Concerning the production of electron-positron pairs from vacuum

V. G. Bagrov, D. M. Gitman, and Sh. M. Shvartsman

Tomsk Institute of Automatic Control Systems and Radio Engineering
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An electron-positron field interacting with an external electromagnetic field is investigated. It is shown that the probability of pair production and the probability that the vacuum remains a vacuum can be expressed in terms of the exact solutions of the Dirac equation in this field. As an illustration of the obtained results, the probability of pair production by a constant electric field of finite duration is found. The derived formulas enable us, in particular, to estimate the effective pair-production time.

1. INTRODUCTION

In recent years a number of articles have appeared devoted to calculations of the probabilities for electron-positron pair production from vacuum by external electromagnetic fields.[1-8] This is explained by projected prospects for the achievement of sufficiently strong fields in which the effect of pair production from vacuum will apparently become observable in practice.

Although the methods used in these articles to calculate the probabilities of pair production are formally different from each other, they have one common feature: all of them utilize, in one way or the other, solutions of the Dirac equation in the external field. This is apparently connected with the fact that, although a noncontradictory single-particle interpretation of the Dirac equation canrot be given for particles in an external electromagnetic field, the exact solutions themselves contain all the information about the scattering processes of the electron-positron field and about processes involving the creation of electron-positron pairs from vacuum. However, it is possible to carry out a consistent extraction of this information from the exact solutions only by utilizing the formalism of quantum field theory.

As is shown in the article by Nikishov,[6] knowledge of the exact solutions of the Dirac equation with the appropriate asymptotic behavior in the fields, which stop pair production for $t = \pm\infty$, provides the possibility to rather simply calculate the probabilities of scattering and pair production. Unfortunately, the majority of exact solutions of the Dirac equation which can be derived pertain to fields which do not vanish asymptotically. At the present time the number of such exact solutions is rapidly increasing, largely due to recent work on the separation of variables in the Dirac equation.[9].

In the present work it is shown that, in spinor electrodynamics with a given external field, the pair-production probabilities and the probability that the vacuum remains a vacuum can be determined with the aid of exact solutions of the Dirac equation. It turns out to be possible to utilize the solutions in fields, which do not have a special asymptotic behavior at infinity. The probability of pair production by a constant electric field of finite duration is found as an illustration of the obtained results. The obtained formulas enable us, in particular, to estimate the effective time for pair production.

2. GENERAL RESULTS

Let us consider a second-quantized, electron-positron Dirac field, perturbed by an external electromagnetic field. Let a field, which does not create pairs, described by the potential $A_1^\mu(\mathbf{r}, t)$ act from $-\infty$ to t_1; a field described by the potential $A_2^\mu(\mathbf{r}, t)$ acts in the intermediate time interval from t_1 to t_2 in which we wish to determine the effect of pair production; a field, which does not create pairs, described by the potential $A_3^\mu(\mathbf{r}, t)$ again acts from t_2 to $+\infty$. The potentials A_1^μ and A_3^μ are introduced in order to guarantee the requisite behavior of the fields at times t_1 and t_2, and also for a possible generalization of the problem to that case when pair production is investigated in a background involving the action of nonpair-producing fields (magnetic fields, plane-wave fields, etc.).

The Hamiltonian of the system under consideration has the following form ($\hbar = c = 1$):

$$\mathscr{H}(t)=\int :\psi^+(\mathbf{r})\,[-i\alpha\nabla+e\alpha A(\mathbf{r},t)+eA_0(\mathbf{r},t)+\beta m]\,\psi(\mathbf{r}):d\mathbf{r},$$

$$A^\mu(\mathbf{r},t)=\begin{cases}A_1^\mu(\mathbf{r},t), & -\infty<t<t_1,\\ A_2^\mu(\mathbf{r},t), & t_1<t<t_2,\\ A_3^\mu(\mathbf{r},t), & t_2<t<\infty,\end{cases}\tag{1}$$

$\psi^+(\mathbf{r})$ and $\psi(\mathbf{r})$ are the field operators of a free electron-positron field, satisfying the well known commutation relations.

