Dynamics of a two-level system in a strong resonant field with variable frequency and amplitude

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Population relaxation of a two-level system in a nonstationary resonant field is investigated. Various types of time dependences of the amplitude r and the frequency ν of the field are pointed out, for which exact solutions exist (in the case of identical as well as nonidentical longitudinal and transverse relaxation times). In particular, the relation is found between the solutions for an arbitrary field $r(t)$, $\nu(t)$, and the time-inverted field $r(-t)$, $\nu(-t)$. An approximate theory of relaxation is given (when exact solutions cannot be found) in the limiting cases of a weak and a strong nonstationary field, and also for a slowly varying and a rapidly varying field. Relations determining the polarization of the system as a function of the population dynamics are presented.

INTRODUCTION

The dynamics of a relaxing quantum system in a strong nonstationary resonant field is an important aspect of the theory of the interaction of a field with matter;[1-3] in this connection sufficiently large fields, such that perturbation theory is not applicable,[1)] are of the greatest interest. Exact solutions for the behavior of the system at exact frequency resonance and for arbitrary variation of the field amplitude were obtained by Lyubimov and Khokhlov[4] and by Faĭn[5, 1] in the absence of relaxation, and by Rautian[2] in the presence of relaxation with identical longitudinal (τ) and transverse (T) relaxation times (at zero equilibrium difference of the populations n_0). For the case $n_0 \neq 0$ and for identical relaxation times ($T = \tau$) the latter problem was investigated by Yankauskas,[6] who found the general solution in quadratures at exact frequency resonance (the integrals obtained in this connection are evaluated in[6] for periodic modulation of the amplitude, and are evaluated in the author's article[3] for various other types of modulation). Zon and Katsnel'son[7] obtained solutions for the case of an exponentially growing amplitude for a fixed frequency difference (in terms of the probability amplitudes[2)]).

In[3], by truncation of the equations for the density matrix, the equation of motion of the populations in a two-level system was derived in the general case of a field with a time-dependent frequency $\nu(t)$ and a time-dependent real amplitude $E(t)$:

$$\frac{r}{\nu}\hat{D}_T\left\{\frac{r}{\nu}\left[\frac{1}{r}\hat{D}_T\left(\frac{1}{r}\hat{D}_\tau(x-1)\right)+x\right]\right\}+\hat{D}_\tau(x-1)=0. \quad (1)$$

Here $x = n/n_0$ is the difference of the populations $n = \sigma_{11} - \sigma_{22}$ of the ground and the excited levels, reduced to the equilibrium value n_0 at the given temperature; $r(t) = \mathbf{d}\cdot\mathbf{E}/\hbar$ is the reduced amplitude of the interaction, having the dimension of a frequency ($\mathbf{d}$ is the dipole moment of a molecule); $\nu(t) = \omega(t) - \omega_0$ is the frequency difference between the instantaneous frequency of the field $\omega(t)$ and the resonance frequency of the system ω_0, and

$$\hat{D}_T=\frac{d}{dt}+\delta_T,\quad \hat{D}_\tau=\frac{d}{dt}+\delta_\tau\quad\left(\delta_T=\frac{1}{T},\ \delta_\tau=\frac{1}{\tau}\right)$$

where T denotes the relaxation time of the polarization and τ is the lifetime of the excited state.[3)] If the system is in equilibrium ($n = n_0$, $\sigma_{12} = \sigma_{21} = 0$) at the instant $t = t_0$ when the field is switched on, the initial conditions are as follows:

$$x(t_0)=1,\ \dot{x}(t_0)=0,\ \ddot{x}(t_0)=-r^2(t_0). \quad (2)$$

An exact theory for a field with a variable amplitude (in the case $\nu \equiv 0$) was developed in[3] for $T = \tau$ and also for $T \neq \tau$.

The goal of the present work is the construction of a theory for the dynamics of a system with simultaneous variation in time of both the amplitude and frequency of the field (for nonidentical, in general, relaxation times).

