indices) we obtain

$$h \sim \frac{G}{c^4}\frac{1}{R_1}\varepsilon V_c. \tag{11}$$

Here R_1 is the "characteristic" distance from the point of observation to the radiator, V_c is the "coherent volume of the radiator." This volume is formed by the elements of the radiator for which taking retardation into account during the integration in (5) does not yet lead to the interference "cancellation" of the field produced by them. We seek the characteristic linear dimensions the product of which will represent V_c in the case under consideration.

In the integration of (5) with respect to z the result will contain L if $L \ll \sqrt{R_1/k}$, and will contain a quantity not greater than $\sqrt{R_1/k}$ if L is of the order of, or much greater than $\sqrt{R_1/k}$. R_1 will enter the result in integration over φ (coherence with respect to the coordinate φ due to the equal separation of the point of observation from the elements of integration). In integration over r a quantity will enter which is not greater than $1/k$, in virtue of the retardation along the line of sight. Thus, $V_c \sim LR_1/k$. Since we have chosen $\Delta R \sim 1/k$ and $L \ll \sqrt{R_1/k}$, then for the points of observation situated in the focal region the whole volume of one radiator operates coherently and we have

$$h \sim \frac{G}{c^4}\frac{1}{R_1}\varepsilon LR_1\Delta R \sim \frac{G}{c^4}\frac{\mathscr{E}}{R_1},$$

where $\mathscr{E}$ is the total energy of the EM field in one resonator, and this coincides with (10).

In calculating the wave gravitational field we assumed that it is created by undamped EM oscillations. In a real situation the gravitational radiation will be maintained at approximately the same level only over an interval of time determined by the Q of the resonator-radiator, $\Delta t \sim Q/\omega$. If this time interval is insufficient to carry out an experiment on detection, it can be increased by supplying additional EM energy to the radiator.

2. DETECTION OF GRAVITATIONAL WAVES

As the detector of gravitational waves we also utilize a EM resonator. The detector is placed in the focal region of the radiator, at the point where the wave gravitational field has the form (7). All information concerning the gravitational field is contained in (7), and we do not need to construct such quantities as the density or the flux of gravitational energy. Nevertheless, it is of interest to note that for (7), just as for all standing waves, the time average of the flux density of the energy (calculated, for example, utilizing the energy-momentum pseudotensor of the gravitational field) is identically equal to zero.

We assume that the resonator-detector has the shape of a cylinder, of height l and of radius R and is oriented along the z axis. The gravitational field (7) changes the state of the EM field existing in the resonator, and this change is, in principle, accessible to detection. In a closed resonator only very definite frequencies and configurations of the EM field are possible. The gravitational wave plays the role of a driving force in the equation for the oscillations. If it satisfies the conditions of resonance, then the amplitude of the EM oscillations is altered particularly strongly, and the magnitude of this alteration is determined by the Q of the resonator.

Since we assume that the frequency of the gravitational wave coincides with, or is close to one of the characteristic frequencies of the EM field, then in this case the walls of the resonator behave as "soft" ones. This means that the elements of the shell of the resonator under the action of the gravitational field move as free particles. Indeed, the lowest characteristic frequencies of the shell of the resonator (ω_S) and of the EM field in it (ω) have the same relation to each other as the velocity of sound and the velocity of light, and consequently $\omega_S \ll \omega - \Omega$, and we can neglect the elastic force compared to the gravitational force. The coincidence of Ω with one of the very high frequencies of the shell does not alter the conclusions with respect to its motion, since the contribution of high frequencies is negligibly small. The boundary conditions for the EM field in the resonator are formulated along the world lines of the elements of its shell. For their description it is convenient to utilize the synchronous system of coordinates, in which they are given by the equation $x^i = \text{const}$. It is precisely because of this that we have carried out a transformation of the wave metric to the synchronous system of coordinates.

