Influence of excitation dragging by phonons on the thermal conductivity of pure superconductors

L. E. Gurevich and E. T. Krylov

A. F. Ioffe Physico-technical Institute, USSR Academy of Sciences, Leningrad
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Dragging of excitations by phonons and its contribution to the thermal conductivity of pure superconductors are investigated. The thermal-conductivity mechanism is compared quantitatively with the familiar mechanisms. It is shown that at temperatures close to the critical the thermal conductivity is determined by the excitation flux, the dominant role in the flux being played by energy exchange between the excitations and phonons. Exchange of momentum between excitations and phonons also contributes to the thermal conductivity but the contribution is negligible at all temperatures. However, at temperatures from 1.5 to 3 times lower than the critical temperature along with phonons, dragging of excitations by them contributes significantly to the thermal conductivity. Qualitative agreement between the theory and experiment is observed in superconducting lead and mercury. An experimental means of observing the effect of excitation dragging by phonons on thermal conductivity is mentioned.

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1. INTRODUCTION

The thermal conductivity of pure superconductors, in which the scattering of phonons and excitations by defects is negligible, can be determined by three phenomena: scattering of the excitations by phonons, scattering of phonons by excitations, and dragging of excitations by phonons. The dragging of phonons by excitations, which is possible in principle, turns out to be very small and can be disregarded.

The first two mechanisms were investigated by Bardeen, Rickayzen, and Tewordt[1] and by Geĭlikman and Kresin[2], and the third is considered by us here. The first two mechanisms consist of two parts, one due to exchange of energy between excitations and phonons, and the other due to momentum exchange. The authors of[1,2] confined themselves to determination of the temperature dependences of the different parts, but did not compare them in magnitude.

Concerning the relative role of these parts, there are still diverging opinions (for example, the opinion that[1] is utterly incorrect, as stated in[3] and also mentioned in[2]). We have therefore estimated quantitatively the roles of all four mechanisms, including the process considered by us, namely the dragging of excitations by the phonons. This estimate shows that the part of the thermal conductivity which was determined in[2] is negligibly small in comparison with the remaining contributions to the thermal conductivity over the entire temperature interval. Near the critical temperature, the main contribution is made by the part obtained in[1].

As the temperature is lowered, the principal role is assumed, first, by the thermal flux of the phonons and, second, by the flux of the excitations dragged by the phonons, which is considered by us. We shall show that these two mechanisms can turn out to be comparable in a certain temperature interval. The latter differs strongly from the situation in the normal metal, where the dragging of the electrons by phonons, according to J. Ziman[4], does not play a significant role in the thermal conductivity. The reason for this difference lies in the fact that in the pure superconductors considered by us the number of excitations (unlike in a normal metal) decreases with decreasing temperature approximately exponentially, so that the phonon mean free path increases just as abruptly, and with it the ability of the "phonon wind," which exists in the presence of a temperature gradient, to drag the excitations.

2. KINETIC EQUATION FOR THE EXCITATIONS; THE THERMAL CONDUCTIVITY DUE TO MOMENTUM EXCHANGE AND TO DRAGGING OF EXCITATIONS BY PHONONS

Measurements of the thermal conductivity of mercury[5] and lead[6] in the superconducting state, as well as the results of other investigations[7,8], lead to the conclusion that the scattering of excitations by phonons plays an essential role in a definite temperature interval. We shall show later on that to explain certain deviations from the simplest theoretical predictions indicated in the cited papers it suffices for the excitations to interact with the phonons. We therefore confine ourselves in the collision integral to allowance for this interaction only. Then the kinetic equation for the excitations takes the form

$$\frac{\partial f_0}{\partial T}\frac{\xi(p)}{\varepsilon(p)}\frac{1}{m}(\mathbf{p},\nabla T)=I_1+I_2, \qquad (1)$$

where the collision integrals I_1 and I_2 are given by

$$I_1=-\Big\{\int UN_0(q)e^{x'}f_0(x)f_0(x')(\varphi'-\varphi)Cq\delta(\varepsilon'-\varepsilon-\hbar\omega)d\Omega$$
$$+\int UN_0(q)e^{x}f_0(x)f_0(x')(\varphi'-\varphi)Cq\delta(\varepsilon'-\varepsilon+\hbar\omega)d\Omega$$
$$-\int VN_0(q)e^{z}f_0(x)f_0(x')(\varphi'+\varphi)Cq\delta(\varepsilon'+\varepsilon-\hbar\omega)d\Omega\Big\} \qquad (2)$$

$$I_2=\Big\{\int UN_1(q)[f_0(x')-f_0(x)]Cq\delta(\varepsilon'-\varepsilon-\hbar\omega)d\Omega$$
$$+\int UN_1(q)[f_0(x')-f_0(x)]Cq\delta(\varepsilon'-\varepsilon+\hbar\omega)d\Omega$$
$$+\int VN_1(q)[1-f_0(x')-f_0(x)]Cq\delta(\varepsilon'+\varepsilon-\hbar\omega)d\Omega\Big\}. \qquad (3)$$

