# Complete integrability and stochastization of discrete dynamical systems

S. V. Manakov

*Institute of Nuclear Physics, Siberian Division, USSR Academy of Sciences*
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We use the inverse scattering method to study a system of particles with exponential interaction (the Toda chain) and a set of equations describing induced scattering of plasma oscillations by ions. We show that a Toda chain with an arbitrary number of particles is completely integrable. We develop a scheme to integrate these systems and study the interaction between solitons. We indicate a class of completely integrable discrete systems, that is, systems which can not be stochastized.

The problem of a possible statistical description of a conservative system leads to the problem of the determination of the stochastization time of that system. The first experimental studies in that direction (by Fermi, Pasta and Ulam [1]) which had as their aim the elucidation of the behavior of a chain of coupled oscillators with a quadratic non-linearity,

$$\ddot{x}_n=(x_{n+1}-x_n)-(x_n-x_{n-1})+{}^1/_2(x_{n+1}-x_n)^2-{}^1/_2(x_n-x_{n-1})^2 \qquad (1)$$

showed already that the stochastization time of that system turned out to be anomalously long, while the motion of the system (1) possesses a strikingly expressed quasi-periodic character over a sufficiently long time. Recently a similar behavior has been observed for some continuous systems (e.g., a system described by the well-known Korteweg-de Vries (KdV) equation [2]). However, in a certain sense the situation turned out to be simpler in the latter case: Zakharov and Faddeev [3] showed that the KdV equation is a completely integrable Hamiltonian system in which stochastization is completely impossible, whereas in subsequent numerical experiments [4] it was, apparently, shown that nevertheless stochastization develops in the system (1), albeit after a very long time.

Zakharov [5] suggested an unexpected explanation of this so strange behavior of the Fermi-Pasta-Ulam chain; his explanation was persuasively based on the hypothesis that the continuum limit of system (1)—the equation of the non-linear string

$$u_{tt}=u_{xx}+(u^2)_{xx}+{}^1/_4u_{xxxx}. \qquad (2)$$

is completely integrable. An immediate consequence of the complete integrability of (2) is the heretofore unexplained fact that the stochastization time of the chain (1) bears no relation whatever to the characteristic time determined by the non-linear term in (1). The stochastization time is thus determined by the "deviation" of the system (1) from its continuum analogue (2), and this was small in typical experiments.

However, the chain (1) demonstrates exactly the same strange behavior even if the initial data for it depend strongly on the number n and thereby do not guarantee that it is possible that it can be replaced by its continuum analog. An even more interesting behavior is shown by a discrete chain with exponential interactions between the particles, introduced by Toda [6]

$$\ddot{x}_n=e^{x_{n+1}-x_n}-e^{x_n-x_{n-1}}, \qquad (3)$$

which, up to terms $\sim(\Delta x)^3$, goes over into (1) for small displacements of the particles. It was noted that the system (3) has a solution of the nature of isolated soliton waves which propagate without a distortion of their form. Moreover, exact solutions of (3) were found [7] which describe collisions of solitons and it was shown that the solitons have practically no interaction with one another—so that as a result of a collision the same solitons are formed as were present before the collision. A completely similar situation occurs in completely integrable continuum systems—kdV, [8,9] in the equations of a non-linear string, [10] and in the non-linear Schrödinger equation, [11] i.e., in systems which can be integrated by means of the inverse scattering method (see, e.g., [12]). In that connection the hypothesis was advanced that the system (3) could also be considered in the framework of that method and is completely integrable, and this is proved in the present paper.

The fact that the Toda chain is integrable shows for an arbitrary chain of non-linear oscillators such as (1) with quadratic and cubic non-linearities that the stochastization time of a chain of oscillators is determined by its deviation from the system (3) and can be very large for well-defined classes of initial conditions.

Another physical important example of an integrable discrete system is the set of equations

$$\dot{N}_k=N_k(N_{k+1}-N_{k-1}). \qquad (4)$$

This set occurs when one studies the fine structure of the spectra of Langmuir oscillations in a plasma. [13] Let us explain its source. It was shown in [14] that for practically any way to excite Langmuir oscillations the stationary spectrum of Langmuir turbulence turns out to be highly anisotropic—the oscillations are concentrated along lines ("jets") in k-space. The transfer of the energy of the Langmuir oscillations along the spectrum occurs due to the induced scattering of the oscillations by ions and is described by the kinetic equation for the number of plasmons:

$$\frac{\partial n_k}{\partial t}=n_k\int T_{kk'}n_{k'}\,dk', \qquad (5)$$

where the kernel $T_{kk'}$ is antisymmetric in its two arguments. By virtue of the above-mentioned jet-like character of the spectrum this equation is one-dimensional. In a plasma with cold ions the main mechanism of stimulated scattering is the excitation of ion sound and the kernel $T_{kk'}$ takes on a δ-function shape: $T_{kk'} = T_0(\delta(k-k'+\kappa)-\delta(k-k'-\kappa))$. Equation (5) then turns into a set of systems like (4). The natural boundary conditions for Eq. (4) are $N_k \to$ constant as $k\to\pm\infty$. The set (4) then describes the propagation of a spectral packet of Langmuir oscillations on the background of thermal noise. Equation (4) also has a soliton-type solution. [13] We show below that this equation can also be studied by using the inverse method. It then turns out that the formalisms of the inverse problems connected with the systems (3) and (4) are in fact the same and this justifies their exposition in the framework of a single paper. These systems are considered everywhere alongside one another.

