Nonlinear theory of the excitation of surface electromagnetic waves in a dielectric medium by a relativistic electron beam

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The problem solved is that of the excitation of nonlinear surface electromagnetic waves on a plane dielectric-vacuum interface by a relativistic electron beam passing over the surface of the dielectric. Nonlinear field equations and their boundary conditions are derived for conditions of strong magnetic confinement of the electron beam. The stationary values of the amplitudes of monochromatic surface waves excited by the beam are found, and the radiative energy fluxes and the efficiency of transformation of beam energy into radiation energy are found for conditions near the saturation threshold.

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1. INTRODUCTION

In previous papers[1,2] we have given a detailed analysis of the problem of the nonlinear interaction of a relativistic electron beam with volume electromagnetic waves in an isotropic dielectric medium with permeability $\epsilon_0(\omega)$; the physical mechanisms have been found which limit the increase of the amplitudes of waves excited by the beam, and also the energy fluxes of the electromagnetic radiation near the saturation threshold.

In the case when the dielectric medium is a plasma and $\epsilon_0(\omega) = 1 - \omega_p^2/\omega^2$ the beam can excite only quasi-longitudinal (almost potential) volume electromagnetic waves, and therefore the flux of radiation is extremely small. In this case the energy loss from the beam is mainly expended in producing intense longitudinal waves in the plasma, which are then dissipated into heat. The situation is different when an electron beam goes past a plasma cylinder and only surface electromagnetic waves are excited in the system.[3] First, the surface waves with large phase velocities which are excited by the beam are strongly nonpotential, and contain a strong radiative flux. Second, under conditions of magnetic confinement of the beam and relatively rarefied plasma a single fundamental axially symmetric mode of surface waves is excited, and therefore we can apply a one-mode approach[1,2] in studying the nonlinear stage.

Incidentally, the assumption of strong magnetic limitation of the transverse motion of the beam is very necessary for excitation of surface electromagnetic waves in the system, since the beam electrons are acted on in such a wave by an average Miller force[4] pushing the beam away from the surface of the plasma. A sufficiently strong longitudinal magnetic field, confining the beam (but not the plasma) neutralizes the action of this force.

In the present paper our purpose is to extend the method expounded in[1] and[2] to the case of excitation of nonlinear surface electromagnetic waves by a beam. Therefore for simplicity we have restricted ourselves to the consideration of a plane dielectric-vacuum interface, with the medium with dielectric constant $\epsilon_0(\omega)$ filling the halfspace $x \le 0$. In the region $x > 0$ an electron beam moves parallel to the surface with velocity $\mathbf{u} \parallel z$, and is confined by a longitudinal magnetic field strong enough so that $\Omega_e \gg \omega_b$, where Ω_e and ω_b are the Larmor and Langmuir frequencies of the beam electrons. This condition allows us to neglect the transverse motion of the electrons in the beam and take into account only the perturbation of their longitudinal velocity by the action of the surface wave excited by the beam.

As before,[1,2] the dielectric medium is assumed linear, and the nonlinearity is entirely due to the action of the electromagnetic wave on the motion of the beam electrons. The thermal motion of the electrons in the beam is neglected (cf.[1]), and the beam is described by the equations of relativistic magnetohydrodynamics[5]

$$\partial n/\partial t+\operatorname{div} n\mathbf{v}=0,$$
$$\left[\frac{\partial}{\partial t}+(\mathbf{v}\nabla)\right]\frac{\mathbf{v}}{(1-v^2/c^2)^{1/2}}=\frac{e}{m}\left\{\mathbf{E}+\frac{1}{c}[\mathbf{v}\times\mathbf{B}]\right\}, \quad (1.1)$$

where n is the mean density and $\mathbf{v}$ the mean velocity of the beam electrons, and $\mathbf{E}$ and $\mathbf{B}$ are the strengths of the electric and magnetic fields, which satisfy Maxwell's equations.