Here

$$d\Omega=d^3q/(2\pi\hbar)^3,\quad d^3q=q^2dq\sin\vartheta d\vartheta d\Psi.$$

The excitation distribution function is

$$f(\mathbf{p})=f_0(x)+f_1(\mathbf{p})=f_0(x)+\frac{\partial f_0}{\partial x}\varphi(x,\beta),$$
$$f_0(x)=\frac{1}{e^x+1},\quad x=\frac{\varepsilon}{T}=\frac{(\xi^2+\Delta^2)^{1/2}}{T},$$

where ϵ and p are the energy and momentum of the excitations, T is the temperature in energy units, β is the angel between the excitation momentum p and ∇T, Δ is the energy gap, which we assume to be independent of the energy and momentum, and $\xi(p)$ is the energy of the electron with momentum p in the normal metal, reckoned from the Fermi level:

$$\xi(p)=(p^2-p_0^2)/2m. \tag{4}$$

The phonon momentum distribution function is

$$N(q)=N_0(q)+N_1(q)=\frac{1}{e^z-1}+N_1(q), \quad z=\frac{sq}{T},$$

where $N_1(q)$ is the nonequilibrium part of the phonon distribution function and s is the speed of sound.

I_1 is the collision integral of the "nonequilibrium" excitations with the "equilibrium" phonons, and U and V are the so-called coherence factors:

$$U=\frac{1}{2}\left(1+\frac{\xi\xi'-\Delta^2}{\varepsilon\varepsilon'}\right), \quad V=\frac{1}{2}\left(1-\frac{\xi\xi'-\Delta^2}{\varepsilon\varepsilon'}\right).$$

The primes denote here and below the values of the corresponding quantities after the collision. The terms of I_1 proportional to U describe the scattering of the excitations by the phonons, while the term proportional to V describes the production of a pair of excitations by a phonon and the inverse process. The quantity $Cq\delta(\ldots)d\Omega$ is the probability of the corresponding transitions per unit time. The integral I_2 describes the scattering of "equilibrium" excitations by "nonequilibrium" phonons or, in short, the dragging of the excitations by the phonons.

We break up the left-hand and right-hand sides of (1) into parts that are even and odd relative to ξ. To this end, we substitute p from formula (4) in the left-hand side. As a result we obtain

$$\frac{\partial f_0}{\partial T}\left(\frac{\xi(p)}{\varepsilon(p)}\frac{p_0}{m}+\frac{\xi^2(p)}{\varepsilon(p)}\frac{1}{p_0}\right)\nabla T\cos\beta=R(\xi)\left(1+\frac{\xi}{2\varepsilon_F}\right)\nabla T\cos\beta;$$

p_0 and ϵ_F are the Fermi momentum and energy in the normal metal. In the right-hand side of (1) we put[1)]

$$\varphi=[\varphi_1(T)+\varphi_2(\xi,T)+\varphi_3(\xi,T)\xi/T]\nabla T\cos\beta,$$

with $\varphi_{2,3}(\xi)=\varphi_{2,3}(-\xi)$.

We use next the relation $d\cos\vartheta\sim(m/pq)d\xi'$; the limits of integration with respect to $d\xi'$ are then

$$\xi+q^2/2m-pq/m, \quad \xi+q^2/2m+pq/m.$$

Within these limits, for all the significant phonon momenta, the δ function that appears under the integral signs twice (at $\xi'>0$ and $\xi'<0$) vanishes, so that these integrals can be extended from $-\infty$ to $+\infty$; this causes the integrand terms that are odd in ξ' to vanish. We recognize also that by virtue of the conservation laws we have

$$\int d\Psi\cos(\mathbf{p}\pm\mathbf{q},\nabla T)\approx 2\pi\cos(\mathbf{p},\nabla T)\left(1-\frac{q^2}{2p_0^2}+\frac{q^2}{p_0^2}\frac{\xi}{2\varepsilon_F}\right),$$

$$\int d\Psi\cos(\mathbf{q},\nabla T)\approx\pm 2\pi\cos(\mathbf{p},\nabla T)\left(-\frac{q}{p_0}+\frac{\xi'-\xi}{p_0 q}\right)m,$$

and furthermore take into account the fact that $p/m\approx v_0(1+\xi/2\epsilon_F)$.