The change of a state with time is determined by the evolution operator

$$U(t,t')=T\exp\left[-i\int_{t'}^{t}\mathscr{H}(\tau)\,d\tau\right].\tag{2}$$

Let us show that if complete sets exist of the exact solutions of the Dirac equation in the fields $\tilde{A}_1^\mu$, $\tilde{A}_2^\mu$, and $\tilde{A}_3^\mu$, which coincide with the fields A_1^μ, A_2^μ, and A_3^μ in the intervals $(-\infty, t_1)$, (t_1, t_2), and $(t_2, +\infty)$, respectively, but are arbitrary in the remaining time regions, then such solutions can be utilized for calculations of the pair-production probabilities. Let us denote the mentioned sets of solutions as follows:

$$\{{}_\pm\varphi_n(\mathbf{r},t)\}\leftrightarrow\tilde{A}_1^\mu,\quad \{\varphi_k(\mathbf{r},t)\}\leftrightarrow\tilde{A}_2^\mu,\quad \{{}^\pm\varphi_m(\mathbf{r},t)\}\leftrightarrow\tilde{A}_3^\mu.\tag{3}$$

Since by assumption the fields A_1^μ and A_3^μ do not create pairs, in their presence one can classify the solutions of the Dirac equation according to the criterion of particle—antiparticle, which is correspondingly reflected by the presence of the indices +, −. In addition, we assume that the combinations (3) form complete orthonormal systems at each instant of time. The subscripts n, k, and m label the solutions and the corresponding integrals of the motion.

Let us proceed to prove our assertion. In the first place, having used the group property of the evolution operator and its explicit form (2) one can write

$$U(t, t')=U_3(t, t_2)U_2(t_2, t_1)U_1(t_1, t'), \quad t_2<t, \quad t'<t_1, \tag{4}$$

where the operators $U_1(t_1, t')$, $U_2(t_2, t_1)$, and $U_3(t, t_2)$ are already defined, each for its own field $\tilde{A}_1^\mu$, $\tilde{A}_2^\mu$, and $\tilde{A}_3^\mu$.

In the second place, knowledge of a complete set of exact solutions of the Dirac equation in an external field enables one, in principle, to determine the explicit form of the evolution operator, and also its commutation relations with an arbitrary system of creation and annihilation operators.

In fact, let us consider the field $\tilde{A}_2^\mu$ and the corresponding evolution operator $U_2(t_2, t_1)$. Let $\psi_2(\mathbf{r}, t)$ and $\psi_2^+(\mathbf{r}, t)$ be the field operators in the Heisenberg representation in the presence of the field $\tilde{A}_2^\mu$, where $\psi_2(\mathbf{r}, t_1) = \psi(\mathbf{r})$ and $\psi_2^+(\mathbf{r}, t_1) = \psi^+(\mathbf{r})$. Then

$$\psi_2(\mathbf{r}, t)=U_2^{-1}(t, t_1)\psi(\mathbf{r})U_2(t, t_1), \quad \psi_2^+(\mathbf{r}, t)=U_2^{-1}(t, t_1)\psi^+(\mathbf{r})U_2(t, t_1). \tag{5}$$

On the other hand, in quantum electrodynamics with a given external field, the Heisenberg equations of motion for the field operators formally have the form of the Dirac equation for an electron in a field. This enables one to always find the solution of the Heisenberg equations, when a complete set of solutions of the Dirac equation is found. In our case the Heisenberg operators, expressed in terms of the solutions, have the form

$$\psi_2(\mathbf{r}, t)=\sum_k c_k\varphi_k(\mathbf{r}, t), \quad \psi_2^+(\mathbf{r}, t)=\sum_k c_k^+\varphi_k^+(\mathbf{r}, t),$$
$$c_k=\int \varphi_k^+(\mathbf{r}, t_1)\psi(\mathbf{r})d\mathbf{r}, \quad c_k^+=\int\varphi_k(\mathbf{r}, t_1)\psi^+(\mathbf{r})d\mathbf{r}. \tag{6}$$