We now turn to the equations of electrodynamics in a curved universe. From the generally covariant Maxwell equations

$$F_{\alpha\beta,\gamma}+F_{\gamma\alpha,\beta}+F_{\beta\gamma,\alpha}=0, \quad F^{\alpha\beta}{}_{;\beta}=-\frac{4\pi}{c}j^{\alpha} \tag{12}$$

it is possible to obtain a generalization of the usual wave equation to the case of a curved space:

$$F_{\mu\nu;\alpha;}{}^{\alpha}+R_{\mu\nu\alpha\beta}F^{\alpha\beta}+R_{\mu}{}^{\alpha}F_{\nu\alpha}+R_{\nu}{}^{\alpha}F_{\alpha\mu}=-\frac{4\pi}{c}j_{[\mu;\nu]},$$

where the square bracket denotes antisymmetrization.

We consider the approximation for a weak gravitational field $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, assuming, that for $h_{\mu\nu}$ the gauge conditions (3) are satisfied. We then obtain the equations

$$F_{\mu\nu,\alpha,}{}^{\alpha}-h^{\alpha\beta}F_{\mu\nu,\alpha,\beta}+(h^{\alpha\beta,\gamma}+h^{\beta\gamma,\alpha}-h^{\alpha\gamma,\beta})\,\eta_{\alpha[\mu}F_{\nu]\beta,\gamma}+h_{\alpha[\mu,\nu],\beta}F^{\beta\alpha}=-\frac{4\pi}{c}j_{[\mu,\nu]}. \tag{13}$$

We consider that the $h_{\mu\nu}$ appearing in (13) are created by the gravitational wave (7). The quantities $F_{\mu\nu}$ represent the sum of the "unperturbed" EM field ${}^{(0)}F_{\mu\nu}$ and of the correction ${}^{(1)}F_{\mu\nu}$ which originates as a result of the action of the gravitational wave. The nature of the initial field ${}^{(0)}F_{\mu\nu}$ and the relationship of the phases of the EM oscillations and the gravitational wave can be quite varied [6]. Depending on their choice different experimental arrangements are realized. We consider two variants in one of which ${}^{(0)}F_{\mu\nu}$ corresponds to a constant magnetic field, and in the second one in addition to the constant magnetic field a weak standing EM wave is present in the resonator.

The electrodynamic problem can be solved most simply in the case of an ideal resonator that has infinitely conducting walls and a nonconducting dielectric. Such a resonator has an infinite Q. But every real resonator has a finite quality factor Q, which, basically, apparently, depends on the conductivity of the walls. We take into account the damping of the alternating EM field in the resonator by setting against the power losses in the wall

$$P = \frac{\omega}{8\pi Q}\int \mathbf{H}^2\,dV$$

at a certain frequency ω the effective conductivity of the dielectric σ producing in it the same magnitude of losses

$$P = \frac{\sigma}{2}\int \mathbf{H}^2\,dV.$$

From this it follows that $\sigma = \omega/4\pi Q$. After such a replacement we can consider the walls of the resonator to be ideal and, consequently, formulate the boundary conditions at them as $F_{0i(\text{tang})} = E_{(\text{tang})} = 0$, and express the currents in (13) in terms of the intensity of the electric field in accordance with the equation $j_i = -\sigma F_{0i}$. Equations (13) can be conveniently solved for the quantities F_{0i}, while the other components F_{ik} can be determined from (12).

Let there be set up in the resonator a constant magnetic field directed along the z axis. The only component different from zero is ${}^{(0)}F_{12} = H = \text{const}$. From Eq. (13) with the indices 0i it follows that the "driving force" is present in the equations for ${}^{(1)}F_{01}$ and ${}^{(1)}F_{02}$. It is convenient to go over to the natural cylindrical components. Then the only nontrivial equation will be the one for $r^{-1}\,{}^{(1)}F_{0\varphi} \equiv E_\varphi$:

$$\Box E_\varphi + \frac{1}{r^2}E_\varphi = Hh_{zz,r,0} = AK^2HJ_1(Kr)\sin(\Omega t+\psi), \qquad (14)$$

while E_r and E_z are identically equal to zero.