Combining the integrand parts that are even and odd in ξ' in the collision integral, we obtain in place of (1) the following two equations:

$$R(\xi)\frac{\xi}{2\varepsilon_F}=I_2^+ + A-\frac{\xi}{2\varepsilon_F}B+R_1,$$

$$R(\xi)=I_2^- -\frac{\xi}{2\varepsilon_F}A+B+R_2, \tag{5}$$

where I_2^+ and I_2^- are the parts of the collision integral I_2 that are even and odd in ξ,

$$A=C\frac{\pi m p_0^2}{(2\pi\hbar)^3}\Bigg\{\left(\frac{T}{p_0 s}\right)^5\varphi_1(T)\Bigg[\int_0^\infty\frac{\varepsilon'}{|\xi'|}\left(1-\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\cdot N_0(z)e^{x+z}f_0(x)f_0(x+z)z^4dz$$
$$+\int_0^{x-b}\frac{\varepsilon'}{|\xi'|}\left(1-\frac{\Delta^2}{\varepsilon\varepsilon'}\right)N_0(z)e^x f_0(x)f_0(x-z)z^4dz$$
$$+\int_{x+b}^\infty\frac{\varepsilon'}{|\xi'|}\left(1+\frac{\Delta^2}{\varepsilon\varepsilon'}\right)N_0(z)e^z f_0(x)f_0(z-x)z^4dz\Bigg]$$
$$-2\left(\frac{T}{p_0 s}\right)^3\Bigg[\int_0^\infty\frac{\varepsilon'}{|\xi'|}\left(1-\frac{\Delta^2}{\varepsilon\varepsilon'}\right)(\varphi_2(x+z)-\varphi_2(x))\cdot N_0(z)e^{x+z}f_0(x)f_0(x+z)z^2dz$$
$$+\int_0^{x-b}\frac{\varepsilon'}{|\xi'|}\left(1-\frac{\Delta^2}{\varepsilon\varepsilon'}\right)(\varphi_2(x-z)-\varphi_2(x))N_0(z)e^x f_0(x)f_0(x-z)z^2dz$$
$$+\int_{b+x}^\infty\frac{\varepsilon'}{|\xi'|}\left(1+\frac{\Delta^2}{\varepsilon\varepsilon'}\right)(\varphi_2(z-x)-\varphi_2(x))N_0(z)e^z f_0(x)f_0(z-x)z^2dz\Bigg]\Bigg\} \tag{6a}$$

$$B=C\frac{2\pi m p_0^2}{(2\pi\hbar)^3}\left(\frac{T}{p_0 s}\right)^3\frac{\xi}{T}\Bigg\{\int_0^\infty\frac{\varepsilon'}{|\xi'|}\Bigg[\frac{\xi'^2}{\varepsilon\varepsilon'}\varphi_3(x')$$
$$-\left(1-\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\varphi_3(x)\Bigg]N_0(z)e^{x+z}f_0(x)f_0(x+z)z^2dz$$
$$+\int_0^{x-b}\frac{\varepsilon'}{|\xi'|}\Bigg[\frac{\xi'^2}{\varepsilon\varepsilon'}\varphi_3(x')-\left(1-\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\varphi_3(x')\Bigg]\cdot$$
$$\cdot N_0(z)e^x f_0(x)f_0(x-z)z^2dz-\int_{x+b}^\infty\frac{\varepsilon'}{|\xi'|}\Bigg[\frac{\xi'^2}{\varepsilon\varepsilon'}\varphi_3(x')$$
$$+\left(1+\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\varphi_3(x)\Bigg]N_0(z)e^z f_0(x)f_0(z-x)z^2dz\Bigg\}; \tag{6b}$$

$b=\Delta(T)/T$, and R_1 and R_2 are corrections that lead to small and therefore immaterial changes of the distribution function. The expressions for them and the corresponding estimates are given in the Appendix.

Neglecting R_1 and R_2, we can obtain for A and B, from (5), equations that are valid with accuracy $(q/p_0)^2\sim(T/\Theta_D)^2$:

$$2R(\xi)\xi/2\varepsilon_F-I_2^+=A, \tag{7}$$

$$R(\xi)-I_2^-=B. \tag{8}$$

In this section we consider the equation for the distribution-function increment that is even in ξ, i.e., Eq. (7). It is seen from (7) that $\varphi_2\sim(T/\Theta_D)^2\varphi_1$, i.e., at the temperatures considered by us we can neglect φ_2 in comparison with φ_1. Since the integrals containing φ_2 are odd with respect to replacement of x' by x, the value of φ_1 can be obtained by integrating (7) with respect to ξ (cf.[2,9]). We then obtain the following expression for φ_1:

$$\varphi_1=\left(2\frac{T}{p_0}F_2(b)+\int d\xi I_2^+\right)\Big/ C\frac{\pi m p_0^2}{(2\pi\hbar)^3}T\left(\frac{T}{p_0 s}\right)^5F_1(b), \tag{9}$$