Substituting (6) into (5), we find

$$U_2^{-1}(t_2, t_1)\psi(\mathbf{r})U_2(t_2, t_1)=\int G(\mathbf{r}, t_2; \mathbf{r}', t_1)\psi(\mathbf{r}')d\mathbf{r}', \tag{7}$$

$$U_2^{-1}(t_2, t_1)\psi^+(\mathbf{r})U_2(t_2, t_1)=\int \psi^+(\mathbf{r}')G(\mathbf{r}', t_1; \mathbf{r}, t_2)d\mathbf{r}',$$
$$G(\mathbf{r}, t; \mathbf{r}', t')=\sum_k\varphi_k(\mathbf{r}, t)\varphi_k^+(\mathbf{r}', t'). \tag{8}$$

It is not difficult to verify that $G(\mathbf{r}, t; \mathbf{r}', t')$ is the anticommutator of the field operators in the Heisenberg representation for different moments of time:

$$[\psi_2(\mathbf{r}, t), \psi_2^+(\mathbf{r}', t')]_+=G(\mathbf{r}, t; \mathbf{r}', t').$$

Furthermore, the function $G(\mathbf{r}, t; \mathbf{r}', t')$ is the complete propagator. It gives the connection between the solutions of the Dirac equation at different moments of time

$$\varphi(\mathbf{r}, t)=\int G(\mathbf{r}, t; \mathbf{r}', t')\varphi(\mathbf{r}', t')d\mathbf{r}'.$$

Finally, the function $G(\mathbf{r}, t; \mathbf{r}', t')$ is the solution of the Dirac equation with the boundary condition

$$G(\mathbf{r}, t; \mathbf{r}', t')|_{t=t'}=\delta(\mathbf{r}-\mathbf{r}').$$

The function (8) was first introduced by Dirac[10].

Expanding the field operators $\psi(\mathbf{r})$ and $\psi^+(\mathbf{r})$ in terms of the functions ${}_\pm\varphi_n(\mathbf{r}, t)$ and ${}^\pm\varphi_m(\mathbf{r}, t)$, we introduce into consideration operators for the creation and annihilation of particles and antiparticles

$$\{\alpha_n(t), \alpha_n^+(t), \quad \beta_n(t), \beta_n^+(t)\}, \quad \{a_m(t), a_m^+(t), \quad b_m(t), b_m^+(t)\}$$

in the presence of the fields $\tilde{A}_1^\mu$ and $\tilde{A}_3^\mu$, respectively:

$$\psi(\mathbf{r})=\sum_n[\alpha_n(t)\,{}_+\varphi_n(\mathbf{r}, t)+\beta_n^+(t)\,{}_-\varphi_n(\mathbf{r}, t)], \tag{9}$$

$$\psi(\mathbf{r})=\sum_m[a_m(t)\,{}^+\varphi_m(\mathbf{r}, t)+b_m^+(t)\,{}^-\varphi_m(\mathbf{r}, t)]. \tag{10}$$

It is not difficult to show that the introduced operators are related to each other at different moments of time by means of the appropriate evolution operators:

$$\alpha_n(t')=U_1^{-1}(t, t')\alpha_n(t)U_1(t, t'), \quad \beta_n(t')=U_1^{-1}(t, t')\beta_n(t)U_1(t, t'),$$
$$a_m(t')=U_3^{-1}(t, t')a_m(t)U_3(t, t'), \quad b_m(t')=U_3^{-1}(t, t')b_m(t)U_3(t, t'). \tag{11}$$

Let us denote the vacuum state for the operators $\alpha_n(t)$ and $\beta_n(t)$ by $|0t\rangle_1$, and let the vacuum state for the operators $a_m(t)$ and $b_m(t)$ be denoted by $|0t\rangle_3$. Then the probability for pair production in the field (1) can be written as follows:

$$W_{m,m'}=\lim_{\substack{t\to\infty\\ t'\to-\infty}}|{}_3\langle 0t|b_{m'}(t)a_m(t)U(t, t')|0t'\rangle_1|^2.$$

A quite obvious simplification follows from Eqs. (4) and (11):

$$W_{m,m'}=|{}_3\langle 0t_2|b_{m'}(t_2)a_m(t_2)U_2(t_2, t_1)|0t_1\rangle_1|^2. \tag{12}$$

