Stationary states of an atom in a quantized electromagnetic field

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(Submitted April 1, 1974)
Zh. Eksp. Teor. Fiz. **67**, 921–925 (September 1974)

The interaction of a two-level atom with a quantized electromagnetic field whose frequency is large in comparison with the separation between the levels is investigated. The energy spectrum and the eigenstates of the system are derived for a field of arbitrary intensity. The transition to the quasiclassical case is considered. The question of oscillations in the intensity of the spontaneous emission as a function of the field amplitude is discussed.

The application of new methods, such as optical pumping with the aid of laser radiation, double resonance, the interference of atomic states in an electric field, and other methods,[1-5] to atomic investigations has determined the successes which have been achieved recently in atomic spectroscopy and has generated interest in the investigation of atomic processes in strong electromagnetic fields. The nonlinear effects arising in this connection require, for their description, going outside the framework of perturbation theory. A review of the fundamental directions of theoretical investigations in the quantum theory of the interaction of electromagnetic fields with atoms is given in article[6] by Stenholm.

For an atom interacting with a field, whose intensity $\mathscr{E}_0$ is small in comparison with the characteristic atomic field $\mathscr{E}_{at} \sim 10^9$ V/cm, it often turns out to be possible to use the model of a two-level system. In this connection the interaction of a specific mode of the field with the two-level system is treated exactly, and the influence of the other states of the atom and modes of the field may be taken into account in perturbation theory.

The change with time of the state vector

$$|\Psi(t)\rangle=\varphi_1|f_1(t)\rangle+\varphi_2|f_2(t)\rangle \tag{1}$$

of the system, consisting of a two-level atom interacting via its dipole moment with a single mode of a quantized electromagnetic field, is determined by the system of equations

$$\begin{aligned} i\frac{\partial}{\partial t}|f_1\rangle&=(H_\omega-\varepsilon)|f_1\rangle+i\frac{\lambda}{2}(c^+-c)|f_2\rangle,\\ i\frac{\partial}{\partial t}|f_2\rangle&=(H_\omega+\varepsilon)|f_2\rangle+i\frac{\lambda}{2}(c^+-c)|f_1\rangle. \end{aligned} \tag{2}$$

Here $\varphi_{1,2}$ and $\mp\epsilon$ denote the wave functions and energy levels of the atom; $H_\omega = (1/2)\omega(cc^+ + c^+c)$ is the Hamiltonian of the field; $|f_{1,2}\rangle$ denotes the state vectors, which depend on the field variables; $\lambda = d(2\omega/V)^{1/2}$ and $d = (\mathbf{d}\cdot\mathbf{e})_{12} = (\mathbf{d}\cdot\mathbf{e})_{21}$ is the dipole matrix element. These equations, and also the corresponding equations for an atom in a classical field, have been investigated by many authors (see, for example, the review[6]). The solution for the case of the one-photon resonance ($\omega \approx 2\epsilon$) is obtained in the article by Jaynes and Cummings[7] in the rotating-phase approximation. Multiphoton resonances were investigated in[8-10]. The solution for a classical field can be expressed in terms of continued fractions.[10] Zaretskiĭ and Kraĭnov[11] calculated the probability for multiquantum excitation in a classical field of low frequency ($\omega \ll \epsilon$). Equations (2) have been solved exactly for degenerate levels ($\epsilon = 0$).[12]

The case when the frequency of the field is large in comparison with the separation between the levels, $\omega \gg \epsilon$, and the parameter characterizing the interaction, $\lambda n^{1/2}/\omega = d\mathscr{E}_0/\omega \gtrless 1$, is investigated in the present article. In this connection, in order that it would be possible to remain within the framework of the two-level approximation, it is necessary that $\omega \ll \omega_{at}$ and $\mathscr{E}_0 \ll \mathscr{E}_{at}$, that is, the question may involve, for example, the interaction of the field with the components of the fine structure.