Although exact solutions can only be found for specific types of relationships between ν and r (Secs. 1, 2), they do, however, allow us to include a broad class of possible dependences of ν and r on the time, dependences which are of immediate physical interest. One of the fundamental merits of these solutions is that they are valid for arbitrary maximal amplitude of the applied field. This permits one to trace the evolution of the system's dynamics following variation of the amplitude from "zero," which is especially important for $\nu \neq 0$ and $T \neq \tau$ (Sec. 2), when the nature of the relaxations changes qualitatively during the passage of the amplitude through a certain value r_{cr}, as though separating a "weak" field region, where perturbation theory still operates, and a "strong" field region where characteristic quantum oscillations should be observed. Finally, the availability of a large number of exact solutions enables one to construct with good accuracy an approximate theory in quite general cases of a field of arbitrary form (Sec. 3), a theory free from the inadequacies of perturbation theory whose series diverge at $r > r_{cr}$.

The results can be used for measurement of the characteristics of matter in experiments that detect the behavior of the populations or polarization in the field of a pulse with a given wave form close to one of those investigated. In this connection, for observations in the regime of a given field it is necessary that the thickness of the layer of matter should not exceed the value

$$l_{cr}\sim c\min(\tau_E,\ 1/r_{max}),$$

where c is the speed of light in matter, τ_E is the duration of the pulse, and r_{max} is the maximum amplitude of the field. If $\tau_E \sim T$, and r_{max} exceeds δ_T by several times (under these conditions, strongly pronounced oscillations should appear), then the thickness is

$l_{cr} \sim 10$–10^2 cm for $T \sim 10^{-9}$–10^{-8} sec. However, if $l > l_{cr}$, it is now necessary to investigate the problem of nonlinear, nonstationary propagation (a special case of which are the π-pulses[10]), where it is necessary to solve Eq. (1) simultaneously with Maxwell's equations.

A natural region of application of the present calculations is the theory of transition processes in lasers, where apparently one can effectively use the approximate solutions of Sec. 3 for a strong field. With their aid one can also investigate the interaction of quantum systems with a strong spontaneous field, which has recently attracted much interest (see, for example,[11-13]). The question of the interaction of a powerful pulse of the field with a resonant medium acquires special value in connection with the observation of self-focusing in such media[12,14] (a mechanism for the formation of a nonlinear component of the dielectric constant due to resonant absorption, and the conditions for self-focusing in the steady-state regime, were discussed in[15,16]) and in connection with the possibility of a strong effect involving the "self-twisting" of light beams.[17,12]

1. THE CASE OF IDENTICAL RELAXATION TIMES (EXACT SOLUTIONS)

For $T = \tau = 1/\delta$, introducing the unknown $y = (x-1)e^{\delta t}$ and the variable $\xi = \int r\,dt$, and omitting the inhomogeneous part of Eq. (1), we reduce it to the form

$$[(y_\xi''+y)/f]'/f+y_\xi'=0,\quad f=\nu/r. \tag{1.1}$$

It is not difficult to verify that all of the real solutions of Eq. (1.1) can be represented in the form

$$y=zz^*-4z_\xi' z_\xi'^*,$$

where z is the general solution of the equation

$$z_\xi''+if(\xi)z_\xi'+\frac{1}{4}z=0. \tag{1.2}$$

Let us indicate certain cases when exact solutions exist.

1) ν/r = const = k. Here the solution of Eq. (1) is trivial to obtain; for the initial conditions (2) it coincides with the solution obtained in[3] for the case $\nu \equiv 0$, $T = \tau$, where instead of r and x it is necessary to take the values

$$r_{\text{eff}}=r(1+k^2)^{1/2},\quad x_{\text{eff}}=x(1+k^2)-k^2.$$

2) $\nu/r = k/\xi$ (k = const). Here the solution of Eq. (1.2) is expressed in terms of Bessel functions of the first kind with complex order $\pm(1/2)(1-ik)$. Let us present a few physically interesting examples of the time dependences r(t) and ν(t) corresponding to this case.

a) For a fixed amplitude of the field (r = const), the frequency difference falls or grows in time like $\nu = k/t$ (Fig. 1a).