Now let us determine the commutation relations of the operator $U_2(t_2, t_1)$ with the operators $a_m(t_2)$ and $b_m(t_2)$. For this purpose, on the right hand side of Eq. (7) we expand the field operator by means of the expansion (9), and on the left hand side—by means of (10) at the moments of time t_1 and t_2, respectively. Then

$$U_2^{-1}(t_2, t_1)a_m(t_2)U_2(t_2, t_1)=\sum_n[G(\overset{+}{m}|\underset{+}{n})\alpha_n(t_1)+G(\overset{+}{m}|\underset{-}{n})\beta_n^+(t_1)], \tag{13}$$

$$U_2^{-1}(t_2, t_1)b_m^+(t_2)U_2(t_2, t_1)=\sum_n[G(\overset{-}{m}|\underset{+}{n})\alpha_n(t_1)+G(\overset{-}{m}|\underset{-}{n})\beta_n^+(t_1)], \tag{14}$$

$$G(\overset{\pm}{m}|\underset{\pm}{n})=\int {}^\pm\varphi_m^+(\mathbf{r}, t_2)G(\mathbf{r}, t_2; \mathbf{r}'t_1)\,{}_\pm\varphi_n(\mathbf{r}', t_1)d\mathbf{r}\,d\mathbf{r}'. \tag{15}$$

With the aid of Eqs. (13) and (14) one can write Eq. (12) in terms of the matrix $G(\overset{+}{m}|\underset{-}{n})$ and the matrix $G^{-1}(\underset{-}{n}|\overset{-}{m})$, which is the inverse of $G(\overset{-}{m}|\underset{-}{n})$:

$$W_{m,m'}=\Big|\sum_n G(\overset{+}{m}|\underset{-}{n})G^{-1}(\underset{-}{n}|\overset{-}{m}')\Big|^2 C_v. \tag{16}$$

We have denoted the probability that the vacuum remains a vacuum by $C_V = |{}_3\langle 0t_2|U_2(t_2, t_1)|0t\rangle_1|^2$.

Now let us trace the arguments enabling us to determine C_V. Let $R(t_2, t_1)$ be the unitary operator of the nonsingular canonical transformation connecting $[\alpha_n(t_1), \beta_n(t_1)]$ with $[a_m(t_2), b_m(t_2)]$:

$$a_m(t_2)=R^{-1}(t_2, t_1)\alpha_m(t_1)R(t_2, t_1), \quad b_m(t_2)=R^{-1}(t_2, t_1)\beta_m(t_1)R(t_2, t_1). \tag{17}$$

Such an operator always exists as soon as we implicitly assume that the vacuum states $|0t\rangle_1$ and $|0t\rangle_3$ lie in one and the same space of states (see[11]). In this connection the vacuum-state vectors themselves are related as follows:

$$|0t_2\rangle_3=R^{-1}(t_2, t_1)|0t_1\rangle_1.$$

If the operator $\Phi(t_2, t_1) = R(t_2, t_1)U_2(t_2, t_1)$ is introduced, in our case the probability that the vacuum remains a vacuum is equal to the square of the expectation value of the operator Φ with respect to the vacuum at the instant of time t_1:

$$C_v=|{}_1\langle 0t_1|\Phi(t_2, t_1)|0t_1\rangle_1|^2. \tag{18}$$

From Eqs. (13), (14), and (17) one can find the commutation relations of the operator $\Phi(t_2, t_1)$ with the operators $\alpha_n(t_1)$ and $\beta_n(t_1)$:

$$\Phi^{-1}(t_2,t_1)\alpha_m(t_1)\Phi(t_2,t_1)=\sum_n[G(\overset{+}{m}|\overset{+}{n})\alpha_n(t_1)+G(\overset{+}{m}|\overset{-}{n})\beta_n^+(t_1)], \quad (19)$$

$$\Phi^{-1}(t_2,t_1)\beta_m^+(t_1)\Phi(t_2,t_1)=\sum_n[G(\overset{-}{m}|\overset{+}{n})\alpha_n(t_1)+G(\overset{-}{m}|\overset{-}{n})\beta_n^+(t_1)]. \quad (20)$$