2. LINEAR THEORY OF EXCITATION OF SURFACE WAVES

Before proceeding to the study of the interaction of the electron beam with the surface oscillations of the dielectric medium, we present the main results of the linear theory. Linearizing the system of equations (1) and the Maxwell equations for perturbations of the form

$$f \sim f(x)\exp(-i\omega t+ik_z z), \quad (2.1)$$

we readily derive the equations for the z component of the E wave[1)]

$$\partial^2E_z/\partial x^2-\varkappa_0^2E_z=0 \quad \text{for } x<0,$$
$$\partial^2E_z/\partial x^2-\varepsilon_b\varkappa^2E_z=0 \quad \text{for } x>0; \quad (2.2)$$

where

$$\varkappa_0^2=k_z^2-\omega^2\varepsilon_0/c^2, \quad \varkappa^2=k_z^2-\omega^2/c^2,$$

and the longitudinal (ϵ_{zz}) component of the dielectric-constant tensor of the magnetically confined beam is

$$\varepsilon_b=1-\omega_b^2\gamma^{-3}/(\omega-k_zu)^2.$$

The other field components of the surface E wave are connected with E_z by the relations

$$E_x=-\frac{ik_z}{\varkappa_0^2}\frac{\partial E_z}{\partial x},\quad B_y=-\frac{i\omega}{c}\frac{\varepsilon_0}{\varkappa_0^2}\frac{\partial E_z}{\partial x}\ \text{for}\ x<0,$$
$$E_x=-\frac{ik_z}{\varkappa^2}\frac{\partial E_z}{\partial x},\quad B_y=-\frac{i\omega}{c}\frac{1}{\varkappa^2}\frac{\partial E_z}{\partial x}\ \text{for}\ x>0. \tag{2.3}$$

The equations (2.2) must be supplemented with the boundary conditions, which are that the tangential field components are continuous:

$$\{E_z\}_{x=0}=\{B_y\}_{x=0}=0. \tag{2.4}$$

The solution of Eq. (2.2) for a surface wave

$$E_z=A_1\exp(\varkappa_0 x)\quad\text{for}\ x<0,$$
$$E_z=A_2\exp\{-(\varepsilon_b\varkappa^2)^{1/2}x\}\quad\text{for}\ x>0 \tag{2.5}$$

exists only for $\epsilon_b\kappa^2>0$ and $\kappa_0^2>0$. Substitution of (2.5) into the boundary conditions (2.4) leads to the following dispersion equation for the E-type surface wave:

$$\varepsilon_0(\omega)/\varkappa_0+\varepsilon_b^{1/2}/\varkappa=0. \tag{2.6}$$

Using the facts that $\kappa_0>0$ and $\kappa>0$, and that ϵ_b is nearly equal to unity, we arrive at the conclusion that E-type surface waves are possible only in a medium with $\epsilon_0(\omega)<0$. In particular, a plasma is such a medium in the frequency range $\omega<\omega_p$. Moreover, it follows from Eq. (2.6) that the beam excites waves whose phase velocity is close to the velocity of the beam, i.e., $\omega_0=k_z u$. Using this fact, we find the frequency spectrum and the growth increment of the surface waves excited by the beam ($\omega=\omega_0+i\delta$):

$$\varepsilon_0(\omega_0)+\gamma^2=0,$$
$$\delta=\frac{-i+\sqrt{3}}{2}\left[\omega_b^2/2\frac{\partial}{\partial\omega_0}\left(\varepsilon_0\frac{\varkappa}{\varkappa_0}\right)\right]^{1/3}\frac{1}{\gamma}, \tag{2.7}$$
$$\gamma=(1-u^2/c^2)^{-1/2}.$$

In the case when the dielectric medium is a plasma, the expressions (2.7) take the well known form[3]:

$$\omega_0^2=\omega_p^2/(1+\gamma^2),$$
$$\delta=\frac{-i+\sqrt{3}}{2}\omega_0\left(\frac{\omega_b^2}{2\omega_p^2}\right)^{1/3}\left(\frac{\gamma}{\gamma^4+1}\right)^{1/3}. \tag{2.8}$$

3. NONLINEAR STATIONARY SURFACE WAVES

Accordingly, small-amplitude surface waves described by the linear theory are unstable and increase with the time. Let us now examine how surface waves of finite amplitude behave; in doing so we shall, as in[1], treat them as stationary. This enables us to ascertain the threshold amplitude values above which there is no instability and the waves do not increase with the time.

