Absolute instability of focused waves in a plasma

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The absolute focused-wave instability due to capture of growing oscillations in the focal region of the pumping wave is considered. The effect of two-dimensional inhomogeneity of the focused pumping wave on the development of parametric instability is investigated.

INTRODUCTION

The theory of nonlinear interaction of large-amplitude waves with plasma is rapidly developing in recent years, owing to a number of applied problems, such as laser and microwave heating of plasma. Much attention is being paid to the investigation of parametric (decay) instabilities (see, e.g.,[1,2]) as the mechanism whereby energy is effectively transferred to a plasma irradiated by high-power electromagnetic waves. As a result of the development of parametric instabilities, intense oscillations with small phase velocities can be excited in the plasma; these interact effectively with the plasma particles thereby ensuring effective absorption of the energy from the external source. Of great importance for parametric instabilities, however, are plasma inhomogeneities. In particular, spatial inhomogeneity introduces a mismatch in the parametric wave-vector resonance, which in turn leads to a decrease of the growth rate and to an increase of the thresholds of the instabilities. In addition, at sufficiently large mismatches, the convective drift of the perturbations out of the parametric-resonance region leads to a finite amplification of the growing fluctuations, as demonstrated in[3,4]. Thus, the inhomogeneities of the plasma can stabilize the instabilities at a low noise-energy level, corresponding to weak absorption of the microwaves. This is why a diligent search is being carried out at present for absolute instabilities that appear in an inhomogeneous plasma as a result of capture (trapping) of the growing perturbations in the region of parametric resonance (in the decay region). Thus, Perkins and Flick[4] have considered the absolute instability of second-order decay $\omega_e(\mathbf{k}) + \omega_e(-\mathbf{k}) = 2\omega_0$ in a plasma of inhomogeneous density. Capture of Langmuir waves occurred near the reflection point, where the growth rate $\gamma_k = \frac{1}{2}\omega_0(v_E/2v_{Te})^2\cos^2\theta_k$ has a maximum. The question of lowering the thresholds in the absolute instability connected with the trapping of the oscillations as a result of their reflection from turning points was considered by Piliya[5]. Absolute instability of first order, which arises in an inhomogeneous plasma following the decay of an electromagnetic wave into two plasmons, was investigated by Silin and Starodub[6], who have shown that the trapping of plasmons is due to their mutual transformation on the boundaries of a parametric resonance region. Other absolute instabilities of electromagnetic waves, connected with the inhomogeneity of the plasma density, were considered in[7-9,2].

In this paper we investigate the absolute instability of focused waves, which exists in a homogeneous and inhomogeneous plasma. This is a sort of universal instability due to the strong inhomogeneity of the pump-wave field, when the coupling of the plasma oscillations via the pump wave leads to the appearance of transformation points near the focus. As a result of the transformation of the third type (see[10] concerning the classification of the types of transformation) the oscillations are trapped in the parametric-resonance region. By the same token, convective outflow of the perturbation energy from the amplification region is prevented, leading to absolute instability. The formation of a local maximum of the pump field is either connected with self-focusing of the pump wave or results from focusing of the wave beam into the plasma from the outside.

To make the exposition compact, the paper is divided into two parts. In Sec. 1 we consider a number of decay processes that result in the development of absolute instabilities when waves are focused in a plasma. We investigate here the simplest case, that of one-dimensional inhomogeneity of the pump-wave field. The parametric instability singularities connected with allowance for the two-dimensional inhomogeneity of the pump wave are considered in Sec. 2, where it is shown that two-dimensional inhomogeneity stabilizes the absolute instability in some cases.

1. EXAMPLES OF ABSOLUTE INSTABILITY OF FOCUSED WAVES IN A PLASMA

Let us investigate the decay (of first order) of a focused electromagnetic wave into two plasmons. In the hydrodynamic approximation, the high-frequency ($\omega \sim \omega_{pe}$) longitudinal oscillations are described by the following equation:

$$\nabla^2\left[\frac{\partial^2}{\partial t^2} + \omega_{pe}^2(1-3\lambda_D^2\nabla^2)\right]\Phi + \left(\nabla^2\frac{\partial}{\partial t}\hat{P} + \hat{P}\nabla^2\frac{\partial}{\partial t}\right)\Phi = 0, \quad (1.1)$$

where Φ is the velocity potential of the electrons and is determined by the relation $\delta \mathbf{v}_e = \nabla\Phi$; λ_D is the Debye radius of the electron; $\hat{P} = \mathbf{v}_0 \cdot \nabla$; $\mathbf{v}_0$ is the oscillatory velocity of the electrons in the field of the pump wave

$$\mathbf{v}_0 = \mathbf{v}_\sim \cos\left[\omega_0 t - \int^z k_0(z')\,dz'\right].$$

We note that Eq. (1.1) enables us to consider the initial-value problem.