We shall enumerate the main results of the present paper.

We prove the complete integrability of the system (3) with periodic boundary conditions, i.e., the integrability of the Toda chain on a ring. We indicate a very broad class of completely integrable systems which, in some sense, are generalizations of Eq. (3). We develop for the infinite discrete chains (3) and (4) the inverse-method formalism which enables us to reduce the solution of the Cauchy problem for these equations to the study of some sets of linear equations. We consider the problem of the interaction of solitons in the chains (3) and (4) which, it turns out, can be solved starting from very general considerations even without including the inverse-method equations, similar to what has been done for some continuum systems.[15,16]

We note, finally, that the continuum analogue of the Toda chain (3) is the non-linear string equation (2). On the other hand, the analog of the system (4) is the KdV equation. From the physical point of view this correspondence is trivial. The mathematical nature of these analogies lies much deeper: the operators which are used to integrate the systems considered change in the continuum limit to the appropriate operators of the non-linear string[5] and the KdV[12] equations.

## 1. L-A PAIRS AND COMPLETE INTEGRABILITY

It is well known (see, e.g.,[12]) that the inverse scattering method can be applied to a non-linear equation if it can be written in the form

$$\partial L/\partial t=[L, A], \tag{6}$$

where L and A are a pair of linear operators which can somehow be constructed from the functions occurring in the equation considered.

The following L—A pair of infinite matrices is connected with the system (3):

$$L_{nm}=ic_n^{1/2}\delta_{n,m+1}-ic_m^{1/2}\delta_{n+1,m}+v_n\delta_{n,m}, \tag{7}$$

$$A_{nm}={}^1/_2 i(c_n^{1/2}\delta_{n,m+1}+c_m^{1/2}\delta_{n+1,m}), \tag{8}$$

where the indices n and m run through all integers, $c_n = \exp(x_n - x_{n-1})$, $v_n = \dot{x}_n$, and $\delta_{n,m}$ is the Kronecker symbol.

The system (4) can be written in the form (6) by means of the operators

$$L_{nm}=iN_n^{1/2}\delta_{n,m+1}-iN_m^{1/2}\delta_{n+1,m}, \tag{9}$$

$$A_{nm}=-{}^1/_2\{(N_nN_{n-1})^{1/2}\delta_{n,\,m+2}-(N_mN_{m-1})^{1/2}\delta_{n+2,\,m}\}, \tag{10}$$

as one can check by direct calculations. We note that the operator (9) is a particular case of the operator L (see (7)) and can be obtained from the latter by putting $v_n \equiv 0$. This fact is the one which makes it possible to consider both problems at the same time.

We shall consider the eigenvalue problem of the operator L given by (7):

$$L\psi=\lambda\psi, \tag{11}$$

where $\psi$ is an infinite column of numbers:

$$ic_n\,\psi_{n-1}-ic_{n+1}\psi_{n+1}+v_n\psi_n=\lambda\psi_n. \tag{11a}$$

The fact that L given by (7) is Hermitian and Eq. (6) guarantee the conservation with time of the spectrum of the operator L, i.e., we have for all $\lambda$ that $\dot{\lambda} = 0$ and all eigenvalues of L turn out to be integrals of motion of the system (3). The same is also true for the operator L given by (9).

For the Toda chain (3) consisting of N particles with periodic boundary conditions ($c_{n+N} = c_n$, $v_{n+N} = v_n$) we shall consider the problem (11) also in the space of periodic functions, putting $\psi_{n+N} = \psi_n$. The infinite matrix (7) then reduces to a Hermitian $N \times N$ matrix with N independent eigenvalues. This matrix has the form

$$L=\begin{vmatrix} v_1 & -ic_2^{1/2} & 0 & . & . & . & . & . & 0 & ic_1^{1/2} \\ ic_2^{1/2} & v_2 & -ic_3^{1/2} & 0 & . & . & . & . & 0 & 0 \\ 0 & ic_3^{1/2} & v_3 & -ic_4^{1/2} & 0 & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & 0 & ic_{N-1}^{1/2} & v_{N-1} & -ic_N^{1/2} \\ -ic_1^{1/2} & 0 & . & . & . & . & . & 0 & ic_N^{1/2} & v_N \end{vmatrix}, \tag{12}$$

and its eigenvalues $\lambda_i$ satisfy the characteristic equation

$$\det\|L-\lambda I\|=0, \tag{13}$$

where I is the unit matrix.

We shall now show that all eigenvalues of L are in an involution, i.e., the Poisson bracket of any pair $\lambda_i$, $\lambda_j$ vanishes (the eigenvalues of L "commute" with one another). The system (3) is Hamiltonian. Its Hamiltonian is

$$H=\sum_n\frac{v_n^2}{2}+\exp\{x_{n+1}-x_n\}, \tag{14}$$

and $v_n$ and $x_n$ are canonically conjugate variables. The Poisson bracket of any pair of variables S and T can in the usual manner in those variables be written as:

$$\{S,T\}=\sum_n\frac{\delta S}{\delta x_n}\frac{\delta T}{\delta v_n}-\frac{\delta S}{\delta v_n}\frac{\delta T}{\delta x_n}. \tag{15}$$