Assuming that all quantities depend on the variable $\xi=t-k_z z/\omega$ and that under conditions of strong magnetic confinement the electrons in the beam move only longitudinally, we find the following integrals of the system (1.1):

$$n=n_b\frac{\omega-k_z u}{\omega-k_z v_z},$$
$$\frac{k_z c}{\omega}(m^2c^2+p_z^2)^{1/2}-p_z+\frac{ek_z}{\omega}\Phi=\frac{k_z c}{\omega}mc\gamma\left(1-\frac{u}{c}\frac{\omega}{k_z c}\right), \tag{3.1}$$

where the function Φ is connected with E_z by the relation

$$E_z=-\frac{\partial\Phi}{\partial z}=\frac{k_z}{\omega}\frac{d\Phi}{d\xi}, \tag{3.2}$$

and $p_z=mv_z/(1-v_z^2/c^2)^{1/2}$.

Using Eqs. (3.1), (3.2), and (2.3), we easily reduce the Maxwell equations in the region $x>0$ to the following nonlinear equation for Φ:

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\varkappa^2}{\omega^2}\frac{\partial^2\Phi}{\partial\xi^2}=-\omega_b^2\frac{m\varkappa^2}{ek_z^2}\left\{\left(1-\frac{2e\Phi k_z^2}{m\gamma^3(\omega-k_z u)^2}\right)^{-1/2}-1\right\}. \tag{3.3}$$

In the region $x<0$ occupied by the medium the field equation is still linear [Eq. (2.2)], and can be written in the same form for the function Φ. Finally, the boundary conditions for Φ can be written in the form

$$\{\Phi\}_{x=0}=0,\quad \left.\frac{\varepsilon_0}{\varkappa_0^2}\frac{\partial\Phi}{\partial x}\right|_{x=0}=\left.\frac{1}{\varkappa^2}\frac{\partial\Phi}{\partial x}\right|_{x=0}. \tag{3.4}$$

Now, having formulated the nonlinear system of equations for the function Φ and its boundary conditions, we can solve the boundary-value problem, using the smallness of the nonlinear terms and applying the Bogolyubov-Krylov approximate method (cf.[1]). To do so we write the required solutions in the form

$$\Phi=\Phi_1 e^{\varkappa_0 x}\cos\omega\xi\ \text{for}\ x<0,$$
$$\Phi=\Phi_2 e^{-kx}\cos\omega\xi\ \text{for}\ x>0. \tag{3.5}$$

Substitution of these solutions in the boundary conditions (3.4) gives $\Phi_1=\Phi_2$ and

$$k=-\varepsilon_0\varkappa^2/\varkappa_0>0. \tag{3.6}$$

(All the requirements of the linear theory, $\kappa>0$, $\kappa_0>0$ and $\epsilon_0(\omega)<0$, remain valid here.) From Eq. (3.3) itself we also obtain

$$\left(\frac{\varepsilon_0^2\varkappa^3}{\varkappa_0^2}-1\right)\Phi_2 e^{-\varkappa x}\cos\Psi=-\frac{\omega_b^2 m}{ek_z^2}\left\{\left(1-\frac{2e\Phi_2 k_z^2 e^{-kx}\cos\Psi}{m\gamma^3(\omega-k_z u)^2}\right)^{-1/2}-1\right\}, \tag{3.7}$$

where $\Psi=\omega\xi$.