We shall first show the possibility of capture of the growing perturbations near the focus of the pump wave. Representing the perturbation as a superposition of two Langmuir waves

$$\Phi = \Phi_1 \exp(i\omega t - i\mathbf{kr}) + \Phi_2 \exp[i(\omega-\omega_0)t - i(\mathbf{k}-\mathbf{k}_0)\mathbf{r}]$$

we obtain by the standard method[1,2] the following dispersion equation:

$$[\omega^2 - \omega_{pe}^2(1+3k^2\lambda_D^2)][(\omega_0-\omega)^2 - \omega_{pe}^2\{1+3\lambda_D^2(\mathbf{k}_0-\mathbf{k})^2\}] = \frac{(\mathbf{kv}_\sim)^2}{4\mathbf{k}^2(\mathbf{k}_0-\mathbf{k})^2}[\mathbf{k}^2(\omega-\omega_0) + (\mathbf{k}_0-\mathbf{k})^2\omega]^2, \quad (1.2)$$

which coincides, as can be readily verified, with that given in Silin's monograph[2]. The dispersion equation (1.2) describes two longitudinal-oscillation modes of a plasma placed in a high-frequency electric field. The mode "intersection" points are the sought points of wave transformation. For sufficiently small pump amplitudes, when the resultant oscillation-frequency shift $\Delta\omega$ is smaller than the dispersion increment $(3/2)\omega_{pe}k^2\lambda_D^2$, the intersection of the oscillation modes in k-space occurs near the surface $\omega_0^2/4 = \omega_{pe}^2(1 + 3k_d^2\lambda_D^2)$.

In the mode "intersection" regions, we put

$$\omega = {}^1/_2\omega_0 + \Delta\omega, \quad \mathbf{k} = \mathbf{k}_d - \Delta\mathbf{k}.$$

It follows then from (1.2) that

$$v_g\Delta k = \pm[\gamma_d^2 + (\Delta\omega)^2]^{1/2}, \tag{1.3}$$

where

$$\gamma_d^2(z) = [\mathbf{k}_d \times \mathbf{v}_\sim(z)]^2(\mathbf{k}_0\mathbf{k}_d)^2/4k_d^4,$$

$v_g = 3\omega_{pe}k_d\lambda_D^2$ is the group velocity of the Langmuir perturbations ($k_d \gg k_0$). For focused waves, the growth rate $\gamma_d(z)$ has a bowl-shaped form; thus, e.g., when the wave beam is focused into the plasma from the outside, the pump field near the focus (in analogy with the case of a converging linear beam in optics[11]) is of the form

$$|E_0|^2 = E_{max}^2 \exp\left[-\frac{r^2/b^2}{1+z^2/a^2}\right]\left(1+\frac{z^2}{a^2}\right)^{-1} \tag{1.4}$$

and consequently we have near the axis

$$|E_0|^2 = E_{max}^2(1+z^2/a^2)^{-1}.$$

Taking this circumstance into account, we see from (1.3) that for perturbations $\Delta\omega = -\gamma$ that increase with time there are located near the focus on the real z axis the points of intersection of the oscillation modes $z_{1,2}$, which are determined from the conditions $\gamma_d^2(z_{1,2}) = \gamma^2$. Between the points $z_{1,2}$ the plasma is "transparent" to the considered oscillations, and the oscillations cannot propagate outside this region. According to[10], a transformation of the third type occurs in the indicated points, as the result of which the perturbations are reflected "backwards," into the region of parametric resonance. Thus, the oscillations become trapped and this leads to absolute instability.

For sufficiently large pump-wave amplitudes, when the contribution of the thermal motion to the plasma dispersion can be neglected, we obtain from the dispersion equation (1.2)

$$k_z/k_\perp = \pm(\sigma \pm \sqrt{\sigma^2 - 1}). \tag{1.5}$$

Here

$$\sigma^2(z) = {}^1/_{16}k_0^2v_\sim^2(z)\,\omega_0^2[\gamma^2\omega_0^2 + ({}^1/_4\omega_0^2 - \omega_{pe}^2)^2]^{-1}$$

and it was assumed that $\omega = \omega_0/2 - i\gamma$. As seen from (1.5), mode "intersection" occurs at the points $\sigma^2(z) = 1$, and consequently the oscillations are trapped in the plasma region where $\sigma^2(z) \geq 1$. The condition $\sigma^2 > 1$, when the intersection points are on the real z axis, can be expressed in terms of the pump-wave energy density W_f at the focus:

$$W_f/mn_0c^2 > {}^8/_3(1 - 4\omega_{pe}^2/\omega_0^2)^2,$$

which imposes, in an inhomogeneous plasma, a limitation on the distance between the focal point and the layer in which $\omega_{pe}(z) = \omega_0/2$. Let us calculate the spectrum of the oscillations trapped in the region of the focus. For perturbations concentrated in the near-axis part of the pump-wave beam, we can derive from (1.1) a fourth-order differential equation similar to that obtained by Silin and Starodub[6]. We confine ourselves below, however, to the quasiclassical case, in which expressions for the wave vectors (1.5) are sufficient for the calculation of the oscillation spectrum. We introduce the notation $k_{1,2} \equiv (\sigma \pm \sqrt{\sigma^2 - 1})k_\perp$ for the modes of the oscillations (1.5).

For the oscillations trapped between two transformations points $z_{1,2}$ ($z_1 < z_2$), the spectrum is determined by the following quantization rule (see, e.g.,[12]):

$$\int_{z_1}^{z_2} dz[k_1(z) - k_2(z)] = \pi(n + {}^1/_2), \quad n = 0, 1, 2, \ldots \tag{1.6}$$

In formula (1.6) it is possible to take into account, in addition to the inhomogeneity of the pump wave, also the inhomogeneity of the plasma density, putting, e.g.,

$$\omega_{pe}^2(z) = {}^1/_4\omega_0^2(1 + \delta + z/L),$$

where $z = 0$ is the focal point. We now consider the case of external focusing of the pump wave. Using (1.4) and assuming $\delta^2 \gg a/k_\perp L^2$ or else $\delta \ll a/k_\perp L^2$ and $(\gamma L/a\omega_0)^2 \gg 1$ (which enables us to neglect the inhomogeneity of the density), we get from (1.6)

$$\frac{1}{\sqrt{1-q^2}}[\mathrm{K}(q) - \mathrm{E}(q)] = \frac{\pi(n + {}^1/_2)}{4ak_\perp}. \tag{1.7}$$

Here $\mathrm{K}(q)$ and $\mathrm{E}(q)$ are complete elliptic integrals. The growth rate of the model numbered n is expressed in terms of q in the following manner:

$$\gamma_n = [\gamma_{max}^2(1 - q^2) - (\omega_0\delta/4)^2]^{1/2},$$

where $\gamma_{max} = k_0v_f/4$, and v_f is the value of $v_\sim$ at the focal point. It follows from (1.7) that with increasing number of the mode the dimension of the capture region increases and the growth rate decreases. The largest growth rate (close to that of the considered instability in the case of a homogeneous pump wave) is possessed by perturbations with $k_\perp a \gg 1$ localized in a narrow region near the focus. For a given $k_\perp$, the detuning of the parametric resonance δ relative to the plasma density determines the number of captured modes. The instability threshold is determined from the condition $2\gamma_n = \nu_{ei}$, where ν_{ei} is the frequency of the electron-ion collisions.

In the case of a self-focusing pump wave, we choose the field in the axial region of the wave beam, following Lugovoĭ and Prokhorov[11], in the form $E^2(z) = E_f^2(1 + z^2/a^2)^{-\alpha}$, where $\alpha = 1/2$ ahead of the focus ($z < 0$) and $\alpha = 1$ at $z > 0$. We then obtain from (1.6)

$$\frac{\pi(n + {}^1/_2)}{2ak_\perp} = \frac{\mathrm{K}(q) - \mathrm{E}(q)}{\sqrt{1-q^2}} + \frac{1}{q\sqrt{1-q^2}}\left[\Pi\left(\varphi_0, -2, \frac{\sqrt{2}}{q}\right) - q^2E\left(\varphi_0, \frac{\sqrt{2}}{q}\right) - (1-q^2)F\left(\varphi_0, \frac{\sqrt{2}}{q}\right)\right], \tag{1.8}$$

where Π, E, and F are incomplete elliptic integrals. For perturbations with $k_\perp \gg 1/a$, the localization region is small in comparison with the inhomogeneity length a, and the difference between the quantization rules (1.7) and (1.8) is immaterial. For example, for the fundamental mode ($n = 0$) we have from (1.8)

$$\gamma^2 = \gamma_{max}^2\left(1 - \frac{\sqrt{2}-1}{ak_\perp}\right) - \left(\frac{\omega_0\delta}{4}\right)^2.$$