We seek the solution of (14) in the form of an expansion in terms of the eigenfunctions of the unperturbed boundary value problem satisfying the boundary conditions $E_\varphi = 0$ for $z = \pm l/2$ and for $r = R$. Since the right hand side of (14) does not depend on φ, we have

$$E_\varphi = \sum_{p,q} B_{pq}(t)J_1(\varkappa_q r)\sin\frac{p\pi}{l}\left(z+\frac{l}{2}\right), \qquad (15)$$

where p is an integer, κ_q is a root of the equation $J_1(\kappa_q R) = 0$. The eigenfrequencies of the EM oscillations are determined from the equation $\omega^2_{pq}/c^2 = k^2_{pq} = \kappa^2_q + (p\pi/l)^2$. We also expand the right hand side of (14) in terms of eigenfunctions:

$$J_1(Kr) = \sum_{p,q} a_p b_q J_1(\varkappa_q r)\sin\frac{p\pi}{l}\left(z+\frac{l}{2}\right), \qquad (16)$$

where $a_p = 2[1-(-1)^p]/p\pi$ and

$$b_q = \frac{2\varkappa_q}{R(K^2-\varkappa_q^2)}\frac{J_1(KR)}{J_0(\varkappa_q R)} \quad \text{for} \quad K\neq\varkappa_q,$$

$$b_q = 1 \quad \text{for} \quad K=\varkappa_q.$$

Substituting (15) and (16) into (14) we obtain the equation for $B_{pq}(t)$:

$$\ddot B_{pq} + \frac{\omega_{pq}}{Q}\dot B_{pq} + \omega_{pq}^2 B_{pq} = A\Omega^2 H a_p b_q \sin(\Omega t+\psi).$$

We are interested in the steady state regime when oscillations occur at the frequency of the driving force. The corresponding solution is of the form

$$B_{pq} = C_{pq}\sin(\Omega t+\psi+\chi),$$

where

$$C_{pq} = A\Omega^2 H a_p b_q\left[(\omega_{pq}^2-\Omega^2)^2+\frac{\omega_{pq}^2\Omega^2}{Q^2}\right]^{-1/2}, \quad \operatorname{tg}\chi = \frac{\omega_{pq}\Omega}{Q(\Omega^2-\omega_{pq}^2)}$$

Evidently the maximum value of C_{pq} is attained when Ω coincides with one of the eigenfrequencies ω_{pq}. Then the resonance value is

$$C_{pq} = AQHa_p b_q, \qquad (17)$$

with C_{pq} being the greater the lower is the eigenfrequency being excited.

From (12) we find the remaining components of the field, and finally for the p,q-harmonic we have:

$$E_\varphi = C_{pq}J_1(\varkappa_q r)\sin\frac{p\pi}{l}\left(z+\frac{l}{2}\right)\sin(\Omega t+\psi+\chi),$$

$$H_\rho = -C_{pq}\frac{p\pi}{Kl}J_1(\varkappa_q r)\cos\frac{p\pi}{l}\left(z+\frac{l}{2}\right)\cos(\Omega t+\psi+\chi),$$

$$H_z = H + C_{pq}\frac{\varkappa_q}{K}J_0(\varkappa_q r)\sin\frac{p\pi}{l}\left(z+\frac{l}{2}\right)\cos(\Omega t+\psi+\chi). \qquad (18)$$

As can be seen, under the action of the gravitational field there appears in the resonator an alternating EM field—a standing EM wave with a characteristic field intensity proportional to AQH.

We obtain the change in the total energy in the resonator. The energy density ϵ is equal to

$$\varepsilon = T_{00} = [E_\varphi^2+H_\rho^2+H_z^2(1-h_{zz})]/8\pi.$$

In zero order approximation ${}^{(0)}\epsilon = H^2/8\pi$, and the total energy realized in the form of a constant magnetic field is given by