where

$$F_1(b)=2\int_0^\infty\frac{z^4dz}{e^z-1}\int_b^\infty dx\frac{x(x+z)-b^2}{(x^2-b^2)^{1/2}[(x+z)^2-b^2]^{1/2}}\frac{1}{(e^x+1)(1+e^{-x-z})}$$
$$+\int_{2b}^\infty\frac{z^4dz}{e^z-1}\int_b^{z-b}dx\frac{x(z-x)+b^2}{(x^2-b^2)^{1/2}[(z-x)^2-b^2]^{1/2}}\frac{1}{(e^x+1)(e^{-x}+e^{-z})} \tag{10a}$$

$$F_2(b)=\int_b^\infty x(x^2-b^2)^{1/2}\frac{\partial f_0}{\partial x}dx=b^2\sum_{k=1}^\infty(-1)^{k+1}K_2(kb), \tag{10b}$$

$K_2(x)$ is the MacDonald function.

The nonequilibrium phonon distribution function $N_1(q)$ can be obtained from the kinetic equation for the phonons[2] and takes the following form:

$$N_1(q)=\frac{1}{4\pi}\frac{(2\pi\hbar)^3}{C}\frac{s}{m^2T^2}zr(z)N_0^2(z)e^z\nabla T\cos\alpha, \tag{11}$$

where

$$\cos\alpha=(q,\nabla T)/|q||\nabla T|,$$

$$r^{-1}(z)=\overline{U}\left(z-\ln\frac{1+e^{z+b}}{1+e^b}\right)+D(z)\overline{V}\left(2b-z+2\ln\frac{1+e^{z-b}}{1+e^b}\right),$$

$$D(z)=0,\quad z<2b,\qquad D(z)=1,\quad z>2b.$$

$\overline{U}$ and $\overline{V}$ are certain mean values of the coherence factors, and change little in the temperature region where a noticeable role is played by the dragging of the excitations by the phonons. For lead and mercury these are respectively the temperatures 3.5–5 and 2.5–3.5°K. In these temperature intervals, $\overline{U}$ and $\overline{V}$ range from 2 to 3–4.

In the calculation of $N_1(q)$ it is easy to show that dragging of phonons by "nonequilibrium excitations" is negligible. Substituting (11) in (9), we obtain ultimately

$$\varphi_1=2\frac{(2\pi\hbar)^3}{C}\frac{p_0^2s^5}{\pi m}\frac{1}{T^5}\frac{F_2(b)+T^2F_3(b)/mp_0s^3}{F_1(b)}, \tag{12}$$

$$F_3=2\int_0^\infty\frac{z^4r(z)\,dz}{e^z-1}\int_b^\infty dx\frac{x(x+z)-b^2}{(x^2-b^2)^{1/2}[(x+z)^2-b^2]^{1/2}}\frac{1}{(e^x+1)(e^{-x}+e^{-z})}$$

$$+\int_{2b}^\infty\frac{z^4r(z)\,dz}{e^z-1}\int_b^{z-b}dx\frac{x(z-x)-b^2}{(x^2-b^2)^{1/2}[(z-x)^2-b^2]^{1/2}}\frac{1}{(e^x+1)(e^{-x}+e^{-z})}. \tag{13}$$

The first term in (12) corresponds to that obtained in[1], and the second term corresponds to dragging of the excitations by the phonons. Obviously, φ_1 is determined by that part of the distribution function which depends only on the momentum direction. It is now easy to find the thermal conductivity connected with the change of the direction of the momentum and with the dragging of the excitations by the phonons:

$$\varkappa_{e1}=\frac{p_0^2s^5}{T^2}\frac{1}{C}\frac{F_2^2(b)}{F_1(b)}, \tag{14}$$

$$\varkappa_{e,ph1}=\frac{p_0s^2}{m}\frac{1}{C}F_2(b)\frac{F_3(b)}{F_1(b)}. \tag{14a}$$

We shall see later on that the thermal conductivity $\kappa_{e,ph1}$ connected with the dragging greatly exceeds κ_{e1}.

3. THERMAL CONDUCTIVITY CONNECTED WITH ENERGY EXCHANGE

In this section we perform the calculation of[1] in a more consistent manner, and furthermore estimate that part of the thermal conductivity which has been omitted from[1].