With increasing transverse wavelength of the perturbations, the growth rate of the absolute instability decreases rapidly. In the limit $ak_\perp \ll 1$, putting $\omega_{pe} = \omega_0/2$, we obtain

$$\gamma_n = \gamma_{max}[ak_\perp/(n+{}^1/_2)]^{1/2}.$$

The capture region is strongly elongated in this case in the direction of $z < 0$, and its dimension is large in comparison with the length a of the pump-wave inhomogeneity. In the case of focusing of the pump wave in the inhomogeneous plasma at the point $\omega_{pe}(z) = \omega_0/2$, we put $\omega_{pe}^2(z) = {}^1/_4\omega_0^2(1 + z/L)$. Then

$$\sigma^2(z) = \frac{(\gamma_{max}/\gamma)^2}{(1+z^2/a^2)^\alpha(1+z^2/L_\gamma^2)}, \qquad (1.9)$$

where $L_\gamma = 4L\gamma/\omega_0$ and $\alpha = 1$ or ${}^1/_2$. Substituting (1.9) in the integral (1.6), we obtain for the perturbations with growth-rate γ of the order of $\gamma_{max}(k_\perp a_{eff} > n + {}^1/_2)$

$$\gamma_n = \gamma_{max}\left[1+\frac{2n+1}{\mu k_\perp a_{eff}}\right]^{-1/2}, \qquad (1.10)$$

where

$$a_{eff} = aL_\gamma(a^2+L_\gamma^2)^{-1/2}, \quad \mu = 1+\left(\frac{a^2+L_\gamma^2}{a^2+\alpha L_\gamma^2}\right)^{1/2} \leqslant 2.$$

As seen from (1.10), the inhomogeneity of the plasma density and the inhomogeneity of the pump wave enter in the growth rate in the form of a single effective inhomogeneity length a_{eff}.

We note in addition that in an inhomogeneous plasma with a group separation between the focus and the point $\omega_{pe} = \omega_0/2$ there can be produced two quasi-independent regions for the capture of the plasma oscillations, one near the focus and one near the point $\omega_{pe} = \omega_0/2$, with an exponentially small interaction between them.

We proceed to consider the absolute instability that arises when a focused electromagnetic wave of frequency ω_0 decays into a plasmon and ion sound in a non-isothermal ($T_e \gg T_i$) plasma. Neglecting the pump wave vector and the dissipative factors, we write down the dispersion equation of the Langmuir and ion-sound oscillations, which are coupled via the pump wave,

$$(k^2-k_s^2)(k^2-k_e^2) = {}^1/_6(\omega_{pe}/\lambda_D v_{Te})^2(\mathbf{k r}_E)^2. \qquad (1.11)$$

Here $k_s = \omega/c_s$; c_s is the speed of sound; r_E is the amplitude of the electron oscillations in the pump-wave field; $k_e^2 = 2[(\omega_0-\omega)^2 - \omega_{pe}^2]/3v_{Te}^2$; k_e is the wave vector of the plasma oscillations. We confine ourselves to the case of small amplitudes of the pump wave

$$W/n_0T_e \ll 8k\lambda_D(m/M)^{1/2},$$

when the instability increment is smaller than the sound frequency. In this case the instability develops in k-space near the surface $k = k_D = \omega_d/c_s$, which is specified by the decay condition $k_e = k_s$. In the region of the model "intersection" we put

$$\mathbf{k} = \mathbf{k}_d + \Delta\mathbf{k}, \quad \omega = \omega_d \pm i\gamma.$$

We then obtain from (1.11)

$$\Delta k = \pm\frac{i\gamma}{2}(c_s^{-1}-v_g^{-1}) \pm \left[\gamma_d^2(c_s v_g)^{-1} - \frac{\gamma^2}{4}(c_s^{-1}+v_g^{-1})^2\right]^{1/2}. \qquad (1.12)$$

Here

$$\Delta k = \frac{\mathbf{k}_d}{k_d}\Delta\mathbf{k}, \quad \gamma_d^2(z) = \frac{1}{16}\omega_i\omega_d\left(\frac{\mathbf{k}_d\mathbf{r}_E}{k_d\lambda_D}\right)^2,$$

v_g is the group velocity of the Langmuir wave. As seen from expression (1.12), for the focused pump wave there exist on the real axis z transformation points defined by the condition

$$2\gamma_d(z) = \gamma\sqrt{v_g/c_s}(1+c_s/v_g).$$

Thus, capture of the oscillations is possible. We present a formula for the growth rate of the absolute instability, assuming for the sake of argument

$$\gamma_d^2(z) = \gamma_{max}^2(1+z^2/a^2)^{-1}.$$

Using (1.12) we obtain with the aid of the quantization rule (1.6)

$$\frac{\pi(n+{}^1/_2)}{4a\gamma_{max}}(c_s v_g)^{1/2} = \mathbf{K}(p) - \mathbf{E}(p), \qquad (1.13)$$

where γ is connected with p by the relation

$$\gamma_n = 2\gamma_{max}\left(\sqrt{\frac{c_s}{v_g}}+\sqrt{\frac{v_g}{c_s}}\right)^{-1}(1-p^2)^{1/2}.$$