Equations (2) have the steady-state solutions

$$|f_{1,2}(t)\rangle=e^{-i\Omega t}|g_{1,2}\rangle. \tag{1'}$$

The eigenfrequencies Ω determine the energy levels of the entire system, i.e., the energy levels of the interacting atom and the field. For what follows it is convenient to change to the canonical variables $\hat{p} = p$ and $\hat{q} = id/dp$ according to the usual formulas

$$c=2^{-1/2}(\hat{q}+i\hat{p}),\quad c^+=2^{-1/2}(\hat{q}-i\hat{p})$$

and to the functions $g_\pm = 2^{-1/2}(g_1 \pm g_2)$, which satisfy the system of equations

$$\begin{aligned} &\left[-\frac{1}{2}\frac{d^2}{dp^2}+\frac{1}{2}(p+p_0)^2-\mu-\frac{1}{2}\right]g_+=\frac{\varepsilon}{\omega}g_-,\\ &\left[-\frac{1}{2}\frac{d^2}{dp^2}+\frac{1}{2}(p-p_0)^2-\mu-\frac{1}{2}\right]g_-=\frac{\varepsilon}{\omega}g_+,\\ &p_0=2^{-1/2}\frac{\lambda}{\omega},\quad \mu=\Omega/\omega+{}^1/_4\lambda^2/\omega^2-{}^1/_2. \end{aligned} \tag{3}$$

Let us expand $g_\pm(p)$ in terms of the functions $\psi_k(p \pm p_0)$:

$$g_+(p)=\sum_{k=0}^\infty a_k\psi_k(p+p_0),\quad g_-(p)=\sum_{k=0}^\infty b_k\psi_k(p-p_0), \tag{4}$$

where $\psi_k(x) = (2^k k!\sqrt{\pi})^{-1/2}\exp(-x^2/2)H_k(x)$ are the harmonic oscillator wave functions.

We note that relations (4) represent expansions of the state vectors $|g_\pm\rangle$ in terms of generalized coherent states.[13] In fact, the function $\psi_k(p + p_0)$, for example, is obtained from $\psi_k(p)$ under the action of the displacement operator $\psi_k(p + p_0) = \hat{A}\psi_k(p)$:

$$\hat{A}=\exp\left\{p_0\frac{d}{dp}\right\}=\exp\left\{-i\frac{\lambda}{2\omega}(c^++c)\right\},$$

which, as is well known, generates a generalized coherent state upon operating on a state with a definite number of photons, $|\psi_k\rangle \equiv |k\rangle$.

After substituting relations (4) into Eqs. (3) we obtain a homogeneous system of equations for the determination of the coefficients a_k, b_k and the eigenvalues μ:

$$a_k(k-\mu)=\frac{\varepsilon}{\omega}\sum_{m=0}^\infty J_{km}(p_0)b_m,\quad b_k(k-\mu)=\frac{\varepsilon}{\omega}\sum_{m=0}^\infty a_mJ_{mk}(p_0). \tag{5}$$

The quantities $J_{km}(p_0)$ are expressed in terms of Laguerre polynomials:

$$J_{km}(p_0)=\int \psi_k(p+p_0)\psi_m(p-p_0)\,dp$$
$$=(m!/k!)^{1/2}(2p_0^2)^{(k-m)/2}e^{-p_0^2}L_m^{k-m}(2p_0^2),\quad k\geqslant m,\quad J_{km}(p_0)=(-1)^{k+m}J_{mk}(p_0). \tag{6}$$

If $\epsilon/\omega \ll 1$, Eq. (5) can be solved according to perturbation theory. The eigenvalues μ are close to integer, nonnegative numbers n. For $k \neq n$ the coefficients a_k and b_k can be expressed in terms of a_n and b_n:

$$a_k=-\frac{\varepsilon}{\omega}\frac{J_{kn}(p_0)}{k-n}b_n,\quad b_k=-\frac{\varepsilon}{\omega}\frac{J_{nk}(p_0)}{k-n}a_n,\quad k\neq n, \tag{7}$$

and Eqs. (5) for k = n then determine the relation between a_n and b_n and the eigenvalues μ:

$$\mu_{1,2}=n\mp\frac{\varepsilon}{\omega}|J_{nn}(p_0)|=n\mp\frac{\varepsilon}{\omega}\exp\left(-\frac{\lambda^2}{2\omega^2}\right)\left|L_n\left(\frac{\lambda^2}{\omega^2}\right)\right|. \tag{8}$$