For the odd part of the distribution function of the excitations we have Eq. (5), which can be symbolically rewritten in the form

$$X=R(\xi)-I_2^-=\hat{L}B. \tag{15}$$

We define the scalar product $\langle\psi\zeta\rangle$ in the usual manner[2)] [7]:

$$\langle\psi\zeta\rangle=\int d^3p\,\psi\zeta. \tag{16}$$

Since the integrands are independent of the angles, we have

$$\int d^3p\ldots=4\pi\int_0^\infty p^2\,dp\ldots\approx4\pi p_0m\int_{-\infty}^{+\infty}d\xi\ldots$$

$$=4\pi p_0mT\left(\int_b^\infty\frac{\varepsilon}{|\xi|}dx(\xi>0)\ldots+\int_b^\infty\frac{\varepsilon}{|\xi|}dx(\xi<0)\ldots\right). \tag{17}$$

We consider the scalar product $\langle\psi\hat{L}\varphi\rangle$; using (16) and (17) and reversing in them the order of integration with respect to z and with respect to x, we obtain

$$\langle\psi\hat{L}\varphi\rangle=C_2\left\{2\int_0^\infty z^2N_0(z)\,dz\int_b^\infty dx\frac{\varepsilon\varepsilon'}{|\xi\xi'|}\right.$$

$$\times U_+(\psi(x+z)-\psi(x))(\varphi(x+z)-\varphi(x))e^{x+z}f_0(x)f_0(x+z)$$

$$+2\int_0^\infty z^2N_0(z)\,dz\int_b^\infty dx\frac{\varepsilon\varepsilon'}{|\xi\xi'|}U_-(\psi(x+z)+\psi(x))$$

$$\times(\varphi(x+z)+\varphi(x))e^{x+z}f_0(x)f_0(x+z)+\int_{2b}^\infty z^2N_0(z)\,dz.$$

$$\cdot\int_b^{z-b}dx\frac{\varepsilon\varepsilon'}{|\xi\xi'|}V_+(\psi(z-x)-\psi(x))(\varphi(z-x)-\varphi(x))e^zf_0(x)f_0(z-x)$$

$$+\int_{2b}^\infty z^2N_0(z)\,dz\int_b^{z-b}dx\frac{\varepsilon\varepsilon'}{|\xi\xi'|}V_-(\psi(z-x)+\psi(x))$$

$$\left.\times(\varphi(z-x)+\varphi(x))e^zf_0(x)f_0(z-x)\right\},$$

$$U_\pm=\left(1\pm\frac{\xi\xi'}{\varepsilon\varepsilon'}-\frac{\Delta^2}{\varepsilon\varepsilon'}\right),\quad V_\pm=\left(1\mp\frac{\xi\xi'}{\varepsilon\varepsilon'}+\frac{\Delta^2}{\varepsilon\varepsilon'}\right), \tag{18}$$

where

$$C_2=8\pi^2m^2C\frac{1}{(2\pi\hbar)^3}\frac{T^4}{s^3}.$$

It is seen from (18) that the operator $\hat{L}$ is self-adjoint and positive-definite. Analogous transformations yield

$$\langle X,\psi\rangle=8\pi p_0^2\int_b^\infty xf_0^2e^x\,dx+4\pi mp_0T\int_b^\infty\psi I_2^-\,dx. \tag{19}$$

We consider the functional $F(\psi)=2\langle\psi X\rangle-\langle\psi\hat{L}\psi\rangle$. Stipulating that its variations vanish, we obtain (8). On the other hand, the solution of (8) causes the variation of $F(\psi)$ to vanish, since it satisfies the obvious condition

$$\langle\psi,X\rangle=\langle\psi\hat{L}\psi\rangle. \tag{20}$$

Thus, the solution of Eq. (8) realizes an extremum $\langle\psi,X\rangle$ of the functional $F(\psi)$[4] (this extremum is a maximum, since $\hat{L}$ is positive-definite). As seen from (19), $\langle\psi R(\xi)\rangle$ differs only by a factor from the thermal conductivity corresponding to the function $f_1=\psi$. Consequently, any approximation satisfying the normalization condition (20) yields a lower bound for the thermal conductivity. We note that the collision integral for a metal in the normal state can be easily reduced to a form that coincides exactly with its form in (6b) or (18) at $T=T_c$, if the collisions are subdivided into those that reverse and do not reverse the sign of ξ. As will be shown below, at $T\sim T_c$ that part of the distribution function which is determined in this section makes the main contribution to the thermal conductivity, and consequently the latter is, to a high degree of accuracy, continuous when the metal goes over into the superconducting state, as is indeed observed in the experiment. The continuity of the left-hand and right-hand sides of the equation in this transition leads to continuity of the nonequilibrium distribution function, which we therefore choose such that it goes over into the "standard" trial function of the normal metal, thereby ensuring[4] an error no larger than 30%. In the case of a superconductor, unlike a metal in the normal state, the choice of φ_3 is not obvious beforehand[3)]. To simplify the calculations we put

$$\varphi_3 = C_1(T)(x-b)/(x^2-b^2)^{1/2}. \quad (21)$$

Substituting (21) in (18) and (19), we obtain

$$\left\langle R-I_2^-, \varphi_3\frac{\xi}{T}\right\rangle = 8\pi p_0^2 C_1(T)\left[F_4(b)+\left(\frac{T}{p_0 s}\right)^3 \frac{ms^2}{T}F_5(b)\right] \quad (22)$$