From this we obtain in the quasiclassical limit of small p

$$\gamma_n^2 = 4\gamma_{max}^2\left(\sqrt{\frac{c_s}{v_g}}+\sqrt{\frac{v_g}{c_s}}\right)^{-2}\left[1-\frac{(2n+1)\sqrt{c_s v_g}}{a\gamma_{max}}\right].$$

In the derivation of (1.13) we did not take into account the detuning of the parametric resonance due to the inhomogeneity of the plasma density, which is valid under the condition

$$(4\gamma_{max}L/a\omega_{pe}) \gg \sqrt{c_s/v_g}.$$

We have investigated above the absolute instability of the focused waves in the plasma using two decays as an example. It is clear, however, that for other decays there also exist absolute instabilities that are due to the trapping of the growing perturbations near the local maximum of the pump field. In particular, this pertains to collapsing waves[13,14], where the role of the absolute instabilities should be the most significant during the initial stage of the development of the collapse, when the plasma density well in the region of the collapse is relatively shallow.

2. INFLUENCE OF TWO-DIMENSIONAL INHOMOGENEITY

In Sec. 1 we considered the absolute instability of focused waves for a one-dimensional inhomogeneity of the pump wave. Under real conditions, however, the inhomogeneity can be essentially two-dimensional. It is therefore necessary to investigate decay singularities that are connected with the influence of two-dimensional inhomogeneity. In particular, the question arises of the conditions under which the growing perturbations are captured near the focus in the case of two-dimensional inhomogeneity of the pump wave. As will be shown below, in the case of two-dimensional inhomogeneity, the oscillations do not always become trapped.

We consider first three-wave decay at sufficiently small pump amplitudes, when the dispersion of the growing plasma oscillations is weakly perturbed. In this case the oscillation amplitudes $\Psi_{1,2}$ satisfy a system of two coupled parabolic equations

$$\hat{L}_{1,2}\Psi_{1,2} = \gamma_d\Psi_{2,1}. \qquad (2.1)$$

Here

$$\hat{L}_n = \frac{\partial}{\partial t} + \mathbf{v}_n\nabla + i\mu_n\left[\frac{\mathbf{v}_n}{v_n}, \nabla\right]^2,$$

$\mathbf{v}_n$ are the group velocities of the interacting wave packets, γ_d is the growth rate of the decay instability, and μ_n are constants that are connected with allowance for the dispersion increments. For example, for the $t \to l + s$ decay they are equal to $\mu_s = c_s^2/2\omega_s$ and $\mu_e = -{}^3/_2\omega_{pe}\lambda_D^2$. For narrow (in k-space) wave packets at noncollinear group velocities, the dispersion increments can be neglected. Then, changing over to oblique-angle coordinates ξ and ζ in accordance with the formulas

$$\mathbf{v}_1\nabla=v_1\frac{\partial}{\partial\xi},\quad \mathbf{v}_2\nabla=v_2\frac{\partial}{\partial\zeta},$$

we transform (2.1) into

$$\left(\frac{\partial}{\partial t}+v_1\frac{\partial}{\partial\xi}\right)\Psi_1=\gamma_d\Psi_2,\quad \left(\frac{\partial}{\partial t}+v_2\frac{\partial}{\partial\zeta}\right)\Psi_2=\gamma_d\Psi_1. \tag{2.2}$$

It is convenient to regard ξ and ζ as the axes of a rectangular coordinate system. Let the pump be a uniform wave beam of thickness 2L, making an angle θ with the ζ axis. We rotate the coordinate system through an angle θ ($0 < \theta < \pi$):

$$x_1=\zeta\cos\theta+\xi\sin\theta,\quad x_2=-\zeta\sin\theta+\xi\cos\theta.$$

In terms of the coordinates x_1 and x_2, the growth rate γ_d is constant in the band $|x_2| < L$, and the projections of the group velocities on the inhomogeneity direction (the x_2 axis) are respectively $v_1\cos\theta$ and $(-v_2\sin\theta)$. We now investigate the spectrum of the oscillations described by Eqs. (2.2). It should be noted that although this problem is one-dimensional, it enables us to reveal a number of decay singularities due to the two-dimensional inhomogeneity, particularly the stopping of the absolute instability.