Multiplying this equation by $e^{-kx}\cos\Psi$ and integrating over x from 0 to ∞ and over Ψ from 0 to π, we get

$$\left(\frac{\varepsilon_0}{\varkappa_0^2}-\frac{1}{\varkappa^2}\right)\Phi_2=\frac{4}{\pi}\frac{m}{e}\frac{\varepsilon_0\omega_b^2}{\varkappa_0 k_z^2}\int_0^\infty dx\int_0^\pi d\Psi\cos\Psi e^{-kx}$$
$$\times\left[1-\frac{2e\Phi_2 k_z^2 e^{-kx}\cos\Psi}{m\gamma^3(\omega-k_z u)^2}\right]^{-1/2}. \tag{3.8}$$

In the small-amplitude case, when

$$m\gamma^3(\omega-k_z u)^2\gg 2e\Phi_2 k_z^2,$$

Eq. (3.8) reduces to the dispersion relation of the linear case, Eq. (2.6), corresponding to increase with time of the oscillations in the frequency range $\omega\approx k_z u$. This instability persists for finite values of the wave amplitude, as long as

$$e\Phi_2<{}^1/_2 m\gamma^3(u-\omega/k_z)^2\approx{}^1/_2 m\gamma^3u^2|\delta^2|/\omega_0^2,$$

where ω_0 and δ are given by the equations of the linear theory, Eq. (2.7).

When this inequality is violated the instability is stabilized and the wave becomes stationary. Equation (3.8) then takes the form

$$\frac{\varepsilon_0^3}{\varkappa_0^2}-\frac{1}{\varkappa^2}=\frac{4m\varepsilon_0\omega_b^2}{\pi e\varkappa_0 k_z^2}\int_0^\infty e^{-kx}dx[2(2\eta^2-1)]^{1/2}[K(\eta)-2E(\eta)], \tag{3.9}$$

$$\eta=\left[\frac{{}^1/_2 m(u-\omega/k_z)^2\gamma^3+e\Phi_2 e^{-kx}}{2e\Phi_2 e^{-kx}}\right]^{1/2}\approx\frac{1}{\sqrt{2}}\left[1+\frac{m\gamma^3(u-\omega/k_z)^2}{4e\Phi_2 e^{-kx}}\right]. \tag{3.10}$$

Using the last relation, which holds when the strong inequality

$$e\Phi_2\gg{}^1/_2 m\gamma^3(u-\omega/k_z)^2,$$

corresponding to complete absence of instability, is satisfied, we finally get

$$\frac{\varepsilon_0^2\varkappa^2}{\varkappa_0^2}-1=2{,}2\frac{m\omega_b^2}{ek_z^2\Phi_2}\left[\frac{m\gamma^3(u-\omega/k_z)^2}{e\Phi_2}\right]^{1/2}. \tag{3.11}$$

This equation determines the amplitude of a mature surface wave excited by a beam, near the threshold of saturation.[2)] Using the results of the linear theory, we now find

$$\frac{e\Phi_2}{mu^2}=\gamma^3\left\{\frac{2{,}2\omega_b^2\gamma}{\omega_0^3(1+\gamma^2)\left[\partial\varepsilon_0/\partial\omega+2\gamma^2(\gamma^2-1)/\omega_0\right]}\right\}^{2/3}. \quad (3.12)$$

In the case in which the dielectric medium is a plasma and $\epsilon_0 = 1 - \omega_0^2/\omega^2$, we have

$$\frac{e\Phi_2}{mu^2}=\gamma^3\left[\frac{1{,}1\omega_b^2\gamma}{\omega_p^2(\gamma^4+1)}\right]^{2/3}. \quad (3.13)$$

4. DISCUSSION OF THE RESULTS. ENERGY FLUX OF THE ELECTROMAGNETIC RADIATION

The relation (3.12) enables us to determine the amplitudes of the electric and magnetic fields in the surface wave inside the dielectric medium as well as outside it. In fact, according to Eq. (3.5)

$$\Phi=\Phi_2\cos\omega_0\xi\begin{cases}\exp(|k_z|\gamma x) & \text{for } x<0\\ \exp(-|k_z|x/\gamma) & \text{for } x>0\end{cases}. \quad (4.1)$$

From this, recalling Eq. (3.2), we have

$$E_z=-\Phi_2 k_z\sin\omega_0\xi\cdot\begin{cases}\exp(|k_z|\gamma x), & x<0\\ \exp(-|k_z|x/\gamma), & x>0\end{cases} \quad (4.2)$$

The components E_x and B_y are found by means of the relations (2.3).