$$\mathscr{E} = H^2V/8\pi, \quad V = \pi R^2 l.$$

In the approximation linear in A we have

$${}^{(1)}\varepsilon = (2H\,{}^{(1)}H_z - H^2 h_{zz})/8\pi.$$

Integration of this quantity over the volume yields zero. (In integrating over h_{zz} we must remember that we represent $h_{zz,r}$ in the form of an expansion in terms of $J_1(\kappa_q r)$, while $J_1(\kappa_q R) = 0$.) Thus, the change in the energy $\Delta\mathscr{E}$ is proportional only to A^2. Of the terms proportional to A^2 the greatest are those which contain Q^2. We know all these terms. For ${}^{(2)}\epsilon$ constructed of these terms we have

$${}^{(2)}\varepsilon = (E_\varphi^2+H_\rho^2+{}^{(1)}H_z^2)/8\pi.$$

After integration over the volume and after taking (17) into account we obtain

$$\Delta\mathscr{E} = VC_{pq}^2J_0^2(\varkappa_q R)/16\pi = {}^1\!/_2\mathscr{E}(AQ)^2(a_p b_q)^2J_0^2(\varkappa_q R).$$

Since $\Delta\mathscr{E}$ is proportional to l, then for a given R it is better to take large values of l, up to $l \sim \sqrt{R_1/k}$. Then for $p\pi R/l \ll \kappa_q R$ we have $b_q \approx 1$. It is advantageous to excite the lowest eigenfrequencies in a resonant manner. As an example we calculate $\Delta\mathscr{E}$ for p = 1, q = 1. We obtain $a_1 = 4/\pi$, $b_1 = 1$, $\kappa_1 R \approx 4$ and finally,

$$\Delta\mathscr{E} = 0.1(AQ)^2\mathscr{E}. \qquad (19)$$

The time for the accumulation of the signals (the time for reaching a steady state regime) is equal in order of magnitude to $\tau \sim 2Q/\Omega$.

We now consider a variant of the detector in which in addition to a constant magnetic field there is also present a weak standing EM wave. The frequency and the configuration of the EM wave are chosen specifically to be of a nature such that they are excited in a resonant manner by the gravitational wave as a result of its interaction with a constant magnetic field (as in the preceding variant). The initial EM wave will be damped in accordance with the Q of the resonator, but the gravitational wave interacting primarily with the intense constant magnetic field will produce an additional EM wave which adds with the initial wave when their phases coincide. As a result the damping of the initial EM wave will be diminished and this amounts effectively to a certain increase in the Q of the resonator or, in other words, to an increase in the "ringing time"—the time required for the energy to be reduced by a factor of e.

Thus, let the unperturbed EM field have the form (18), where C_{pq} should be replaced by $D\exp(-\Omega t/2Q)$. We consider that $D \ll H$, but nevertheless that $D \gg AQH$, since in the opposite case we would be brought back to the variant already considered. The additional EM field produced by the gravitational wave also has the form (18) (without the term H in the expression for H_z), where C_{pq} should be replaced by $AQHa_pb_q[1-\exp(-\Omega t/2Q)]$. The resulting alternating field is the sum of the initial and the additional EM waves.

The total energy contained in the resonator in the form of an unperturbed alternating EM field is given by

$$\tilde{\mathscr{E}}=\frac{1}{16\pi}VD^2J_0^2(\varkappa_qR)e^{-\Omega t/Q}.$$

We obtain $\Delta\mathscr{E}$—the correction to $\tilde{\mathscr{E}}$ linear in A. If in the calculation of ${}^{(1)}\epsilon$ in the unperturbed EM field we retain only H, then we obtain the term which vanishes after integration over the volume. In comparing the remaining terms appearing in ${}^{(1)}\epsilon$ we should keep in mind that the factor $Q(1-e^{-\Omega t/2Q})$ becomes greater than unity already at a value of t which exceeds several periods (although we can still have $t \ll Q/\Omega$). Therefore the principal terms in ${}^{(1)}\epsilon$ are those that are proportional to $DAQHe^{-\Omega t/2Q}(1-e^{-\Omega t/2Q})$. After integration of ${}^{(1)}\epsilon$ over the volume we obtain

$$\Delta\mathscr{E}=2\tilde{\mathscr{E}}AQ\frac{H}{D}a_pb_qe^{\Omega t/2Q}(1-e^{-\Omega t/2Q}).$$