Upon adiabatic switching off of the interaction ($\lambda^2 \to 0$) $L_n(\lambda^2/\omega^2) \to 1$ and the eigenvalues $\mu_{1,2} \to n \mp \epsilon/\omega$ correspond to those states of the system in which the field contains n quanta and the atom is found in the lower (upper) state. If one formally lets $\lambda^2 \to \infty$, then $J_{nn} \to 0$ exponentially and $\mu \to n$, which corresponds to the interaction of the field with a degenerate two-level system. In virtue of the inequality [14] $e^{-x/2}|L_n(x)| \leq 1$ for Laguerre polynomials, the separation between the levels satisfies the inequality $\Delta\mu \leq 2\epsilon/\omega$. Finally, $L_n(\lambda^2/\omega^2) = 0$ at certain values of the field parameters (the polynomial $L_n(x)$ has n positive zeros), as a consequence of which the terms intersect. Near points of intersection it is necessary to take corrections of the next order in ϵ/ω into consideration. One can show that the splitting $\Delta\mu \sim \epsilon^3/\omega^3$ at the "intersection" point itself.

In the quasiclassical case for $n \gg 1$ one can use the asymptotic expression [14] for Laguerre polynomials in terms of a Bessel function

$$e^{-x/2}L_n(x)\approx J_0(2((n+1/2)x)^{1/2}).$$

Then, the eigenvalues μ have the following form for $n \gg 1$:

$$\mu_{1,2}\approx n\mp\frac{\varepsilon}{\omega}\left|J_0\left(2\left(\frac{\lambda^2}{\omega^2}\left(n+\frac{1}{2}\right)\right)^{1/2}\right)\right|. \tag{8'}$$

The argument of the Bessel function $2[\lambda^2(n + 1/2)/\omega^2]^{1/2} = 2d\,\mathscr{E}_0/\omega$, where $\mathscr{E}_0$ is the amplitude of the classical field. Formula (8′) is analogous to the result [2,15] for the renormalization of the atomic g-factor in the presence of a radiofrequency field. For a weak field ($\lambda \to 0$) one can replace the Bessel function by two terms of its series expansion. Then

$$\mu_{1,2}\approx n\mp\frac{\varepsilon}{\omega}\pm\frac{\varepsilon}{\omega}\left(\frac{d\mathscr{E}_0}{\omega}\right)^2,$$

and the last term in this expression represents the usual correction to the energy in second-order perturbation theory (under the condition $\omega \gg \epsilon$).

We present expressions for the normalized functions (4):

$$g_+(p)=\frac{1}{\sqrt{2}}\left\{\psi_n(p+p_0)+\eta\frac{\varepsilon}{\omega}\sum_k{}'\frac{J_{kn}(p_0)}{k-n}\psi_k(p+p_0)\right\},$$
$$g_-(p)=\frac{1}{\sqrt{2}}\left\{\eta\psi_n(p-p_0)+\frac{\varepsilon}{\omega}\sum_k{}'\frac{J_{nk}(p_0)}{k-n}\psi_k(p-p_0)\right\}, \tag{9}$$

where $\eta = \pm\,\mathrm{sign}\, J_{nn}(p_0) = \pm\,\mathrm{sign}\, L_n(\lambda^2/\omega^2)$ and the signs are taken simultaneously with the signs in (8).

In order to change to the occupation number representation, we expand $g_\pm(p)$ in terms of the functions $\psi_s(p)$. The coefficients determine the expansion of the state vectors of the field into states containing a definite number of photons:

$$|g_\pm\rangle=\sum_{s=0}^{\infty}C_s^{(\pm)}|s\rangle,\quad |g_{1,2}\rangle=\sum_{s=0}^{\infty}C_s^{(1,2)}|s\rangle,\quad C_s^{(1,2)}=\frac{(C_s^{(+)}\pm C_s^{(-)})}{\sqrt{2}},$$
$$C_s^{(+)}=(-1)^{s+n}C_s^{(-)}=\frac{1}{\sqrt{2}}\left\{J_{ns}\left(\frac{p_0}{2}\right)+\eta\frac{\varepsilon}{\omega}\sum_k{}'\frac{J_{kn}(p_0)J_{ks}(p_0/2)}{k-n}\right\}. \tag{10}$$

From these expressions, in particular, it follows that, for example, for $\eta = 1$ the coefficient $C_s^{(1)}$ differs from zero for even values of n + s, and $C_s^{(2)}$ differs from zero for odd values.