Let $0 < \theta < \pi/2$. Then the components of the group velocities of the perturbations transverse to the pump wave beam have opposite signs, and consequently the oscillations can be captured as a result of their mutual transformations at the boundaries of the pump beam. For the oscillations trapped inside the pump beam, we put $\Psi_{1,2} \sim \exp(\gamma t + kx_1)$. Solving the eigenvalue problem, we obtain the quantization condition in the form

$$\gamma/\gamma_\theta+k/k_\theta=\cos\varphi_\lambda, \tag{2.3}$$

where

$$k_\theta=2\gamma_d\left(\frac{\sin\theta\cos\theta}{v_1v_2}\right)^{1/2}$$

$$\gamma_\theta=2\gamma_d(v_1v_2\sin\theta\cos\theta)^{1/2}(v_1\cos\theta+v_2\sin\theta)^{-1}.$$

Furthermore

$$\sin\varphi_\lambda=\varphi_\lambda/\lambda,\quad \lambda=2\gamma_dL(v_1v_2\sin\theta\cos\theta)^{-1/2}. \tag{2.4}$$

As seen from (2.3) and (2.4), the spectrum of the trapped oscillations is determined by the value of the parameter λ, which is equal to the logarithm of the amplitude gain as the perturbation moves across the pump beam. From (2.4) it is easy to establish that there exists a sequence of monotonically increasing values of $\lambda_n > 1$ ($n = 1, 2, \ldots$), satisfying the equation $\cos(\lambda^2 - 1)^{1/2} = -1/\lambda$ such that Eq. (2.4) has exactly 2n real roots φ_λ in the interval $\lambda_n < \lambda < \lambda_{n+1}$. Further, the condition

$$\tfrac{1}{2}\pi(2m+1)<\lambda<\tfrac{1}{2}\pi(2m+5) \tag{2.5}$$

determines the interval of variation of λ in which Eq. (2.4) has exactly m roots φ_λ satisfying the instability condition $\cos\varphi_\lambda > 0$. Thus, the inequality (2.5) specifies the interval of variation of λ as a function of the number m of captured modes.

Formula (2.3) establishes the connection between the growth rates of the perturbations in space and in time for arbitrary initial data. For example, in the stationary case $\gamma = 0$ the perturbation is trapped between the "walls" $x_2 = \pm L$ and drifts in the positive x_1 direction, increasing exponentially with a spatial growth rate $\kappa = k_0\cos\varphi_\lambda$. From (2.3) we get the following expression for the velocity v_θ of the convective drift of the unstable oscillations along the x_1 axis

$$v_0 = v_1v_2/(v_1\cos 0 + v_2\sin 0).$$

The drift velocity reaches a minimum value $v_1v_2(v_1^2 + v_2^2)^{-1/2}$ at $\theta = \tan^{-1}(v_2/v_1)$.

In the case $\pi/2 < \theta < \pi$, the components of the group velocities of the perturbations along the direction x_2 of the pump inhomogeneity have equal signs. Consequently, the transformation of the oscillations on the pump-beam boundary $x_2 = -L$ cannot cause reflection of the energy "backward," into the interior of the pump beam. Trapping of the oscillations is therefore impossible. Thus, the convective outflow of the perturbations from the pump beam stops the absolute instability. The amplitude gain of the wave packet as a result of the intersection of the pump beam does not exceed the value $\cosh|\lambda|$.

It follows from the foregoing that in the case of decay interaction of wave packets of the type (2.1) with noncolinear group-velocity vectors, the two-dimensional instability of the pump wave stops the absolute instability, this being due to the convective outflow of the perturbations in directions corresponding to the internal part of the angle between the group-velocity vectors. It should be noted here that in one particular case, that of almost antiparallel group velocities of the perturbations, the convective outflow of the oscillation energy from the region of parametric resonance is appreciably slowed down. The gain of the perturbations then reaches values so large that the mechanism of stabilization of the absolute instability by the two-dimensional inhomogeneity of the pump wave does not have time to go into effect.