In the variables $c_q$, $v_n$ (15) becomes

$$\{S,T\}=\sum_n c_n\left(\frac{\delta S}{\delta c_n}\frac{\delta T}{\delta v_n}-\frac{\delta S}{\delta v_n}\frac{\delta T}{\delta c_n}\right)-c_{n+1}\left(\frac{\delta S}{\delta c_{n+1}}\frac{\delta T}{\delta v_n}-\frac{\delta S}{\delta v_n}\frac{\delta T}{\delta c_{n+1}}\right). \tag{16}$$

We shall now evaluate the variational derivatives $\delta\lambda_i/\delta c_n$ and $\delta\lambda_i/\delta v_n$. To do this we use the well known formula of perturbation theory which determines the change in the eigenvalues of an operator if the latter is slightly changed:

$$\delta\lambda=\langle\psi|\delta L|\psi\rangle. \tag{17}$$

Here $\psi$ is a normalized eigenfunction of L. Using (11a) and (17) we have

$$\begin{gathered}\delta\lambda_j/\delta c_n={}^1/_2 i c_n^{-1/2}(\psi_n^*(j)\psi_{n-1}(j)-\dot{\psi}_{n-1}(j)\psi_n(j)),\\ \delta\lambda_j/\delta v_n=\psi_n^*(j)\psi_n(j),\end{gathered} \tag{18}$$

where $\psi(j)$ is the eigenfunction of L for $\lambda = \lambda_j$.

We note further that for any two eigenfunctions of the operator L the relation

$$\begin{gathered}(\lambda_i-\lambda_j)\psi_n^*(j)\psi_n(i)=W_n(\psi^*(j)\psi(i))-W_{n+1}(\psi^*(j)\psi(i)),\\ W_n(\psi^*(j)\psi(i))=ic_n^{1/2}(\psi_n^*(j)\psi_{n-1}(i)-\dot{\psi}_{n-1}(j)\psi_n(i)).\end{gathered} \tag{19}$$

holds which follows immediately from (11a).

If we now evaluate $\{\lambda_i, \lambda_j\}$ using (16), (18), and (19) we verify easily that

$$\{\lambda_i,\lambda_j\}=\frac{1}{\lambda_i-\lambda_j}\sum_{n=1}^{N}|W_{n+1}(\psi^*(i)\psi(j))|^2-|W_n(\psi^*(i)\psi(j))|^2$$

$$= \frac{1}{\lambda_i - \lambda_j}\{|W_{N+1}(\psi^*(i)\psi(j))|^2 - |W_1(\psi^*(i)\psi(j))|^2\}.$$

When there are periodic boundary conditions imposed upon the system this expression vanishes. We have thus shown that the Hamiltonian system (3) with N degrees of freedom has N integrals of motion which commute with one another and this means by virtue of the well known Liouville theorem (see, e.g., [17]) the complete integrability of the system considered.[1)]

We note further that the Hamiltonian (14) can be expressed in terms of the trace of the square of the matrix L given by (12):

$$H = \frac{1}{2}\operatorname{Tr} L^2 = \frac{1}{2}\sum_{n=1}^{N}\lambda_n^2. \tag{20}$$

According to the above-mentioned Liouville theorem one can choose the quantities $\lambda_i$ as the canonical actions of the system (3). Formula (20) expresses the Hamiltonian (14) in those variables. We now draw attention to the following circumstance: the fact that all eigenvalues of L given by (12) commute is in no ways connected with the actual form of the system (3) but follows directly from the form of the Poisson brackets (16). We can thus assert that any system of the form

$$\dot{x}_n = \delta H/\delta v_n, \quad \dot{v}_n = -\delta H/\delta x_n, \tag{21}$$

is completely integrable, if the Hamiltonian H is some function of the eigenvalues of the matrix (12), or, what amounts to the same, some function of the coefficients of the characteristic equation (13). A particular case of a system of the form (21) with H given by Eq. (20) is the Toda chain. Stochastization is impossible for all dynamical systems of this type.

## 2. THE INVERSE SCATTERING PROBLEM

Let us now turn to a study of the infinite chains (3) and (4). We shall first consider the detailed properties of the operator L given by (7) under the assumption that $c_n \to 1$ and $v_n \to 0$ as $n \to \pm\infty$, and we shall also assume that the sequences $c_n$ and $v_n$ converge sufficiently rapidly to their limits. The operator L given by (7) has then a finite number of discrete eigenvalues and a continuous spectrum occupying the section $-2 \le \lambda \le 2$ of the real axis.

We note first of all that if $\psi_n$ is a solution of (11a) with real $\lambda$, $\tilde{\psi}^n = (-1)^n\psi_n^*$ is also a solution of (11a) with the same $\lambda$. We note also that for any pair of solutions of (11a), $\psi_n^{(1)}$ and $\psi_n^{(2)}$ with the same $\lambda$ the quantity

$$W(\psi^{(1)}, \psi^{(2)}) = (-1)^{n+1}c_n^{1/2}(\psi_n^{(1)}\psi_{n-1}^{(2)} - \psi_{n-1}^{(1)}\psi_n^{(2)}) \tag{22}$$

is independent of the number n. If $W(\psi^{(1)}, \psi^{(2)}) = 0$, the functions $\psi_n^{(1)}$, $\psi_n^{(2)}$ are linearly dependent. This means, in particular, that the discrete spectrum of L with eigenfunctions which tend to zero as $n \to \pm\infty$ is non-degenerate. However, the continuous spectrum of L is, generally speaking, twofold degenerate.