$$F_4(b) = \int_b^\infty x(x-b)f_0^2 e^x\,dx = b\ln(1+e^{-b}) + 2\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k^2}e^{-kb}, \quad (23)$$

$$F_5(b) = \int_0^\infty \frac{z^4 r(z)\,dz}{e^z-1}\int_b^\infty \frac{\varepsilon\varepsilon'}{|\xi\xi'|}e^{x+z}f_0(x)f(x+z)\,dx$$
$$+\int_{2b}^\infty \frac{z^3 r(z)\,dz}{e^z-1}\int_b^{z-b}\frac{\varepsilon\varepsilon'}{|\xi\xi'|}e^z f_0(x)f_0(z-x)\,dx, \quad (24)$$

and

$$\left\langle \varphi_3\frac{\xi}{T}, \hat{L}\varphi_3\frac{\xi}{T}\right\rangle = C_1^2(T)C_2F_6(b), \quad (25)$$

$$F_6(b) = 2\int_0^\infty \frac{z^4\,dz}{e^z-1}\int_b^\infty dx\frac{\varepsilon\varepsilon'}{|\xi\xi'|}U_+e^{x+z}f_0(x)f_0(x+z)$$
$$+2\int_0^\infty \frac{z^2\,dz}{e^z-1}\int_b^\infty dx\frac{\varepsilon\varepsilon'}{|\xi\xi'|}U_-(2x-2b+z)^2e^{x+z}f_0(x)f_0(x+z)$$
$$+\int_{2b}^\infty \frac{z^2\,dz}{e^z-1}\int_b^{z-b}dx\frac{\varepsilon\varepsilon'}{|\xi\xi'|}V_+(z-2x+2b)^2e^z f_0(x)f_0(z-x)$$
$$+\int_{2b}^\infty \frac{z^4\,dz}{e^z-1}\int_b^{z-b}dx\frac{\varepsilon\varepsilon'}{|\xi\xi'|}V_-e^z f_0(x)f_0(z-x). \quad (26)$$

To find the extremum of $F(\varphi_3)$, we take the derivative with respect to C_1 or, equivalently, we stipulate satisfaction of the normalization condition (20). This yields

$$C_1 = 8\pi p_0^2 F_4(b)/C_2F_6(b). \quad (27)$$

We can now determine directly the thermal conductivities κ_{e2} and $\kappa_{e,ph2}$ corresponding to the function φ_3:

$$\varkappa_{e2} = \frac{p_0^4 s^3}{m^2}\frac{1}{T^2}\frac{1}{C}\frac{F_4^2(b)}{F_6(b)},$$

$$\varkappa_{e,ph2} = \frac{p_0 s^2}{m}\frac{1}{C}\frac{F_4(b)F_5(b)}{F_6(b)}. \quad (28)$$

The total thermal conductivity due to the dragging of the excitations by the phonons is $\kappa_{e,ph} = \kappa_{e,ph1} + \kappa_{e,ph2}$.

4. COMPARISON OF THE DIFFERENT THERMAL-CONDUCTIVITY MECHANISMS

Let us now compare four different contributions made to the expression for the thermal conductivity of a superconductor (we note that these contributions are additive): the contribution κ_{e1} (14) corresponding to momentum exchange between the electrons and phonons, the contribution κ_{e2} (28) corresponding to energy exchange between the electrons and phonons, the contribution $\kappa_{e,ph}$ (14a), (28) determined by the thermal flux of the excitations dragged by the "nonequilibrium" phonons, and finally the contribution connected with the thermal flux of the phonons themselves:

$$\varkappa_{ph} = \frac{1}{3}\frac{T^2}{m^2 s}\frac{1}{C}F_7(b),$$

$$F_7(b) = \int_0^\infty z^4\frac{e^z}{(e^z-1)^2}r(z,b)\,dz, \quad (29)$$

where $r(z, b)$ is given by (11).

It is convenient to use the asymptotic values of the functions $F_1(b)$ to $F_7(b)$ at temperatures close to critical ($b \ll 1$) and temperatures noticeably lower than critical ($e^b \gg 1$). In the former case we have

$$F_1(0) = F_6(0) = 120\zeta(4), \quad F_2(0) = F_4(0) = \pi^2/6,$$
$$F_3(0) \approx 30, \quad F_5(0) \approx 20, \quad F_7(0) \approx 10,$$

and therefore

$$\text{a)}\ \frac{\varkappa_{e1}(T_c)}{\varkappa_{e2}(T_c)} \sim 10^{-1}, \quad \text{b)}\ \frac{\varkappa_{e,ph}(T_c)}{\varkappa_{e2}(T_c)} \sim 10^{-2},$$
$$\text{c)}\ \frac{\varkappa_{ph}(T_c)}{\varkappa_{e2}(T_c)} \sim 2\cdot 10^{-3}. \quad (30)$$