In the case of antiparallel group velocities of the perturbations (with allowance for the dispersion increments, this imposes the requirement $\mu_1/\mu_2 = -v_1/v_2$), the two-dimensional inhomogeneity does not stop the absolute instability; for any direction of the field gradient of the pump wave, the components of the drift velocities of the perturbations along the inhomogeneity have opposite signs, so that transformation of the oscillations on the pump boundary leads to reflection of the oscillations towards an increasing pump field, thus preventing convective outflow of the perturbations. An example of such a decay is the decay of an electromagnetic wave into a plasmon and phonon, which was investigated in Sec. 1.

We now discuss the case of large amplitudes of the pump wave, when the dispersion of the growing perturbations is determined by the high-frequency pump field. By way of example, we consider the decay of an electromagnetic wave into two plasmons. In the approximation where the inhomogeneity is smooth, we obtain from (1.1) the following model equation:

$$\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial y^2}-2\sigma\frac{\partial^2}{\partial z\,\partial y}\right)\Phi=0, \tag{2.6}$$

where $\sigma(z, y)$ coincides with the parameter of formula (1.5). We note that in the geometrical-optics approximation we obtain from (2.6) the dispersion equation (1.5). We write down the equation of the characteristics

$$dz/dy=\sigma\pm\sqrt{\sigma^2-1}. \tag{2.7}$$

We see therefore that Eq. (2.6) is of the hyperbolic type in the region where $\sigma > 1$ and of the elliptic type at $\sigma < 1$. Thus, the degeneracy line is given by the equation $\sigma^2(z, y) = 1$. We shall henceforth assume $\sigma_{max} > 1$. For focused waves, the lines $\sigma^2 = \text{const}$ are closed curves, and near the focus they differ little from ellipses. The characteristics (2.7) have in this case two singular points of the focused type, located on the degeneracy line and defined by the condition $dz/dy = 1$. It can be shown that as a result of reflection from the degeneracy line the characteristics, unwinding from one singular point, then wind themselves on the other singular point. At the singular points, the characteristics are tangent to the degeneracy line.

Let us consider now the question of quantization of the two-dimensional equation (2.6). First, in the elasticity region ($\sigma^2 < 1$). we can construct solutions that decrease at infinity and assume certain values $f(s)$ (the function $f(s)$ is continuous) on a contour located near the degeneracy line. Then, continuing this solution into the hyperbolicity region, we find that the solution should have singularities on the degeneracy line at the singular point of the characteristics. Let us investigate the solution near a singular point. In the case $(\sigma_{max} - 1) \ll 1$, Eq. (2.6) near the singular point can be transformed into

$$\left(y-\frac{x^2}{2}\right)\frac{\partial^2\Phi}{\partial y^2}+\alpha\frac{\partial\Phi}{\partial y}-\frac{\partial^2\Phi}{\partial x^2}=0, \qquad (2.8)$$

where α is a constant. For (2.8) the degeneracy line is $y = x^2/2$, and the hyperbolicity region is $y > x^2/2$. We note that an equation of the type (2.8) was considered earlier by Piliya and Fedorov[15] in connection with an investigation of the singularities of the wave field in an anisotropic plasma with two-dimensional inhomogeneity. The one-parameter family of solutions of Eq. (2.8) that decrease far from the singular point is expressed in terms of the hypergeometric function

$$\Phi_\beta(x, y)=x^{-2\beta}F(\beta;\ \beta+{}^1/_2;\ \gamma_\beta;\ R), \qquad (2.9)$$

where the parameter β is subject to the requirement $\mathrm{Re}\,\beta > 0$,

$$\gamma_\beta=\frac{3}{4}+\beta+i\left(\frac{3}{4}+\beta-\alpha\right)\mathrm{ctg}\,\theta,\quad R=\frac{1}{2}+\frac{i}{2}\left(1-\frac{8y}{x^2}\right)\mathrm{ctg}\,\theta$$

and $\theta = \tan^{-1}(7^{1/2})$. We note that with the aid of the variables $\xi = 2\ln x$ and $\zeta = y/x^2$ Eq. (2.8) is transformed into an equation with one-dimensional inhomogeneity along the coordinate ζ. Along the normal curve y/x^2 = const the solution (2.9) varies in proportion to $x^{-2\beta}$. Near the origin, at $|y/x^2| \gg 1$, the solution has the asymptotic form $\Phi_\beta \sim (8y - qx^2)^{-\beta}$, where $q = 1 - i\tan\theta$. Thus, for solutions that decrease at infinity, a singularity exists at the singular point $x = 0$, $y = 0$. The solutions are limited at the singular point because of the small imaginary increment of part of the function $\sigma(z, y)$, which arises as a result of the difference between the plasmon frequencies.