We shall select a special class of eigenfunctions of the operator L—the Jost functions. Let $\psi_n(\xi)$ and $\varphi_n(\xi)$ be solutions of the system (11a) for $\lambda = 2\sin\xi$, determined by the asymptotic behavior

$$\begin{aligned}\psi_n(\xi) &\to e^{i\xi n}, \quad n \to +\infty,\\ \varphi_n(\xi) &\to e^{i\xi n}, \quad n \to -\infty.\end{aligned} \tag{23}$$

One can show that the Jost function $\psi_n(\xi)$ is analytic in the upper half-plane of the complex $\xi$ variable, and $\varphi_n(\xi)$ in the lower half-plane. The function

$$\tilde{\varphi}_n(\xi) = (-1)^n\varphi_n^*(\xi^*) \tag{24}$$

is then analytical in the region Im $\xi > 0$.

The functions $\varphi_n(\xi)$ and $\tilde{\varphi}_n(\xi)$ with $\xi$ real form a complete set of linearly independent solutions of (11a) so that

$$\psi_n(\xi) = \alpha(\xi)\varphi_n(\xi) + \beta(\xi)\tilde{\varphi}_n(\xi). \tag{25}$$

Evaluating $W(\psi(\xi)$ and $\tilde{\psi}(\xi))$ from (22) as $n \to \pm\infty$, we can check that

$$|\alpha(\xi)|^2 - |\beta(\xi)|^2 = 1. \tag{26}$$

Moreover, we see easily that

$$\alpha(\xi) = W(\psi(\xi), \varphi(\xi))/2\cos\xi. \tag{27}$$

We shall now find the time-dependence of $\alpha(\xi)$ and $\beta(\xi)$. Differentiating (11) with respect to time for fixed $\lambda$, we get

$$(L-\lambda)(\partial\psi/\partial t + A\psi) = 0,$$

i.e., $\partial\psi/\partial t + A\psi$ is also an eigenfunction of L with the same $\lambda$ as $\psi$. The requirement that the definition of the Jost function $\psi(\xi)$ given by (23) remains the same with time enables us to find the vector $\partial\psi/\partial t + A\psi$:

$$\partial\psi/\partial t + A\psi = i\cos\xi\psi.$$

Substituting into that expression $\psi(\xi)$ from (25) and taking the limit as $n \to -\infty$ we get

$$\begin{aligned}&\partial\alpha(\xi)/\partial t = 0, \quad \partial\beta(\xi)/\partial t = 2i\cos\xi\beta(\xi),\\ &\beta(\xi, t) = \beta(\xi, 0)e^{2i\cos\xi t}.\end{aligned} \tag{28}$$

The points in the upper $\xi$-half-plane where $\alpha(\xi) = 0$ correspond by virtue of (27), (23), and (24) to the discrete spectrum of the operator L. Since L is Hermitian, i.e., $\lambda = 2\sin\xi$ is real, they lie on the lines Re $\xi = \frac{1}{2}\pi + 2\pi n$ and Re $\xi = -\frac{1}{2}\pi + 2\pi n$, where n is an integer. We note that by virtue of the obvious periodicity of all Jost functions (e.g., $\psi_n(\xi + 2\pi) = \psi_n(\xi)$) we can restrict ourselves to the band[2)] $-\pi \le$ Re $\xi \le \pi$, Im $\xi \ge 0$ in the upper half-plane. In that band the zeroes of $\alpha(\xi)$ can lie only on the lines Re $\xi = \pm\frac{1}{2}\pi$. We denote the zeroes of $\alpha(\xi)$ by $\zeta_k^\pm$: $\zeta_k^\pm = \pm\frac{1}{2}\pi + i\eta_k$, $\eta_k > 0$. By virtue of (27) we have for them[3)]

$$\psi_n(\zeta) = C_\zeta\tilde{\varphi}_n(\zeta). \tag{29}$$

We can also easily find the time-dependence of the $C_\zeta$:

$$C_{\zeta\pm}(t) = C_{\zeta\pm}(0)e^{\pm 2\operatorname{sh}\eta t}. \tag{30}$$

The set of quantities $\zeta_k$, $C_{\zeta_k}$, $\alpha(\xi)$, $\beta(\xi)$ form the "scattering data" and we shall show below that they determine completely the matrix of the operator L given by (7), i.e., $x_n$ and $v_n$, and the scattering data depend simply on the time: $\partial\zeta/\partial t = \partial\alpha/\partial t = 0$, while the time-dependence of $\beta(\xi)$ and $C_\zeta$ is given by (28) and (30). This fact enables us to solve the Cauchy problem for the Toda chain using the classical inverse method:

$$\begin{aligned}x_n(0), v_n(0) &\xrightarrow{L\psi=\lambda\psi} \zeta, C_\zeta(0), \alpha(\xi), \beta(\xi, 0) \xrightarrow{(28),(30)} \zeta, C_\zeta(t), \alpha(\xi), \beta(\xi, t)\\ &\to v_n(t), x_n(t).\end{aligned} \tag{31}$$

In this scheme the first and last stages are non-trivial. In the first stage one solves for the eigenvalues of the operator L, and in the last stage one solves the inverse spectral problem; we shall now turn to that problem.