The number of photons does not have a definite value in the state $|\Psi(t)\rangle$ described by Eq. (1). With Eqs. (1′) and (10) taken into consideration, the average value and the fluctuation in the number of quanta in this state are given by

$$\bar{N}=n+\frac{1}{4}\frac{\lambda^2}{\omega^2}-\eta\frac{\varepsilon}{\omega}\exp\left(-\frac{\lambda^2}{2\omega^2}\right)\times\left\{L_n\left(\frac{\lambda^2}{\omega^2}\right)+nL_{n-1}\left(\frac{\lambda^2}{\omega^2}\right)-(n+1)L_{n+1}\left(\frac{\lambda^2}{\omega^2}\right)\right\},$$
$$\overline{(\Delta N)^2}=\frac{\lambda^2}{2\omega^2}\left(n+\frac{1}{2}\right)-\eta\frac{\varepsilon}{\omega}\frac{\lambda^2}{4\omega^2}\exp\left(-\frac{\lambda^2}{2\omega^2}\right)\times\left\{3L_n\left(\frac{\lambda^2}{\omega^2}\right)-nL_{n-1}\left(\frac{\lambda^2}{\omega^2}\right)+(n+1)L_{n+1}\left(\frac{\lambda^2}{\omega^2}\right)\right\}. \tag{11}$$

These expressions simplify considerably for $n \gg 1$ ($\lambda^2 n/\omega^2$ is bounded):

$$\bar{N}\approx n,\quad \overline{(\Delta N)^2}=\frac{\lambda^2}{2\omega^2}\left(n+\frac{1}{2}\right)=\frac{1}{2}\left(\frac{d\mathscr{E}_0}{\omega}\right)^2, \tag{11'}$$

i.e., in the quasiclassical case the quantum number n coincides with the average number of photons, and the relative fluctuation $\delta \sim d\,\mathscr{E}_0/n\omega$.

The populations of the atomic levels significantly depend on the magnitude of the field. Thus, for example, if prior to switching on the interaction $\lambda^2 \to 0$ the atom is found in the lower level, then in the stationary state (1) the probability of observing the atom in the upper level is given by

$$w_2=\langle g_2|g_2\rangle=\frac{1}{2}\{1-|J_{nn}(p_0)|\}-\frac{\varepsilon}{\omega}\sum_k{}'\frac{J_{nk}^2(p_0)}{k-n}. \tag{12}$$

In the limiting case of a classical field ($n \gg 1$) the second term in (12) turns out to be negligible, and the probability

$$w_2=\frac{1}{2}\left\{1-\left|J_0\left(2\frac{d\mathscr{E}_0}{\omega}\right)\right|\right\} \tag{12'}$$

oscillates as a function of the field intensity.

Let the pair of levels under consideration correspond to the excited state of an atom; the upper level is coupled to the ground state by a dipole transition so that the frequency of this transition, ω_{at}, is considerably greater than the frequency ω of the intense field ($\omega_{at} \gg \omega$). We shall assume that the dipole transition from the lower level to the ground state is forbidden. Then the intensity of the spontaneous emission associated with the transition to the ground state will be proportional to expression (12′) for the probability w_2 of finding the atom in the upper level, i.e., it will oscillate as a function of the intensity of the strong field $\mathscr{E}_0$. The first maximum corresponds to the value of the parameter $d\,\mathscr{E}_0/\omega \approx 1.2$, when the Bessel function vanishes.

In conclusion the author expresses his gratitude to S. P. Goreslavskiĭ for helpful discussions and also thanks the participants in M. I. Ryazanov's seminar for a discussion of the work.

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Translated by H. H. Nickle
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