To illustrate the quantization singularities in the two-dimensional case, which are connected with the presence of the degeneracy line and of the aingular points of the characteristics, we consider the equation

$$\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial y}\right)^2\Psi+\mathrm{sgn}(r-b)\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)^2\Psi=0, \qquad (2.10)$$

where $r = \sqrt{z^2 + y^2}$ and b is a constant ($b > 0$). An equation of this type can be obtained from (2.6) by assuming that the function $\sigma(z, y)$ is a step function.

Equation (2.10) is of the hyperbolic type inside the circle $r < b$, and is a two-dimensional Laplace equation outside the circle. In the hyperbolic region the characteristics are parallel to the coordinate axes, and their trajectories are closed and form perimeters of rectangles inscribed in the circle $r = b$. For Eq. (2.10), the quantization problem can be solved completely. In the ellipticity region, the solution in terms of the polar coordinates r and θ takes the form

$$\Psi=\sum_{n=1}^{\infty}\left(\frac{b}{r}\right)^n(F_n\sin n\theta+P_n\cos n\theta),\quad r>b. \qquad (2.11)$$

In the hyperbolic region we have

$$\Psi=U\left(\frac{r}{b}\cos\theta\right)+W\left(\frac{r}{b}\sin\theta\right),\quad r<b, \qquad (2.12)$$

where U and W are arbitrary functions to be determined. Matching the solutions (2.11) and (2.12) on the degeneracy line $r = b$, we obtain, from the conditions that the functions and their first derivatives be continuous, equations for the determination of the arbitrary functions U and W. It is convenient to introduce the basis solutions $\Psi^{(1,2)}(r, \theta)$. We specify them in the ellipticity region by means of the formulas

$$\Psi_n^{(1)}=\left(\frac{b}{r}\right)^n\sin n\theta,\quad \Psi_n^{(2)}=\left(\frac{b}{r}\right)^n\cos n\theta,\quad r>b. \qquad (2.13)$$

As a result of matching on the degeneracy line, the solutions (2.13) generate a family of functions $U_n^{(1,2)}$ and $W_n^{(1,2)}$. We present some of the solutions Ψ_n in the hyperbolic region:

$$\Psi_2^{(2)}=\frac{r^2}{b^2}\left(3-\frac{2r^2}{b^2}\right)\cos 2\theta,$$
$$\Psi_1^{(2)}=\frac{r}{b}\left(1-\frac{2r^2}{3b^2}\cos^2\theta\right)\cos\theta+\frac{2}{3}\left(1-\frac{r^2}{b^2}\sin^2\theta\right)^{3/2},$$
$$\Psi_1^{(1)}=\frac{r}{b}\left(1-\frac{2r^2}{3b^2}\sin^2\theta\right)+\frac{2}{3}\left(1-\frac{r^2}{b^2}\cos^2\theta\right)^{3/2}.$$

We call attention to the fact that the second derivatives of the solution $\Psi_1^{(2)}$ and $\Psi_1^{(1)}$ increase without limit as they approach the degeneracy line from the hyperbolic region at the singular points $\theta = 0, \pi$ and $\theta = \pi/2; 3\pi/2$. The indicated points are remarkable for the fact that one of the characteristics is tangent to the degeneracy line at these points. Thus, although the solutions of (2.10) that decrease at infinity are bounded on the entire (x, y) plane, nevertheless the presence of singular points on the degeneracy line leads to singularities of the higher derivatives of the solutions at these points.

We note also the singularities of the solution of the initial-value problem in the case of strong pump fields. At large pump-wave amplitudes, the plasma is similar to a strongly anisotropic dielectric, and therefore singularities of the type of resonant cones, which are considered, e.g., in[16], can be observed in the distribution of the perturbation field.

CONCLUSION

We present a brief summary of the results.

1. We have demonstrated the existence of absolute instabilities of focused waves in a plasma; these instabilities are due to capture of growing oscillations due to focus of the pump wave. Since the focusing region is determined by the external conditions, it is possible to shift the region of buildup of the oscillations over the volume of the plasma by changing the position of the focal point.

2. The class of absolute parametric instabilities increases appreciably for the focused waves, and this is of particular importance in the case of the interaction of powerful radiation with an inhomogeneous plasma.

3. Attention is called to the need for taking into account two-dimensional inhomogeneity in the investigation of parametric instabilities. The conditions are indicated for the stabilization of the absolute instability by two-dimensional inhomogeneity of the pump field.

4. It is shown that when the two-dimensional inhomogeneity is taken into account, a strong anisotropy can arise in the distribution of the fields of the trapped plasma oscillations. The plasma perturbations trapped in the two-dimensional well are localized near certain singular points of the characteristics of the wave equation.

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