b) The field pulses increase or decrease exponentially (Fig. 2a):

$$r=r_0\exp(\delta_E t),\quad \nu=\nu_0/[1+C\exp(-\delta_E t)]\quad (k=\nu_0/\delta_E);$$

here r_0, ν_0, C, and δ_E are arbitrary constants (the particular case $\nu = \nu_0$ = const, i.e., C = 0, was investigated in[7] in terms of probability amplitudes).

c) The field pulses are bell-shaped with arbitrary width and amplitude (Fig. 3):

$$r=r_0/\operatorname{ch}\delta_E t,\quad \nu=\nu_0/[\operatorname{arc\,tg}(\operatorname{sh}\delta_E t)+C]\operatorname{ch}\delta_E t.$$

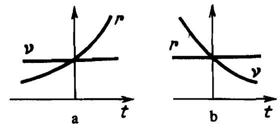

FIG. 1

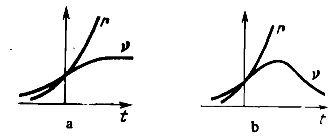

FIG. 2

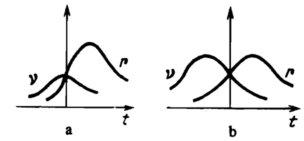

FIG. 3

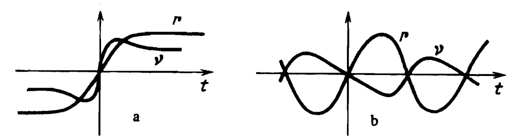

FIG. 4

d) The amplitude of the field passes through zero (Fig. 4a):

$$r=r_0\operatorname{th}\delta_E t,\quad \nu=\nu_0\operatorname{th}\delta_E t/\ln|C\operatorname{th}\delta_E t|\quad (k=\nu_0/\delta_E).$$

e) The amplitude and frequency vary periodically (Fig. 4b):

$$r=r_0\sin\Omega t,\quad \nu=\nu_0\sin\Omega t/(\cos\Omega t+C)\quad (k=\nu_0/\Omega).$$

In conclusion we note that Eq. (1.1) is invariant under the substitutions

$$t\to -t,\quad r(t)\to r(-t),\quad \nu(t)\to\nu(-t).$$

Hence it follows that if the solution of the homogeneous equation (1.1) is found with regard to any pair of functions $\nu(t)$ and r(t), then for the time-inverted field the solution is also determined by simply inverting the sign of the time, that is, $y_1(t) = y(-t)$. In particular, if relaxation is absent or if the field pulse is very short ($\tau_E \ll T$), then what has been said pertains directly to the population x(t) itself.

2. THE CASE OF UNEQUAL RELAXATION TIMES (EXACT SOLUTIONS)

For $T \neq \tau$ Eq. (1) reduces to a second-order equation if the frequency modulation and the amplitude modulation are related to each other by one of the following relations:

$$\nu/r=\text{const}\cdot\exp[\pm(\delta_T-\delta_\tau)t]. \tag{2.1}$$

By selection of the origin of the time measurements, one can set the constant equal to unity. Then, in the case of the negative sign in the exponential, the first integral of Eq. (1) is given by

$$(\hat D_\tau-\dot\nu/\nu)\hat D_\tau(x-1)+(r^2+\nu^2)x=\nu^2[1+C\exp(-\delta_\tau t)].$$

For the initial conditions (2) we have C = 0; introducing the new variables

$$\varphi=\int\nu dt,\quad v(\varphi)=(x-1)\exp(\delta_\tau t),$$

we obtain

$$v_\varphi''+v[1+\varepsilon^2(\varphi)]=-\varepsilon^{\delta_\perp/\delta_p},\quad \varepsilon=e^{\delta_p t(\varphi)};$$
$$\delta_p=\delta_T-\delta_\tau,\quad \delta_\perp=2\delta_T-\delta_\tau. \tag{2.2}$$

Let us indicate certain forms of ν(t) and r(t) which are of physical interest, for which one can find the exact solution of Eq. (2.2).