Knowing the fields, we can calculate the flux of electromagnetic energy (the Poynting vector) and thus find the efficiency of the transformation of the energy of the beam into the energy of the electromagnetic surface wave which it excites. It is easily shown that in the case we are considering, the problem with a plane geometry, the only component of the energy-flux vector of the field which is nonzero when averaged over a period is the longitudinal component

$$P_z=\frac{c}{4\pi}\int_{-\infty}^{+\infty}\overline{E_xB_y}\,dx=P_{z1}+P_{z2}. \quad (4.3)$$

Here P_{z1} is the flux of radiation in the dielectric medium (i.e., in the region $x < 0$), and P_{z2} is that outside the medium (i.e., with $x > 0$). Using the relations (2.3), we find

$$P_{z\,1,2}=\frac{\omega_0}{16\pi}\Phi_2^2\cdot\begin{cases}\gamma^{-1}, & x<0\\ \gamma^3, & x>0\end{cases}, \quad (4.4)$$

where Φ_2 is given by Eq. (3.12).

It follows from Eq. (4.4) that the radiation outside the medium is larger than that in the medium by a factor γ^4. This result can be understood if we use the fact that in the strongly relativistic case the phase velocity of the surface waves excited by the beam is close to the velocity of light; these waves are decidedly nonpotential in nature and are easily radiated out of the medium.

The ratio of the quantity P_z to the flux of kinetic energy of the electron beam gives the efficiency of the transformation of the energy of the beam into energy of electromagnetic radiation. In the case we are considering, when the beam occupies the half-space $x > 0$, the total flux of kinetic energy in the beam obviously diverges. However, the surface wave penetrates into the region $x > 0$ only to a distance of the order of

$$\gamma/|k_z|\approx\gamma u/\omega_0,$$

and only the part of the beam that fills this layer interacts with the wave; the rest of the beam has practically no effect in strengthening the wave and can be discarded. Therefore in determining the radiative efficiency we are to divide the quantity P_z by

$$n_b mc^3(\gamma-1)\gamma u^2/\omega_0.$$

The result is

$$\frac{P_z}{n_b mc^2(\gamma-1)\gamma u^2/\omega_0}=\eta_1+\eta_2; \quad (4.5)$$

$$\eta_{1,2}=\frac{0.55}{\omega_0^2}\frac{\gamma+1}{(\gamma^2+1)^{4/3}}(2.2\omega_b^2\gamma)^{4/3}\left[\frac{\partial\varepsilon_0}{\partial\omega_0}+\frac{2\gamma^2(\gamma^2-1)}{\omega_0}\right]^{-4/3}\cdot\begin{cases}\gamma^3, & x<0\\ \gamma^7, & x>0\end{cases} \quad (4.6)$$

For the case of a plasma, in which $\epsilon_0 = 1 - \omega_p^2/\omega^2$, the expressions (4.6) take the form

$$\eta_{1,2}=0.27\frac{(1{,}1\omega_b^2\gamma)^{4/3}(\gamma+1)}{\omega_p^{8/3}(\gamma^2+1)(\gamma^4+1)^{4/3}}\cdot\begin{cases}\gamma^3, & x<0\\ \gamma^7, & x>0\end{cases}. \quad (4.7)$$

In conclusion we note that the efficiency of radiation into surface waves, precisely as in the case of excitation of transverse electromagnetic waves in a spatially unlimited medium,[1,2] is proportional to $j_b^{1/3}$, where j_b is the current density in the electron beam.

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1)A surface wave of the B type, in which the components B_z, B_x, and E_y are nonzero, does not interact with the beam and therefore is always stable.

2)We note that an electromagnetic wave excited by a beam never becomes rigorously stationary. As has been shown [6] with the example of an electrostatic instability in a plasma-beam system, the amplitude of the wave, after it has reached a certain threshold value, will oscillate slowly with time. It is this threshold value of the amplitude which is determined in our treatment.

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