We note first of all that by virtue of the periodicity of the Jost functions $\psi_n(\xi)$, $\varphi_n(\xi)$ they can be written in the form of Fourier series:

$$\psi_n(\xi)=\sum_m K_{nm}e^{i\xi m}, \quad (32)$$

$$\varphi_n(\xi)=\sum_m A_{nm}e^{i\xi m}. \quad (33)$$

However, the analyticity and boundedness of $\psi_n(\xi)e^{-i\xi n}$ in the upper $\xi$-half-plane means that the series (32) breaks off for $m < n$, i.e., we have for $\psi_n(\xi)$ the triangular representation

$$\psi_n(\xi)=\sum_{m\geq n} K_{nm}e^{i\xi m}. \quad (34a)$$

Completely similarly we have

$$\varphi_n(\xi)=\sum_{m\leq n} A_{nm}e^{i\xi m}. \quad (34b)$$

Substituting (34) into (11a) we get

$$c_n^{1/2}=\frac{K_{nn}}{K_{n-1,n-1}}=\frac{A_{n-1,n-1}}{A_{nn}},$$
$$v_n=i\left(\frac{K_{n,n+1}}{K_{nn}}-\frac{K_{n-1,n}}{K_{n-1,n-1}}\right)=i\left(\frac{A_{n+1,n}}{A_{n+1,n+1}}-\frac{A_{n,n-1}}{A_{nn}}\right) \quad (35)$$

It is now obvious that $K_{nn}$ and $A_{nn}$ are real and positive, and also that $A_{nn} = \gamma/K_{nn}$. The constant $\gamma$ which occurs here can be expressed in terms of $\alpha(i\infty)$.

Substituting

$$\tilde{\varphi}_n(\xi)=(-1)^n\sum_{m\leq n} A_{nm}^* e^{-i\xi m} \quad (36)$$

and (34a) into (27), taking the limit as $\xi \to i\infty$, and using (35) we find that

$$A_{nn}=\alpha(i\infty)/K_{nn}. \quad (37)$$

Dividing now Eq. (25) by $\alpha(\xi)$, multiplying the ensuing equation by $e^{-i\xi m}$, where $m \leq n$, and integrating the result over $\xi$ from $-\pi$ to $\pi$, we get

$$\int_{-\pi}^{\pi}\frac{\psi_n(\xi)e^{-i\xi m}}{\alpha(\xi)}d\xi=\int_{-\pi}^{\pi}\varphi_n(\xi)e^{-i\xi m}d\xi+\int_{-\pi}^{\pi}\frac{\beta(\xi)}{\alpha(\xi)}\tilde{\varphi}_n(\xi)e^{-i\xi m}d\xi. \quad (38)$$

we can write the integral on the left-hand side of (38) in the form

$$\int_\Gamma\frac{\psi_n(\xi)e^{-i\xi m}}{\alpha(\xi)}d\xi+\lim_{\eta\to\infty}\int_{-\pi+i\eta}^{\pi+i\eta}\frac{\psi_n(\xi)e^{-i\xi m}}{\alpha(\xi)}d\xi, \quad (39)$$

where $\Gamma$ is a closed contour consisting of the section $-\pi \leq \xi \leq \pi$ of the real axis, the lines $\mathrm{Im}\,\xi > 0$, $\mathrm{Re}\,\xi = \pm 1/2\pi$, being closed at imaginary infinity. As the integrand $\psi_n(\xi)e^{-i\xi m}/\alpha(\xi)$ is analytical in the domain enclosed by the contour $\Gamma$ except where $\alpha(\xi) = 0$, when $\psi_n(\xi)e^{-i\xi m}/\alpha(\xi)$ has simple poles, this integral is equal to the sum of the residues.

Because of (34a) and (37) the second integral in (39) is

$$\lim_{\eta\to\infty}\int_{-\pi+i\eta}^{\pi+i\eta}\frac{\psi_n(\xi)e^{-i\xi m}}{\alpha(\xi)}d\xi=\frac{\delta_{nm}}{A_{nn}}, \quad m\leq n.$$

The left-hand side of (38) thus has the form

$$2\pi i\sum_k\frac{C_{\zeta_k}\tilde{\varphi}_n(\zeta_k)e^{-i\zeta_k m}}{\alpha'(\zeta_k)}+\frac{\delta_{nm}}{A_{nn}}.$$

If we now substitute into (38) the triangular representations (34b) and (36) and the representation for $\tilde{\varphi}_n(\zeta_k)$ which follows from (36) when $\xi = \zeta_k$, we get a set of equations to determine the kernel $A_{nm}$ of the triangular representation (34b):

$$A_{nm}-\frac{\delta_{nm}}{A_{nn}}+(-1)^nF_{n+m}A_{nn}+(-1)^n\sum_{m_1<n}A_{nm_1}^*F_{m_1+m}=0,$$

where

$$F_n=-i\sum_k\frac{C_{\zeta_k}e^{-i\zeta_k n}}{\alpha'(\zeta_k)}+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\beta(\xi)}{\alpha(\xi)}e^{-i\xi n}d\xi. \quad (40)$$

Putting $A_{nm} = G_{nm}A_{nn}$, $m < n$, we get a set of linear equations for the quantities $G_{nm}$ and an expression for $A_{nn}$:

$$G_{nm}+(-1)^nF_{n+m}+(-1)^n\sum_{m_1<n}G_{nm_1}^*F_{m_1+m}=0, \quad (41)$$

$$A_{nn}^2=\left\{1+(-1)^nF_{2n}+(-1)^n\sum_{m<n}G_{nm}^*F_{m+n}\right\}^{-1}. \quad (42)$$

The matrix of the operator L given by (7) can be expressed in terms of the solution of the set (41) (see (35)):

$$v_n=i(G_{n+1,n}-G_{n,n-1}), \quad c_n=A_{n-1,n-1}^2/A_{nn}^2. \quad (43)$$

The set (41) thus enables us to reconstruct the operator L completely from the scattering data for it, i.e., it is the complete set of linear equations of the inverse problem for the operator L given by (7).