In the latter case

$$F_1(b) \sim 50\bar{U}\ln(1+e^{-b}), \quad F_2(b) \sim b^{3/2}e^{-b},$$
$$F_3(b) \sim 2e^{-b}F_7(b), \quad F_4(b) \sim (b+2)e^{-b}, \quad F_5(b) \sim e^{-b}F_7(b), \quad (31)$$
$$F_6(b) \sim 2F_1(b), \quad F_7(b) \sim 10e^b,$$

and therefore

$$\text{a)}\ \frac{\varkappa_{e,ph}}{\varkappa_{e2}} \sim \frac{e^b}{2b^{5/2}} \quad \text{b)}\ \frac{\varkappa_{ph}}{\varkappa_{e2}} \sim \frac{e^{2b}}{2b^6}. \quad (32)$$

It is important to note that $\kappa_{e,ph2}$, which does not depend on the form of the trial function, is of the order of unity at $b \sim 2-5$.

These estimates lead to the following conclusions:

1. Near the critical temperature, the principal role is played by the thermal conductivity κ_{e2} due to energy exchange between the excitations and the phonons. This thermal conductivity is continuous as the metal goes over into the superconducting state, and the phonon thermal conductivity is equally continuous.

2. With decreasing temperature, the thermal conductivity which is due near the critical temperature to exchange of energy between the excitations and the phonons decreases, and the contributions made to the thermal conductivity by the excitations by the phonons increases; as to the part due to the momentum exchange, investigated in[2], it remains small over the entire temperature interval.

3. Rosenberg[6] measured the thermal conductivity of lead from the critical temperature (7.2°K) to temperatures of the order of 1.8°K. It was shown quite convincingly that the sample was so pure and large that only the scattering of excitations by phonons played any role. The plot of $\kappa(T)$ obtained in[6] drops to 5.5°K, is practically horizontal from 5.5 to 4.5°K, has an appreciable maximum at approximately 3°K, and then again decreases rapidly.

Olsen and Renton[7] investigated the same sample at temperatures below 1.5°K. They obtained, at sufficiently low temperatures, a thermal conductivity that was almost proportional to T^3, i.e., brought about by phonon scattering from the boundaries. Mendelssohn and Olsen[10] investigated the thermal conductivity of lead with beryllium impurities. From the experiments of[10]; knowing the temperature dependence of the thermal conductivity of the phonons and of the excitations scattered by the impurities[1], it is possible to estimate the phonon thermal conductivity. From these data we can estimate the thermal resistance of the phonons due to their scattering by the boundaries and due to their scattering by excitations in the region of the thermal-conductivity maximum. These estimates show that the phonon thermal conductivity does not exceed 0.7—0.8 W/cm-deg, i.e., it amounts to approximately one-quarter of the measured maximum.

The theory developed above explains the entire curve. In the decreasing region, i.e., in the interval from the critical temperature to 5.5°K, the thermal conductivity κ_{e2} connected with energy exchange between the excitations and phonons predominates. Below 4.5°K, the terms κ_{ph} and $\kappa_{e,ph}$ connected with the phonon flux and with the dragging of the excitations by the phonons predominate. The aforementioned conclusion of Olsen, Renton, and Mendelssohn proves that in the region of maximum thermal conductivity the contribution due to the dragging is comparable with the contribution due to the phonon flux, so that it can be assumed that the thermal conductivity due to dragging plays the principal role over the entire interval temperature up to the maximum, where κ_{e2} is insignificant.

In the region where κ_{e2} predominates, the theoretical curve drops somewhat more slowly than the experimental one. This disparity can be eliminated by using a more complicated trial function in the variational principle, but this complication leads to very cumbersome expressions that are hardly justified at the accuracy attained both in the observations and in the theory[4)].

Hulm[5] investigated the thermal conductivity of superconducting mercury and obtained a $\kappa(T)$ plot that decreased from the critical temperature $T_c = 4.12$ to 3.5°K, was flat to 2.5°K, and then decreased further. This variation can also be explained on the basis of the developed theory. The thermal conductivity κ_{e2} predominates in the first interval, and the contributions $\kappa_{e,ph}$ and κ_{ph} begin to assume roles comparable with that of κ_{e2} in the second interval. The absence of a thermal-conductivity maximum is due to the fact that the dimensions of the sample were too small, so that phonon scattering by the boundaries set in before the phonon-induced growth of the thermal conductivity began.

The contribution made to the thermal conductivity by the dragging of the excitations by the phonons can be determined in the following manner: When the number of defects increases the mean free path of the excitations decreases much more rapidly than the mean free path of the phonons. The thermal conductivity should then decrease much more rapidly than in the presence of only the contribution connected with the phonons.