1) The frequency difference is constant, $\nu = \text{const} = \nu_0$, and the amplitude grows exponentially: $r = \nu_0 \exp(\delta_p t)$ (Fig. 1a). With this as an example, let us consider the oscillatory properties of the solution.

Solving Eq. (2.2) for this case and again changing to x, we have

$$x=1-m^2\varepsilon^{1-\mu}[S_{\mu,\,im}(m\varepsilon)+CJ_{im}(m\varepsilon)+C^*J_{-im}(m\varepsilon)], \tag{2.3}$$

where $m = \nu_0/\delta_p$, $\mu = \delta_T/\delta_p$, and $S_{\mu,im}$ is a Lommel's function.[18] The solution (2.3) has an oscillatory component whose frequency and damping constant are different at different stages of the process, where the oscillation has, essentially, a different physical nature. For small amplitudes r(t), when the argument $|m\epsilon|$ of the Bessel function is small in comparison with the quantity max $(1, |m|)$ (i.e., provided that $r^2 \ll r_{cr}^2 \equiv \nu^2 + \delta_p^2/4$), in relation (2.3) one can write

$$CJ_{im}(m\varepsilon)+C^*J_{-im}(m\varepsilon)\approx C(m\varepsilon/2)^{im}/\Gamma(1+im)+\text{c.c.}$$
$$=A\cos(m\ln m\varepsilon+\psi)=A\cos(\nu_0 t+\psi),$$

where A and ψ are determined from the initial conditions. Thus, for $r^2 \ll r_{cr}^2$ a component

$$\exp(-\delta_\tau t)\cos\nu_0 t,$$

may be present in the relaxation of the populations, due to the usual linear beats between the frequency $\omega(t)$ of the weak field and the intrinsic polarization at the resonant frequency ω_0 of the still "undistorted" system (the damping constant δ_τ is produced as the resultant between the damping constant δ_T of the polarization and the growth increment of the field (here δ_p), that is, $\delta_\tau = \delta_T - \delta_p$).

The usual asymptotic expansion is valid for the function J_{im} for large values of the argument $m\epsilon$ (for $r^2 \gg r_{cr}^2$); hence

$$CJ_{im}(m\varepsilon)+C^*J_{-im}(m\varepsilon)\propto\varepsilon^{-1/2}\cos(m\varepsilon+\psi),$$

i.e., here oscillations

$$\sim\varepsilon^{1/2-\mu}\cos m\varepsilon=\exp(-\delta_\Sigma t/2)\cos\left(\int r\,dt\right)\qquad\delta_\Sigma=\delta_T+\delta_\tau$$

are present in the population (a sort of averaging of the reciprocal relaxation times occurs, and the new damping constant does not depend on the rate of growth of the field). Thus, for $r^2 \gg r_{cr}^2$ the oscillations approach those which occur in the case $\nu \equiv 0$ for $r^2 \gg \delta_p^2/4$,[3] which now corresponds specifically to the oscillator regime in a strong field.

We note that the "strong field condition $r^2 \gg r_{cr}^2$, which is necessary for the excitation of such oscillations, does not in general coincide with the saturation condition. In the quasistationary case we have from Eq. (1)

$$x_{st}=1/(1+r^2/r_{sat}^2),\quad r_{sat}^2=\delta_T\delta_\tau(1+\nu^2/\delta_T^2), \tag{2.4}$$

that is, for $T \sim \tau$ and $\nu^2 < \delta_T^2$ (the frequency of the field is within the limits of the resonance line) the critical "oscillatory" amplitude r_{cr} may be below the saturating value r_{sat}, whereas for $\tau \gg T$ one will have $r_{cr}^2 \gg r_{sat}^2$.