As the operator L given by (9) for the chain (4) is a particular case of the operator L given by (7), all results obtained can immediately be applied also to that case. One must then, however, bear in mind that as the time-dependence of the scattering data is determined by the operator A which for the chain (4) is very different from (8), Eqs. (28) and (30) will in this case also look different. Repeating the corresponding calculations for the operator (10) we get

$$\alpha(\xi,t)=\alpha(\xi,0), \quad \beta(\xi,t)=\beta(\xi,0)e^{2i\sin 2\xi t},$$
$$C_\zeta(t)=C_\zeta(0)e^{2\,\mathrm{sh}\,2\eta t}. \quad (44)$$

Moreover, as the operator (9) is substantially simpler than the operator L given by (7), the scattering data for L given by (9) must also look somewhat simpler. Indeed, considering the problem (11a) with $v_n \equiv 0$, we can convince ourselves that all Jost functions, and at the same time, the quantities $\alpha(\xi)$, $\beta(\xi)$, are periodic with period $\pi$. Moreover, one sees easily that $\alpha(-\xi) = \alpha^*(\xi)$, $\beta(-\xi) = \beta^*(\xi)$. In particular, putting $\zeta = 1/2\pi + i\eta$, we see that if $\alpha(\zeta) = 0$, we have also $\alpha(\zeta - \pi) = 0$, i.e., $\zeta' = 1/2\pi + i\eta$ also corresponds to an eigenvalue of the operator L given by (9) and, furthermore, the corresponding quantities $C_\zeta$ given by (29) are the same in the points $\zeta$ and $\zeta'$: $C_\zeta = C_{\zeta'}$.

All this means that the quantities $F_n$ given by (40) which occur in the equations of the inverse problem are the form

$$F_{2k+1}=0,$$
$$F_{2k}=-2i(-1)^k\sum_m\frac{C_m e^{2\eta_m k}}{\alpha'(\zeta_m)}+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\beta(\xi)}{\alpha(\xi)}e^{-2ik\xi}d\xi, \quad \zeta_m=\pi/2+i\eta_m. \quad (45)$$

Equation (41) then shows that $G_{nm} = 0$, if $n + m$ is odd: this means in particular that $G_{n,n-1} = 0$; this guarantees us that the quantities $v_n$ vanish (see (43)). Denoting $G_{n,n-2k}$ by $M_{nk}$ we get for them from (41) and (42)

$$M_{nk}+(-1)^nF_{2(n-k)}+(-1)^n\sum_{k'>0}M_{nk'}F_{2(n-k-k')}=0, \quad (46)$$

$$A_{nn}^2=\left\{1+(-1)^nF_{2n}+(-1)^n\sum_{k>0}M_{nk}F_{2(n-k)}\right\}^{-1}. \quad (47)$$

The solution $N_k(t)$ of the set (4) is again given by (43):

$$N_k=A_{k-1,k-1}/A_{kk}^2. \quad (48)$$

## 3. SOLITON COLLISIONS

As in the case of other systems which can be solved using the inverse scattering method, the set (41) can be solved, if $\beta(\xi) \equiv 0$. Such solutions are completely determined by giving the N zeroes of $\alpha(\xi)$ and the corresponding quantities C and they describe the N-soliton solutions of the system (3).

We shall consider the simplest situation when $\alpha(\xi)$ has only a single zero which lies, e.g., on the line Re $\xi = -\frac{1}{2}\pi (\xi = -\frac{1}{2}\pi + i\eta)$. The condition that $\alpha(\xi)$ be periodic and Eq. (26) which in the case considered means $|\alpha(\xi)|^2 = 1$, enables us to reconstruct $\alpha(\xi)$ for real $\xi$:

$$\alpha(\xi)=\sin\frac{\xi+\pi/2-i\eta}{2} \Big/ \sin\frac{\xi+\pi/2+i\eta}{2}. \tag{49}$$

The function $F_n$ determined, using (40), then has the form

$$F_n=2C \operatorname{sh}\eta e^{i\pi n/2}e^{\eta n}.$$

We look for the solution of (41) in the form

$$G_{nm}=G_n e^{i\pi m/2}e^{\eta m}.$$

After a simple calculation we find

$$G_n^*=-\gamma e^{i\pi n/2}e^{\eta n} \Big/ \left(1+\gamma\frac{e^{2\eta n}}{e^{2\eta}-1}\right), \quad \gamma=2C\operatorname{sh}\eta,$$

whence we get at once

$$\frac{1}{A_{nn}^2}=1+\gamma e^{2\eta n} \Big/ \left(1+\frac{e^{2\eta n}}{e^{2\eta}-1}\right).$$