Near the critical temperature, there exists a certain temperature interval ΔT in which the kinetic equation (1) no longer holds, and consequently the conclusions drawn from it are likewise not valid. This temperature interval can be estimated in the following manner. The kinetic equation (1) leads to the following estimate for the excitation relaxation time relative to energy exchange with the phonons (at $T \approx T_c$):

$$\tau \sim \frac{\varepsilon_F}{T}\left(\frac{ms^2}{T}\right)^2 \cdot 10^{-2}\frac{Ma^2}{\pi\hbar},$$

where M and a are the mass and linear dimension of the unit cell. The uncertainty in the excitation energy $\Delta\epsilon \sim \hbar/\tau$ then turns out to be comparable with the energy gap $\Delta \sim T_c(1 - T/T_c)^{1/2}$ at a temperature T that differs from the critical one by an amount $T_c - T = \Delta T \sim 10^{-2}T_c$; this estimate agrees with the result of Éliashberg[11]. At a temperature closer to critical, the concept of excitations and the kinetic equation become meaningless.

APPENDIX

The expressions for R_1 and R_2 are

$$R_1 = C\frac{2\pi\cos(\mathbf{p},\nabla T)}{(2\pi\hbar)^3} m p_0^2 \left(\frac{T}{p_0 s}\right)^5 \frac{T}{\varepsilon_F}$$

$$\times\left\{\int_0^\infty z^4\,dz\frac{\varepsilon'|\xi'|}{T^2}\left(1-\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\varphi_3(x+z)N_0(z)f_0(x)f_0(x+z)e^{x+z}\right.$$

$$+\int_0^{x-b} z^4\,dz\frac{\varepsilon'|\xi'|}{T^2}\left(1-\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\varphi_3(x-z)N_0(z)f_0(x)f_0(x-z)e^z$$

$$\left.+\int_{b+x}^\infty z^4\,dz\frac{\varepsilon'|\xi'|}{T^2}\left(1-\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\varphi_3(z-x)N_0(z)f_0(x)f_0(z-x)e^z\right\};$$

$$R_2 = C\frac{2\pi\nabla T\cos(\mathbf{p},\nabla T)}{(2\pi\hbar)^3} m p_0^2\left(\frac{T}{p_0 s}\right)^5\frac{\xi}{\varepsilon_F}\varphi_1(T)$$

$$\times\left\{\int_0^\infty z^4\,dz e^{x+z} f_0(x)f_0(x+z)\frac{|\xi'|}{\varepsilon}N_0(z)+\int_0^{z-b} z^4\,dz e^x f_0(x)f_0(x-z)\frac{|\xi'|}{\varepsilon}N_0(z)\right.$$

$$\left.-\int_{b+x}^\infty z^4\,dz e^z f_0(x)f_0(z-x)\frac{|\xi'|}{\varepsilon}N_0(z)\right\}.$$

To estimate the errors due to neglecting R_1 and R_2, we substitute in these expressions the obtained values of φ_1 and φ_3 and compare the results with the left-hand sides of the corresponding equations. We find as a result that the relative errors in the determination of φ_1 and φ_3 are of the following order of magnitude:

$$\frac{\Delta\varphi_1}{\varphi_1} \sim \left(\frac{T}{p_0 s}\right)^2 \sim \left(\frac{T}{\Theta_D}\right)^2, \quad \frac{\Delta\varphi_3}{\varphi_3} \leqslant 10^2\frac{T}{\varepsilon_F}\left(\frac{T}{\Theta_D}\right)^3 e^{\Delta_0/T}.$$

The error of φ_3 contains a factor that increases exponentially with decreasing temperature, but we are not interested in temperatures below $0.3T_c$, in which case $\Delta\varphi_3/\varphi_3 < 10^{-3}$.

1)In the article of Landau and Pomeranchuk [9], the nonequilibrium part of the distribution function is broken up into energy-dependent and energy-independent parts; the former turns out to be of no consequence for the authors' purposes, since the crossing coefficients of interest to us, which satisfy the Onsager principle, are determined only by the second part. We, on the other hand, are interested in the thermal conductivity, which is determined by the entire nonequilibrium distribution function, so that the subdivision proposed in the text is more convenient for our purposes.

2)All the functions considered in this section are odd relative to the Fermi surface, inasmuch as Eq. (8) is valid only for such functions.

3)Thus, a function of the type $(\xi/T)(|\xi|/\epsilon)^n$ satisfies the condition indicated above for arbitrary n, so that the choice of the best approximation leads to cumbersome numerical calculations.

4)The increased thermal conductivity κ_{e2} obtained in [1] via the variational principle and used by Geĭlikman and Kresin [2] as a basis for discarding the entire term κ_{e2} may actually be due to the insufficient accuracy of the calculations and to the insufficiently good choice of the trial function.

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145