2) The frequency difference and the amplitude vary according to the law

$$\nu=\nu_0\varepsilon^2/(1+\varepsilon^2)^\alpha,\quad r=\nu_0\varepsilon^3/(1+\varepsilon^2)^\alpha;\quad \nu_0=\text{const},\quad\alpha=\text{const} \tag{2.5}$$

(for the case $\alpha = 1$, see Fig. 2a; for $\alpha = 3/2$, see Fig. 5a; for $\alpha = 2$, see Fig. 3a). Here the exact solutions of the homogeneous equation, corresponding to Eq. (2.2), are expressed in terms of Bessel functions (of the first and second kind) of order $m = (1-\alpha)/(3-2\alpha)$ (except for the case $\alpha = 3/2$, when (2.2) reduces to the Euler equation). With the aid of the functions (2.5), by changing the value of α one can describe a number of different physical situations. Even in the case when the amplitude and frequency vary in time like

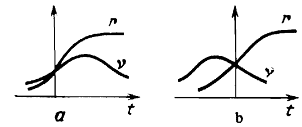

FIG. 5

$$r=r_0\varepsilon^3/(A^2+\varepsilon^2)^{1/2},\quad\nu=r_0\varepsilon^2/(A^2+\varepsilon^2)^{1/2},\quad A=\text{const}. \tag{2.6}$$

the solution can also be described in terms of the Bessel function of order $m = (1/2)(1-(2r_0/\delta_p)^2)^{1/2}$. For A = 0 this is a field with a constant amplitude (Fig. 1b–cf. Item 1); for $0 < A^2 < 1$–this field has the form shown in Fig. 5b; and for $A^2 > 1$ the curves (2.6) are analogous to Fig. 5a, where now the position of the maximum of the curve $\nu(t)$ can be varied by changing the value of A.

Let us return to formula (2.1). In the case of the positive sign in the exponential, by introducing the new variables

$$\varphi=\int\nu\,dt=\int r\varepsilon\,dt,\quad u(\varphi)=r^{-1}\exp(\delta_T t)\hat D_\tau(x-1),$$

we reduce (1) to the equation

$$u_\varphi''+(1+\varepsilon^{-2})u=-\delta_\tau\nu^{-1}\exp[\delta_\tau t(\varphi)]. \tag{2.7}$$

It is easy to see that the solution v of Eq. (2.2) (without the right hand part), corresponding to any field (i.e., to some $\nu(t)$ and r(t)), is simultaneously a solution u(t) of Eq. (2.7) (without the right hand part), corresponding to the time-inverted field (that is, $\nu_1(t) = \nu(-t)$ and $r_1(t) = r(-t)$).

3. APPROXIMATE SOLUTIONS IN VARIOUS LIMITING CASES

For arbitrary variations of the field in time and for arbitrary ratios of the relaxation times, Eq. (1) can be approximately solved with good accuracy in the limiting cases of a weak field,

$$r^2\ll\max(r_{cr}^2, r_{sat}^2)$$

and a strong field,

$$r^2\gg r_{cr}^2=\nu^2+\delta_p^2/4,$$

and also for arbitrary amplitude–in the cases of slowly varying or, conversely, rapidly varying fields.

1) In a weak field ($r^2 \ll r_{sat}^2$) the population deviates slightly from the equilibrium value ($1 - x \ll 1$); therefore, Eq. (1) can be written in the form

$$\frac{1}{\nu}\hat D_T\left\{\frac{1}{\nu}\hat D_T\left[\frac{1}{r}\hat D_\tau(x-1)\right]\right\}+\frac{1}{r}\hat D_\tau(x-1)$$
$$=-\frac{1}{\nu}\hat D_T\left(\frac{r}{\nu}\,x\right)\approx-\frac{1}{\nu}\hat D_T\left(\frac{r}{\nu}\right). \tag{3.1}$$

Having set

$$\varphi=\int\nu\,dt,\quad u(\varphi)=r^{-1}\exp(\delta_T t)\hat D_\tau(x-1),$$

we bring Eq. (3.1) to the form

$$u_\varphi''+u\approx\frac{d}{d\varphi}\left[\frac{r}{\nu}\exp(\delta_T t(\varphi))\right], \tag{3.1'}$$

from where, taking the definition of $u(\varphi)$ into account, we obtain

$$1-x\approx\exp(-\delta_\tau t)\int^{t}\int^{t'}\exp(\delta_\tau t')\exp[\delta_T(t''-t')]r(t')r(t'')\cos\left(\int_{t''}^{t'}\nu\,dt\right)dt'\,dt''. \quad (3.2)$$