By using the formula[12]

$$\exp(-\tau b)\,a\exp(\tau b)=\sum_{n=0}^{\infty}\frac{(-\tau)^n}{n!}\underbrace{[b\ldots[b,}_{n}a]\ldots], \quad (21)$$

it is not difficult to show that relations of the type (19) and (20) can always be satisfied, having chosen $\Phi(t_2, t_1)$ in the following form:

$$\Phi(t_2,t_1)=\exp(\alpha^+A\alpha+\alpha^+B\beta^++\beta C\alpha+\beta D\beta^+), \quad (22)$$

since the conditions imposed on the matrices A, B, C, and D, guaranteeing the unitarity of $\Phi(t_2, t_1)$, and the corresponding equations, following from Eqs. (19) and (20), do not contradict each other.

On the other hand, it is not difficult to show that the totality of all commutators of the four quadratic forms $\alpha^+A\alpha$, $\alpha^+B\beta^+$, $\beta C\alpha$, and $\beta D\beta^+$ forms a set of quadratic forms of the same type. In this case it is possible to represent (21) in the following form which is convenient for the determination of C_v:[12]

$$\Phi(t_2,t_1)=\exp(\alpha^+\tilde{B}\beta^+)\exp(\alpha^+\tilde{A}\alpha)\exp(\beta\tilde{D}\beta^+)\exp(\beta\tilde{C}\alpha). \quad (22a)$$

Substituting (22a) into Eq. (18), we see that

$$C_v=|\exp\operatorname{Sp}\tilde{D}|^2.$$

One can find the relation between the matrix $\tilde{D}$ and the matrices $G(\overset{\pm}{m}|\overset{\pm}{n})$. In order to do this, we substitute (22a) into (20) and utilize formula (21) again. Then $G(\bar{m}|\bar{n}) = (\exp\tilde{D})_{m,n}$. As a result we obtain

$$C_v=|\exp\operatorname{Sp}\ln G(\bar{m}|\bar{n})|^2=|\det G(\bar{m}|\bar{n})|^2. \quad (23)$$

We see that the probability that the vacuum remains a vacuum is expressed in terms of the infinite determinant of the matrix $G(\bar{m}|\bar{n})$. The properties of such determinants have been rather extensively investigated in Schwinger's work[13].

3. PAIR PRODUCTION IN AN ELECTRIC FIELD OF FINITE DURATION

Using the derived formulas, let us determine the probability for pair production in a constant electric field of finite duration:

$$E=(0,0,E(t)),\quad H=0,\quad E(t)=\begin{cases}0, & -\infty<t<t_1,\\ E_0, & t_1<t<t_2,\\ 0, & t_2<t<\infty.\end{cases}$$

Let us choose the potential as follows:

$$A_i^\mu(0,0,0,f_i),\quad f_1=-cE_0t_1,\quad f_2=-cE_0t,\quad f_3=-cE_0t_2$$

A complete set of solutions of the Dirac equation in a field described by the potentials A_2^μ may be found[14] in the form

$$\varphi_{\mathbf{k},\zeta,j}(\mathbf{r},t)=L^{-3/2}\exp(i\mathbf{kr})[v_{1\zeta}\chi_1^{(j)}(t)+v_{2\zeta}\chi_2^{(j)}(t)], \quad (24)$$
$$\zeta=\pm1,\quad j=1,2.$$

The bispinors $v_{1\zeta}$ and $v_{2\zeta}$ and the functions $\chi_1^{(j)}(t)$ and $\chi_2^{(j)}(t)$ have the form

$$v_{1\zeta}=\begin{vmatrix}c_1\\c_3\\-c_1\\c_3\end{vmatrix}\qquad v_{2\zeta}=\begin{vmatrix}c_2\\-c_4\\c_2\\c_4\end{vmatrix}$$