By the time the characteristic time instant $t_c = 2Q/\Omega$ is reached and for p = q = 1 the accumulated energy $\Delta\mathscr{E}$ will amount to

$$\Delta\mathscr{E}_c\approx 5AQ\frac{H}{D}\tilde{\mathscr{E}}_c, \tag{20}$$

where $\tilde{\mathscr{E}}_c$ is the value of $\tilde{\mathscr{E}}$ at $t = t_c$, $\tilde{\mathscr{E}}_c \approx 10^{-4}VD^2/3$. For $1/\Omega \ll t \ll Q/\Omega$ we can write

$$\tilde{\mathscr{E}}+\Delta\mathscr{E}\approx\frac{1}{16\pi}VD^2J_0^2(\varkappa_qR)\exp\left[-\frac{\Omega}{Q}\left(1-\frac{AQH}{D}a_pb_q\right)t\right],$$

and this means an increase in Q of $\Delta Q = Q(AQH/D)a_pb_q$, or, what is equivalent to this, an increase in the "ringing time" of the resonator by $\Delta\tau = \tau(AQH/D)a_pb_q$. We recall the condition of applicability of the formulas obtained above: $AQH/D \ll 1$.

In conclusion we note that the toroidal resonator-radiator is situated in the field of its own gravitational radiation, as a result of which there arises a possibility in principle of utilizing it also as a detector. Indeed, the radiated convergent cylindrical wave after passing through the axis of symmetry is converted into a divergent wave, which passes through the radiator. If in the radiator there was excited an EM oscillation of frequency ω, then the frequency of the resulting gravitational wave is 2ω. The interaction of the gravitational wave of frequency 2ω with the EM field of the radiator leads to the production of additional EM fields at frequencies of $2\omega-\omega=\omega$ and $2\omega+\omega=3\omega$. With a special choice of the dimensions of the radiator one can achieve the situation that the frequency 3ω would also be an eigenfrequency of the resonator. In this case there will occur a resonant excitation of oscillations of this frequency in analogy to the situation that was described above in the first variant of the detector. At the frequency ω, in analogy to the case described in the second variant of the detector, addition of the initial and the additional EM fields will occur for a suitable relationship between their phases (which can always be achieved by a suitable choice of the radius R_1).

3. POSSIBILITIES OF PERFORMING AN EXPERIMENT

The resonant action of the gravitational wave on the detector leads to an increase in its total EM energy, and this can be described as the appearance of new "quanta" of the EM field. The number of these new quanta is $N_1 = \Delta\mathscr{E}/\hbar\Omega$ where $\hbar$ is the Planck constant. If in the resonator there is present an initial alternating EM field, then the number of quanta already present is $N = \mathscr{E}/\hbar\Omega$. We assume that the noise in the detector is determined by the number of quanta $\sqrt{N}$. The gravitational signal can be regarded as having been detected if $N_1 = 1$ against the background of the constant EM field (as in the first variant of the detector) or if $N_1 = \sqrt{N}$ when N quanta are already present (as in the second variant).

Utilizing (19) we can write down the equation for the possibility of detection in the form

$$7\cdot10^{-2}\,AQHV^{1/2}(\hbar\Omega)^{-1/2}=1 \tag{21}$$

and, utilizing (20), in the form

$$3\cdot10^{-2}\,AQHV^{1/2}(\hbar\Omega)^{-1/2}=1. \tag{22}$$

As we can see, the conditions for the possibility of detection practically coincide in the two variants. We note that approximately the same equation is obtained in the variant of the detector containing only the initial standing EM wave without the constant magnetic field. In this case the role of H is played by the characteristic value of the intensity of the alternating EM field.

The condition for the possibility of detection imposes requirements on the parameters of the detector for a given A and Ω. Since A and Ω themselves are determined by the parameters of the radiator it follows that from this one can obtain certain general limitations on the properties of the system as a whole. For preliminary estimates as to orders of magnitude we consider the limiting case of the close position of the detector at the boundary of the wave zone. At the same time the dimensions of the radiator and the detector are also of the order of magnitude of the wavelength of the gravitational wave λ[2].