In particular, in the limiting case of a rarefied gas ($T = 2\tau$) we have

$$1-x\approx\frac{\exp(2\delta_T t)}{2}\left[\left(\int r\exp(\delta_T t)\sin\varphi\,dt\right)^2+\left(\int r\exp(\delta_T t)\cos\varphi\,dt\right)^2+C\right],\quad \varphi=\int\nu\,dt. \quad (3.3)$$

However, if $r_{sat}^2 < r^2 < r_{cr}^2$ (this may occur for $\tau \gg T$, see Sec. 2, Item 1), then in the first approximation it is necessary to use the quasistationary solution $x_0 \approx x_{st}$ given by Eq. (2.4) in the right hand side of Eq. (3.1) instead of $x_0 \approx 1$. In connection with this, the right hand side of Eq. (3.1′) will look like

$$\frac{d}{d\varphi}\left[\frac{r}{\nu}\exp(\delta_\tau t)\Big/\left(1+\frac{r^2}{r_{sat}^2}\right)\right].$$

2) In a strong field $\nu^2/r^2 \ll 1$; therefore, the solution of Eq. (1) should differ slightly from the solution of the degenerate equation, corresponding to the case $\nu \equiv 0$[3]

$$r^{-1}\hat{D}_T[r^{-1}\hat{D}_\tau(x-1)]+x=0. \quad (3.4)$$

(The smallness of the difference in the complex eigenfrequencies of the oscillations is sufficient for finding an approximation solution, and this smallness can be verified by investigating the exact solutions for the cases cited in Secs. 1 and 2.) In connection with this, let us here write Eq. (1) in the form of an integro-differential equation

$$\frac{1}{r}\hat{D}_T\left[\frac{1}{r}\hat{D}_\tau(x-1)\right]+x=-\frac{\nu}{r}\hat{D}_T^{-1}\left[\frac{\nu}{r}\hat{D}_\tau(x-1)\right]\equiv Q, \quad (3.5)$$

where $\hat{D}_T^{-1}$ is the operator which is the inverse of $\hat{D}_T$, that is, $\hat{D}_T\hat{D}_T^{-1} = 1$, from which

$$\hat{D}_T^{-1}(z)=\exp(-\delta_T t)\int r\exp(\delta_T t)\,dt,$$

and we shall seek an approximate solution, considering (3.5) as a second-order differential equation with a given right hand part $Q(t)$, for which the unknown $x(t)$ appearing in it can, for $\nu^2/r^2 \ll 1$, be chosen (in the first approximation) in the form of a solution of the initial equation (3.4) which is valid for $\nu \equiv 0$.

The solutions of Eq. (3.4) are found in[3] for a number of specific kinds of functions $r(t)$ in the case when $T \neq \tau$, and for arbitrary $r(t)$ when $T = \tau$. In the general case the approximate solution of the homogeneous equation, corresponding to (3.4), is given by

$$\bar{x}\approx\exp\left[-\delta_\Sigma t/2\pm i\int(r^2-\delta_p^2/4)^{1/2}\,dt\right];$$

with its aid one can obtain an approximate solution of Eq. (3.4) itself:

$$x_0\approx-\frac{\delta_p^2}{4\varkappa^2}+\int^{t}\exp\left[\frac{\delta_\Sigma(t'-t)}{2}\right]\hat{D}_\Sigma\left(\frac{r^2}{\varkappa^2}\right)\cos\left(\int_{t'}^{t}\varkappa\,dt\right)dt'; \quad (3.6)$$

$$\varkappa=(r^2-\delta_p^2/4)^{1/2},\quad \hat{D}_\Sigma=d/dt+\delta_\Sigma/2,\quad \delta_\Sigma=\delta_T+\delta_\tau.$$

This solution is exact for $T = \tau$ for an arbitrary function $r(t)$, or for $T \neq \tau$ it is exact for r = const. Examination shows that (3.6) is close to the exact solution for those $r(t)$ for which the exact solutions were found in[3]; this specifically pertains to values $r^2 \gg \delta_p^2/4$, when $r \sim \kappa$.