$$2c_j=\zeta^{1+j}[1+\zeta k_0(K^2-k_3^2)^{-1/2}]^{1/2},$$
$$2c_{j+2}=(-\zeta)^j[1-\zeta k_0(K^2-k_3^2)^{-1/2}]^{1/2};$$
$$\chi_1^{(1)}(t)=2(\lambda)^{1/2}\exp(-\pi\lambda)D_{-\nu-1}(y),\quad \chi_1^{(2)}(t)=\exp(-\pi\lambda)D_\nu(iy).$$
$$\chi_2^{(1)}(t)=\exp(-\pi\lambda+i\pi/4)D_{-\nu}(y),$$
$$\chi_2^{(2)}(t)=-2(\lambda)^{1/2}\exp(-\pi\lambda+i\pi/4)D_{\nu-1}(iy),$$
$$K^2=k_0^2+k^2,\quad k_0=mc\hbar^{-1},\quad 8eE_0\lambda=c\hbar(K^2-k_3^2),$$
$$\nu=-4i\lambda,\quad \beta=(eE_0c\hbar^{-1})^{1/2},\quad y=(i-1)(\beta t-ck_3\beta^{-1}),$$

and $D_\nu(y)$ denotes the parabolic cylinder function.

Let us select complete sets of free solutions of the Dirac equation with constant potentials A_1^μ and A_3^μ in the following convenient form:

$${}_\pm\varphi_{\mathbf{k},\zeta}(\mathbf{r},t)=L^{-3/2}\exp i[\mathbf{kr}\mp p_0(t_1)ct][v_{1\zeta}\chi_1^{(\pm)}(t_1)+v_{2\zeta}\chi_2^{(\pm)}(t_1)],$$
$${}^\pm\varphi_{\mathbf{k},\zeta}(\mathbf{r},t)=L^{-3/2}\exp i[\mathbf{kr}\mp p_0(t_2)ct][v_{1\zeta}\chi_1^{(\pm)}(t_2)+v_{2\zeta}\chi_2^{(\pm)}(t_2)],$$
$$\chi_1^{(\pm)}(t)=\left[\frac{\hbar p_0(t)\mp(\hbar k_3-eE_0t)}{2\hbar p_0(t)}\right]^{1/2},\quad \chi_2^{(\pm)}(t)\pm\left[\frac{\hbar p_0(t)\pm(\hbar k_3-eE_0t)}{2\hbar p_0(t)}\right]^{1/2}$$
$$\hbar p_0(t)=[\hbar^2K^2-\hbar^2k_3^2+(\hbar k_3-eE_0t)^2]^{1/2}. \quad (25)$$

With the aid of Eqs. (24) and (25), one can calculate the matrix elements of the type (15) and find, according to formulas (16) and (23), the total probability of pair production during the time T of the electric field's action. It is convenient to set $t_2 = -t_1 = T/2$.

$$W_{\mathbf{k}\zeta,\mathbf{k}'\zeta'}=\delta_{\mathbf{kk}'}\delta_{\zeta,\zeta'}\frac{p_+(t_2)}{p_-(t_2)}\left|\prod_{\mathbf{p}}\left[\frac{(p_\perp^2+k_0^2)c^2\hbar p_-(t_2)p_0(t_1)p_0(t_2)}{32p_-(t_1)(eE_0)^2\operatorname{sh}[\pi(p_\perp^2+k_0^2)c\hbar/2eE_0]}\right]\right.$$
$$\times[{}_+\Phi_{\mathbf{k}}^{(1)}(t_1){}_+\Phi_{\mathbf{k}}^{(2)}(t_2)-{}_+\Phi_{\mathbf{k}}^{(2)}(t_1){}_+\Phi_{\mathbf{k}}^{(1)}(t_2)]\prod_{\mathbf{p}\neq\mathbf{k}}[{}_+\Phi_{\mathbf{p}}^{(1)}(t_1){}_-\Phi_{\mathbf{p}}^{(2)}(t_2)$$
$$\left.-{}_+\Phi_{\mathbf{p}}^{(2)}(t_1){}_-\Phi_{\mathbf{p}}^{(1)}(t_2)]\right|^2,\quad p_\pm(t)=p_0(t)\pm\left(k_3-\frac{eE_0t}{\hbar}\right),$$
$${}_\pm\Phi_{\mathbf{k}}^{(j)}(t)=\left[1\mp\frac{i}{cp_0(t)}\frac{d}{dt}\right]D_{\nu-1}[(-1)^jiy].$$