Specifically we assume that $R_1 = 2\lambda$, then $l = \sqrt{\lambda R_1/2} = \lambda$. Since $\Omega \approx 4c/R$, then $R \approx 2\lambda/3$.

Substituting these data into (21) we rewrite the condition for the possibility of detection:

$$AQH\lambda^2(\hbar c)^{-1/2}\approx 32. \tag{23}$$

For the parameters of the radiator we have $L \approx \sqrt{\lambda R_1/2} = \lambda$ and from condition (9) we obtain $\Delta R = 3\lambda/2$, and from this $R_2 = 7\lambda/2$. On the basis of (10) the amplitude is determined from the equation

$$A=\frac{G}{c^4}\frac{\pi(R_2^2-R_1^2)}{R_1}L\frac{E^2}{4\pi},$$

where E is the characteristic intensity of the EM field in the resonator-radiator. Utilizing the parameters of the radiator introduced above we arrive at the relation

$$A\approx\frac{G}{c^4}E^2\lambda^2. \tag{24}$$

Substituting (24) into (23) we obtain finally

$$\lambda^4E^2HQ\approx 32\frac{c^4}{G}(\hbar c)^{1/2}\approx 2\cdot10^{42}\ \text{g}^{3/2}\text{cm}^{5/2}\text{sec}^{-3}. \tag{25}$$

This relation imposes very severe requirements on the quantities appearing in it and, probably, can not be satisfied on the basis of the present level of technology. It could be, for example, satisfied using the presently unattainable set of values $E \sim H \sim 3 \times 10^5$ G, $Q \sim 7 \times 10^{13}$ and $\lambda \sim 10^3$ cm, which assumes a total volume for the whole system of $V_t \sim 25 \times 10^9$ cm^3 and an accumulation time for the signal of $\tau \sim 4 \times 10^5$ sec. At the same time the energy of one quantum is $\hbar\Omega = 2 \times 10^{-19}$ erg, which corresponds to a temperature of $T \sim 10^{-3}$ K. The volume of the whole system could be reduced by increasing E, H and Q, and conversely, a decrease in the product E^2HQ requires an increase in λ and in the volume of the system. It is clear that the realization of such an experiment requires the overcoming of tremendous difficulties, but doubtlessly such expenditures will be compensated by a broadening in principle of our knowledge of nature and, in future, by a possible utilization of gravitational waves for practical purposes.

Note added in proof (March 20, 1975). The metric (7) is a linear approximation, written in the TT-gauge, to the exact Einstein-Rosen vacuum solution (cf., for example, [10]). This metric can be obtained from the usual form of the Einstein-Rosen solution by means of standard transformations ([9], p. 950).

1)The Greek indices take on the values 0, 1, 2, 3 and the Latin ones take on the values 1, 2, 3. The signature of the metric is given by: +---.

2)If we assume that Newtonian gravitational theory is valid in the wave zone, then we can obtain from the Poisson equation the alternating gravitational potential the instantaneous value of which is determined by the distribution of mass in the resonator-radiator. From the known potential we obtain the gravitational force at the points of space occupied by the detector. The EM field in the detector can be described as an elastic medium of density $\rho = \epsilon/c^2$ (where ϵ is the energy density of the EM field), and with the speed of sound comparable to the speed of light. Then the "Newtonian" gravitational force, just as the "Einsteinian" force, leads to the appearance of an alternating EM field in the resonator, but the magnitude of this field is smaller by a factor of $(\lambda/R_1)^2$. As the radiator and the detector are made to approach each other to a distance of the order of λ the "Newtonian" effect becomes comparable to the "Einsteinian" effect. An exact quantitative calculation must indicate that optimal distance between the detector and the radiator at which the relativistic effect is, say, still greater than the "Newtonian" effect by a factor of 10. Probably, for this it is sufficient to choose a distance equal to several wavelengths.

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