Now, with the aid of (3.6), it is not difficult to calculate the quantity $Q(t)$ in Eq. (3.5). From here, by again using the approximate solution $\bar{x}$ of the homogeneous equation corresponding to (3.4), we obtain from (3.5) the correction Δx, resulting from the calculated value of $Q(t)$ (for a strong field $r^2 \gg r_{cr}^2$), to the solution x_0 given by Eq. (3.6):

$$\Delta x=x-x_0=\int^{t}\exp[\delta_\Sigma(t'-t)/2]r(t')Q(t')\sin\left(\int_{t'}^{t}r\,dt\right)dt'. \quad (3.7)$$

With the aid of these results one can describe the resonant interaction of the quantum system with a field possessing an arbitrary spectrum, whose width may be either smaller or larger than the resonant linewidth; it is only necessary that the field be sufficiently strong ($\overline{r^2} \gg \overline{r_{cr}^2}$).

3) Now let us separately discuss the case of a strong field $r = r_0 + \Delta(t)$ with an almost constant amplitude (r_0 = const), such that the deviations of the frequency and amplitude in time are arbitrary but small in comparison with r_0 (Δ^2, $\nu^2 \ll r_0^2$; however, the relation between ν and δ_T is arbitrary). One can show that in this connection the behavior of the correction $\Delta x = x - x_0$ to the steady-state value of the population is described by the following equation with constant coefficients:

$$\Delta\ddot{x}+\delta_\Sigma\Delta\dot{x}+(r_0^2+r_{sat}^2)\Delta x=\frac{\delta_\tau}{1+r_{sat}^2/r_0^2}[q_\Delta(t)+q_\nu(t)]; \quad (3.8)$$

where

$$q_\Delta(t)=\frac{\ddot{\Delta}+2\delta_T\dot{\Delta}}{r_0},\quad q_\nu(t)=\nu\exp(-\delta_T t)\int\nu\exp(\delta_T t')\,dt'.$$

Thus, if the amplitude is modulated by the periodic oscillation $\Delta = \Delta_m \sin\Omega t$, then a resonance should be observed at $\Omega \sim 4_0$, where the halfwidth of the resonance curve is $\Delta\Omega \sim \delta_E/2$.[3] In order to obtain resonance with the aid of a modulation $\nu \sim \nu_m \sin\Omega t$ of the field's frequency, the frequency of the modulation must be twice as small ($\Omega \sim r_0/2$) since, as is clear from Eq. (3.8), the effect is quadratic in the frequency. However, if the frequency difference has a constant component ν_0, a resonance also occurs at $\Omega \sim r_0$, and its amplitude is proportional to $\nu_0\nu_m$.

4) In a number of cases the approximation results from the value of the characteristic time τ_E of variation of the field. In a quasistationary field ($\tau_E \gg \tau$) the population is determined by relation (2.4), which follows from Eq. (1) for

$$\hat{D}_T\approx\delta_T,\quad \hat{D}_\tau\approx\delta_\tau.$$

If the field pulse is much shorter than the lifetime, but exceeds the reciprocal of the linewidth ($\tau \gg \tau_E \gg T$), which is frequently realized in laser technology,[10] then, having set

$$\hat{D}_\tau\approx d/dt,\quad \hat{D}_T\approx\delta_T,$$

in Eq. (1), we obtain

$$\dot{x}+\delta_T r^2(\nu^2+\delta_T^2)^{-1}x=0, \quad (3.9)$$

from which (for $t - t_0 \lesssim \tau_E$)

$$x\approx\exp\left[-\delta_T\int\frac{r^2}{\nu^2+\delta_T^2}\,dt\right].$$

Finally, if the pulse is very short ($\tau_E \ll T, \tau$), then one can take $\hat{D}_T \approx \hat{D}_\tau \approx d/dt$ in Eq. (1), which leads to Eq. (1.1) where it is necessary to substitute the value x of the population directly in place of y.