We have the following expression asymptotically as $T\to\infty$:

$$W_{\mathbf{k}\zeta,\mathbf{k}'\zeta'}=\delta_{\mathbf{k},\mathbf{k}'}\delta_{\zeta,\zeta'}\frac{1}{2}(\operatorname{cth}4\pi\lambda-1)\exp\left(-\frac{e^2E_0^2L^3T}{4\pi c\hbar^2}\sum_{n=1}^{\infty}\frac{\exp(-\pi E_kn/E_0)}{n^2}\right),$$
$$E_k=m^2c^3(e\hbar)^{-1}.$$

Following Nikishov's terminology[6], one can find the absolute probability of pair production in a given state: $\omega_{\mathbf{k}\zeta} = |G(\mathbf{k}^-\zeta|\mathbf{k}^+\zeta)|^2$. Let us analyze the dependence of this probability on the duration of the field's action. It is sufficient to do this for the probability of pair production in the lowest energy state, i.e., for $\mathbf{k} = 0$. The general formula has the form

$$\omega_{0\zeta}=\left|\frac{c\hbar p_0(T)}{4eE_0}\exp(-6\pi\lambda)[p_-(T)|{}_+\Phi_0^{(2)}(t_1)|^2-p_+(T)|{}_+\Phi_0^{(1)}(t_2)|^2]\right.$$
$$\left.+8\lambda\left(\frac{\pi\hbar c}{2eE_0}\right)^{1/2}\exp(-2\pi\lambda)p_0(T)\left[\operatorname{Re}(1+i)\frac{{}_+\Phi_0^{(1)}(t_2){}_+\Phi_0^{(2)}(t_1)}{\Gamma(1+\nu)}\right]\right|^2.$$

It can be seen that the duration T of the field's action enters into this formula in the dimensionless combination $T(eE_0c/\hbar)^{1/2}$. The formulas for asymptotic expansions of the parabolic cylinder functions[15] become applicable for $T \gg T_0$, where $T_0 = (\hbar/mc^2)(E_k/E_0)^{3/2}$. This gives

$$\omega_{0\zeta}=\exp\left(-\frac{\pi E_k}{E_0}\right)\left[1-13\left(\frac{T_0}{T}\right)^2\frac{E_0}{E_k}+101\left(\frac{T_0}{T}\right)^4\left(\frac{E_0}{E_k}\right)^2+\ldots\right]. \quad (26)$$

As $T \to \infty$, $\omega_{0\zeta} \to \exp(-\pi E_k/E_0)$, which agrees with the result of article[1]. From the cited conditions and from formula (26) it is seen that the probability is automatically stabilized for $T \gg T_0$. This is a specific estimate of the time for pair production. Thus, $T_0 \sim 10^{-9}$ sec if $E_0 \sim 10^8$ V/cm.

A different expansion is valid for $T \ll T_0 (E_0/E_k)^{1/2}$:

$$\omega_{0\zeta} = \left(\frac{T}{T_0}\right)^2 \frac{E_k}{E_0} - \frac{4}{3}\left(\frac{T}{T_0}\right)^4 \left(\frac{E_k}{E_0}\right)^3 + \ldots$$

4. CONCLUSION

The fundamental conclusion, that knowledge of the exact solutions of the Dirac equation enables one to obtain information with regard to the behavior of a second-quantized electron-positron field interacting with an external electromagnetic field, follows from the results of Sec. 2. Expressions are found for the pair-production probabilities and for the probability that the vacuum remains a vacuum in terms of the propagator, which is explicitly expressed in terms of a complete set of solutions. In the assumed formulation of the problem concerning pair production, it turns out to be possible in the calculations to utilize solutions in fields which do not have a special asymptotic behavior. Furthermore, the cited formulas allow one to investigate pair production with arbitrary quantum numbers in the final state.

The treatment under consideration, based on switching the field creating the particles on and off at definite moments of time corresponds, in our opinion, most closely to the possible experimental situation. Here it is also possible to make an analysis of the effect of the duration of the field's action on the corresponding probabilities. Thus, the considered example of the calculation of the probability for pair production by a constant electric field of finite duration, has revealed the existence of a characteristic time T_0 that determines where the probability reaches its asymptotic value, so that this time can be interpreted as the effective time for pair production.

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45