Using now (43) and the fact that $c_n = \exp(x_n - x_{n-1})$ we find finally

$$x_n=\ln\frac{\operatorname{ch}\eta(n-x_0+1/2)}{\operatorname{ch}\eta(n-x_0-1/2)}+\text{const}, \tag{50}$$

$$v_n=\frac{\operatorname{sh}^2\eta}{\operatorname{ch}\eta(n-x_0+1/2)\operatorname{ch}\eta(n-x_0-1/2)}, \tag{51}$$

$$x_0=\frac{1}{2\eta}\ln\frac{1}{C},$$

where $x_0$ is the coordinate of the center of the soliton. If we now substitute in (51) the time-dependence of C, we get

$$x_0(t)=x_0(0)+\frac{\operatorname{sh}\eta}{\eta}t,$$

whence it follows that (50) describes a solution of (3) which propagates along the Toda chain without distortion of its form with a velocity $v = \eta^{-1}\sinh\eta$, i.e., we obtain a soliton. Similar calculations show that the zero of $\alpha(\xi)$ on the line Re $\xi = \pi/2$ corresponds to a soliton moving in the opposite direction.

In the more general case when $\alpha(\xi)$ has N zeroes in the band $-\pi <$ Re $\xi < \pi$ in the upper $\xi$-half-plane, of which $N_+$ lie on the line Re $\xi = \pi/2$ and $N_-$ on the line Re $\xi = -\pi/2$, one can show that the corresponding solution of the system (3) is asymptotically, as $t \to \pm\infty$, a set of solitons of which $N_-$ have positive velocities and $N_+$ negative velocities, i.e., they describe N-soliton collisions. One can also check that the amplitudes (and hence also the velocities) of the solitons occurring as $t \to +\infty$ are exactly the same as the same quantities for the case as $t \to -\infty$, which is completely obvious from the point of view of the method considered here as these soliton parameters are determined by the eigenvalues of the operator L which are conserved in time. The whole effect of the soliton collisions reduces thus to a change in the quantities $x_0(t)$ in (51) which refer to some time, say, t = 0. We shall now give a simple method to calculate the change in these quantities.

To fix our ideas we shall consider the collision of two solitons moving in the positive direction. Such a solution corresponds to two zeroes of $\alpha(\xi)$ on the line Re $\xi = -\pi/2$ in the points $\zeta_1 = -\pi/2 + i\eta_1$, $\zeta_2 = -\pi/2 + i\eta_2$. We put $\eta_2 > \eta_1$; as the velocity of the second soliton is larger than that of the first one, the first soliton will then be to the right of the second one as $t \to -\infty$; as $t \to +\infty$ the arrangement is the opposite one. We shall consider the behavior of the eigenfunction $\psi_n(\zeta_2)$ of the operator L in these cases. We denote the quantities $x_0(t)$ for the first and the second soliton by $x_0(1)$, $x_0(2)$. As $t \to -\infty$ we have $x_0^-(2) \ll x_0^-(1)$. In the region $n \gg x_0^-(1)$ the function $\psi_n(\zeta_2)$ has the form $\psi_n(\zeta_2) \approx \exp(i\zeta_2 n)$. When the soliton 1 has passed through, the function $\psi_n(\zeta_2)$ changes by virtue of (25) into $\alpha_1(\zeta_2)\exp(i\zeta_2 n)$, where, because of (49), $\alpha_1(\zeta_2)$ is

$$\alpha_1(\zeta_2)=\operatorname{sh}\frac{\eta_2-\eta_1}{2} \Big/ \operatorname{sh}\frac{\eta_2+\eta_1}{2}.$$

For the second soliton the function $\exp(i\zeta_2 n)$ is the asymptotic form of its eigenfunction to the right of it. In the region $n \ll x_0^-(2)$ the function $\psi_n(\zeta_2)$ (see (29)) is thus equal to

$$\psi_n(\zeta_2)=C_2^-\tilde{\varphi}_n(\zeta_2)\operatorname{sh}\frac{\eta_2-\eta_1}{2} \Big/ \operatorname{sh}\frac{\eta_2+\eta_1}{2}, \tag{52a}$$

where $C_2^-$ is connected with $x_0^-(2)$ through (51).

Let now t tend to $+\infty$. In that case $x_0^+(1) \ll x_0^+(2)$. When $n \ll x_0^+(1)$ the function $\psi_n(\zeta_2)$ is by virtue of (29)

$$\psi_n(\zeta_2)=C_{\zeta_2}\tilde{\varphi}_n(\zeta_2) \tag{52b}$$

(this refers also to the case as $t \to -\infty$). Taking the complex conjugate of (25) and solving the resulting set for $\tilde{\varphi}_n(\xi)$ we find

$$\tilde{\varphi}_n(\xi)=\alpha(\xi)\tilde{\psi}_n(\xi)-\beta^*(\xi)\psi_n(\xi).$$