4. DYNAMICS OF THE POLARIZATION

The resonant part of the polarization of a two-level system is given by

$$\mathbf{P}=\mathbf{d}(\sigma_{12}+\sigma_{21})=\tfrac{1}{2}in_0\mathbf{d}(Ye^{-i\omega_0 t}-Y^*e^{i\omega_0 t}),$$

where the following truncated equations are valid for Y:[3]

$$\hat{D}_T(Y)=-rxe^{i\varphi},\quad \hat{D}_\tau(x-1)=\tfrac{1}{2}r(Ye^{-i\varphi}+\mathrm{c.c.})$$
$$\left(\varphi=\int \nu\, dt\right). \tag{4.1}$$

If the regime of the populations $x(t)$ is found with the aid of Eq. (1) for a given field $r(t)$ and $\nu(t)$, the polarization is also uniquely determined from Eq. (4.1):

$$Y=e^{i\varphi}\left\{\frac{1}{r}\hat{D}_\tau(x-1)-i\frac{r}{\nu}\left[x+\frac{1}{r}\hat{D}_T\left(\frac{1}{r}\hat{D}_\tau(x-1)\right)\right]\right\}. \tag{4.2}$$

Use of integration, allowing for Eq. (1), gives

$$Y=e^{i\varphi}\left\{\frac{1}{r}\hat{D}_\tau(x-1)+i\hat{D}_T^{-1}\left[\frac{\nu}{r}\hat{D}_\tau(x-1)\right]\right\}. \tag{4.3}$$

At exact frequency resonance ($\nu \equiv 0$) and for the initial conditions (2), from Eq. (4.3) we have the following result for the reduced amplitude of the polarization $p = P_{\mathrm{ampl}}/n_0 d$:

$$p=|Y|=r^{-1}\hat{D}_\tau(x-1), \tag{4.4}$$

and the phase of the polarization is $\psi = \varphi + \pi/2$. If the field is sufficiently slowly varying ($\tau_E \gg T$), then $\hat{D}_T^{-1} \approx T$, whence under conditions (2) it follows from Eq. (4.3) that

$$Y\approx e^{i\varphi}(1+i\nu T)\hat{D}_\tau(x-1)/r, \tag{4.5}$$

i.e., the phase of the polarization follows the frequency and phase of the field in a quasistationary manner.

In the general case the real amplitude of the polarization $p \equiv |Y|$ and the difference x of the populations are related by a simple relationship, not containing the field parameters and following immediately from Eq. (4.1):

$$p\hat{D}_T(p)+x\hat{D}_\tau(x-1)=0. \tag{4.6}$$

If relaxation is absent or if the field pulse is very short ($\tau_E \ll \tau$, T), then (4.6) leads to the following simple invariant of the motion:

$$p^2+x^2=\mathrm{const.}$$

Under the initial conditions (2), the constant is equal to unity, from which $p_{max} = 1$ (for x = 0) or $(P_{\mathrm{ampl}})_{max} = |n_0 d|$. For the quasistationary regime ($\tau_E \gg T, \tau$) we have

$$p^2=T\tau^{-1}x(1-x),$$

i.e., $p^2_{max} = T/4\tau$ (for x = 1/2) or $(p^2_{max})_{max} = 1/2$ (for $T = 2\tau$). If $\tau_E \gg T$ (the phase of the polarization follows the phase of the field), then

$$p^2=-Tx\hat{D}_\tau(x-1).$$

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1)See, for example, [2,8,9] for calculations with the aid of perturbation theory or its modifications.

2)We note that, in connection with the phenomenologically introduced relaxation times, the transition from a description with the aid of a wave function defined by probability amplitudes [2,7] to the density matrix (and, consequently, to populations) is possible only for $T = \tau$ and $n_0 = 0$.

3)The limiting relations between T and τ are as follows: in a rarefied gas $T = 2\tau$; for strongly broadened lines $T \ll \tau$, which is often observed in the optical band; and in the radio band usually $T \sim \tau$.

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