Using that relation and (52b) we get an expression for $\psi_n(\zeta_2)$ in the region $x_0^+(1) \ll n \ll x_0^+(2)$:

$$\psi_n(\zeta_2)=C_{\zeta_2}\alpha_1(\zeta_2)(-1)^n e^{-i\zeta_2 n}.$$

However, $(-1)^n\exp(-i\zeta_2 n)$ is the "left-hand" asymptotic form of the eigenfunction of the second soliton. We must thus have for $n \gg x_0^+(2)$

$$\psi_n(\zeta_2)=C_{\zeta_2}\alpha_1(\zeta_2)\psi_n(\zeta_2)/C_2^+, \tag{52c}$$

where $C_2^+$ is also connected with $x_0^+(2)$ through (51). It thus follows from (52c) that

$$C_2^+=C_{\zeta_2}\alpha_1(\zeta_2). \tag{52d}$$

Comparing Eqs. (52a) and (52b) we get

$$C_2^-=C_{\zeta_2}/\alpha_1(\zeta_2), \tag{52e}$$

whence, after eliminating the time-dependence from Eqs. (52d) and (52e), we get

$$C_2^+/C_2^-=\alpha_1^2(\zeta_2), \tag{53}$$

which, because of (51), means that

$$\Delta x_0(2)=x_0^+(2)-x_0^-(2)=\frac{1}{2\eta_2}\ln\frac{\operatorname{sh}^2 1/2(\eta_1+\eta_2)}{\operatorname{sh}^2 1/2(\eta_1-\eta_2)}. \tag{54a}$$

Similarly we have for the slow soliton

$$\Delta x_0(1)=-\frac{1}{2\eta_1}\ln\frac{\operatorname{sh}^2 1/2(\eta_1+\eta_2)}{\operatorname{sh}^2 1/2(\eta_1-\eta_2)}. \tag{54b}$$

If, on the other hand, one of the solitons moves in the opposite direction (let this be the second soliton, to fix the ideas), it follows immediately from (53) and (51) that

$$\Delta x_0(1)=\frac{1}{2\eta_1}\ln\frac{\operatorname{ch}^2 1/2(\eta_1+\eta_2)}{\operatorname{ch}^2 1/2(\eta_1-\eta_2)}$$
$$\Delta x_0(2)=-\frac{1}{2\eta_2}\ln\frac{\operatorname{ch}^2 1/2(\eta_1+\eta_2)}{\operatorname{ch}^2 1/2(\eta_1-\eta_2)} \tag{55}$$

Equations (54) and (55) give the complete solution of the problem of the collisions of solitons in the Toda chain.

We shall apply the procedure expounded here also to the general case of an N-solution collision. It then turns out that only pair collisions occur, i.e., the displacement of any soliton equals the sum of the displacements arising when this soliton collides with all the other ones separately.

Let us now study solitons in the chain (4). Assuming, as before, that $\beta(\xi) \equiv 0$ and that $\alpha(\xi)$ has only one pair of zeroes in the band $-\pi \leq \xi \leq \pi$ on the lines $\mathrm{Re}\,\xi = \pm\pi/2$ with $\mathrm{Im}\,\xi = \eta$, we can find $\alpha(\xi)$:

$$\alpha(\xi)=\frac{\sin(\xi-\pi/2-i\eta)}{\sin(\xi-\pi/2+i\eta)}. \qquad (56)$$

In that case we have from (45)

$$F_{2k}=2C\,\mathrm{sh}\,2\eta(-1)^k e^{2\eta k}.$$

One can easily solve the set (46) with these F:

$$M_{nk}=-\frac{\gamma e^{2\eta n}}{1+\gamma e^{2\eta n}/(e^{4\eta}-1)}(-1)^k e^{-2\eta k}, \qquad \gamma=2C\,\mathrm{sh}\,2\eta.$$

If we then determine $A_{nn}$ from (47) and substitute it in (48) we get

$$N_n=\frac{\mathrm{ch}\,\eta(n-x_0-2)\,\mathrm{ch}\,\eta(n-x_0+1)}{\mathrm{ch}\,\eta(n-x_0-1)\,\mathrm{ch}\,\eta(n-x_0)}, \qquad (57)$$

$$x_0=\frac{1}{2\eta}\ln\frac{1}{C}, \qquad (58)$$

where $x_0$ is the coordinate of the soliton center.

Substituting in (58) the time-dependence of C from (44), we find

$$x_0(t)=x_0(0)-\frac{\mathrm{sh}\,2\eta}{\eta}t. \qquad (59)$$

Hence it follows that all solitons in the chain (4) move in the same direction.

The picture of the scattering of solitons in this chain is completely similar to that described above, i.e., only the coordinates of the soliton centers are changed as the result of collisions; that change can be found immediately from Eq. (53); using (56) we get

$$\Delta x_0(1)=\frac{1}{2\eta_1}\ln\frac{\mathrm{sh}^2(\eta_1+\eta_2)}{\mathrm{sh}^2(\eta_1-\eta_2)},$$

$$\Delta x_0(2)=-\frac{1}{2\eta_2}\ln\frac{\mathrm{sh}^2(\eta_1+\eta_2)}{\mathrm{sh}^2(\eta_1-\eta_2)}, \qquad \eta_2>\eta_1.$$

The quantities $\eta_1$ and $\eta_2$ are connected with the soliton velocities through (59).

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[1)]The existence of the necessary number of independent integrals for the system (3) was proved by Henon (private communication) using a completely different technique.

[2)]We note by the way that it follows immediately from (11a), (23), and (25) that, e.g., $\psi_n(\pm\frac{1}{2}\pi+\xi)=\tilde{\psi}_n(\pm\frac{1}{2}\pi-\xi)$, i.e., $\alpha(\pm\frac{1}{2}\pi+\xi)=\alpha^*(\pm\frac{1}{2}\pi-\xi)$, $\beta(\pm\frac{1}{2}\pi+\xi)=\beta^*(\pm\frac{1}{2}\pi-\xi)$.

[3)]We note that the quantities $C_\zeta$ occurring in (29) are real.


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