The theory of the solution of two near-ideal Bose gases

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(Submitted February 4, 1974)
Zh. Eksp. Teor. Fiz. 67, 263–272 (July 1974)

A solution of two near-ideal Bose gases at a temperature close to absolute zero is considered. It is found that two independent superfluid motions of the components can exist with different velocities in such a solution. It is shown that two Goldstone elementary-excitation branches with a linear dispersion law at small momenta exist in the solution. It is established that in the dilute-solution limit the sound velocities corresponding to these branches coincide with the first- and third-sound velocities defined in the hydrodynamics of solutions of two superfluid liquids. Also considered is the problem of the velocity dependence of the elementary-excitation energies in the presence in the solution of two independent motions, and the Landau condition for such a solution to be superfluid is found.

The discovery[1] that liquid He^3 undergoes a phase transition at a temperature of a few millikelvin has placed a second superfluid liquid at the disposal of experimenters. It is also well known that the solution of liquid He^3 in liquid He^4 does not, at He^3 concentrations of up to 6%, stratify into pure components even at T = 0°K[2]. This allows us to hope that a similar phase transition of He^3 into the superfluid state will be discovered in solutions of He^3 in superfluid He^4, i.e., that it will be possible to produce a solution of the two superfluid liquids.

In[3,4] Khalatnikov derives, from phenomenological arguments, hydrodynamic equations for such solutions and investigates the problem of sound propagation in them (see also[5]). In the present paper we study the microscopic model problem of the solution of two superfluid, near-ideal Bose gases. Let us note at once, however, that there apparently does not exist on this planet a solution of two Bose liquids such that both components do not solidify before the λ transition. The only other candidate besides He^4, namely He^6, is radioactive (half-life 0.8 sec). But besides possible astrophysical applications, such a simple microscopic model is of interest, since it may help us understand the general laws typical of solutions of two superfluid liquids. For example, the presence of two Bose branches in the spectrum of the elementary excitations is typical of both Bose + Bose and Fermi + Bose superfluid mixtures.

In the first section of the present paper we study the problems connected with the energy spectrum of a solution of near-ideal Bose gases. In the second section we consider a solution in which the components move with independent velocities. This turns out to be possible owing to the invariance of the Hamiltonian of the system under two independent groups of gauge transformations of the first kind. In other words, on account of the conservation of the number of particles of each of the components of the solution taken separately, we can introduce two independent phases of two superfluid motions and, consequently, two superfluid velocities. Also, in this section, we establish the Landau condition for the existence of superfluidity in the solution under consideration.

1. A STATIONARY SOLUTION OF TWO NEAR-IDEAL BOSE GASES

The Hamiltonian of a mixture of two Bose gases in the case of point interaction between the particles has, in second-quantization representation, the following form:

$$\hat{H}=\sum_{\mathbf{p}}\frac{\mathbf{p}^2}{2m_1}a_{\mathbf{p}}^{+}a_{\mathbf{p}}+\sum_{\mathbf{p}}\frac{\mathbf{p}^2}{2m_2}b_{\mathbf{p}}^{+}b_{\mathbf{p}}+\frac{g_1}{2V}\sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4}a_{\mathbf{p}_1}^{+}a_{\mathbf{p}_2}^{+}a_{\mathbf{p}_3}a_{\mathbf{p}_4}$$
$$+\frac{g_2}{2V}\sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4}b_{\mathbf{p}_1}^{+}b_{\mathbf{p}_2}^{+}b_{\mathbf{p}_3}b_{\mathbf{p}_4}+\frac{g_3}{V}\sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4}a_{\mathbf{p}_1}^{+}b_{\mathbf{p}_2}^{+}b_{\mathbf{p}_3}a_{\mathbf{p}_4}. \qquad (1.1)$$

Here $a_{\mathbf{p}}^{+}$, $a_{\mathbf{p}}$, $b_{\mathbf{p}}^{+}$, and $b_{\mathbf{p}}$ are the particle creation and annihilation operators for the first and second components of the solution respectively, the operators satisfying the standard Bose commutation relations:

$$a_{\mathbf{p}}a_{\mathbf{p}'}-a_{\mathbf{p}'}a_{\mathbf{p}}=0,\quad a_{\mathbf{p}}a_{\mathbf{p}'}^{+}-a_{\mathbf{p}'}^{+}a_{\mathbf{p}}=\delta_{\mathbf{pp}'},$$
$$b_{\mathbf{p}}b_{\mathbf{p}'}-b_{\mathbf{p}'}b_{\mathbf{p}}=0,\quad b_{\mathbf{p}}b_{\mathbf{p}'}^{+}-b_{\mathbf{p}'}^{+}b_{\mathbf{p}}=\delta_{\mathbf{pp}'},$$
$$a_{\mathbf{p}}b_{\mathbf{p}'}-b_{\mathbf{p}'}a_{\mathbf{p}}=0,\quad a_{\mathbf{p}}b_{\mathbf{p}'}^{+}-b_{\mathbf{p}'}^{+}a_{\mathbf{p}}=0;$$

m_1 and m_2 are the masses of the particles of the components; g_1, g_2, and g_3 are respectively the constants of the interaction between the particles of the first component, between the particles of the second component, and between the particles of the first and second components. The summation in the last three sums in (1.1) is carried out under the condition that the momentum-conservation law, $\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_3+\mathbf{p}_4$, is fulfilled. As usual, it is assumed that the conditions for "near ideality" of the gas

$$\frac{m_1g_1}{4\pi\hbar^2}\left(\frac{N_1}{V}\right)^{1/3}\ll 1,\quad \frac{m_2g_2}{4\pi\hbar^2}\left(\frac{N_2}{V}\right)^{1/3}\ll 1,\quad \frac{m_1m_2g_3}{2\pi\hbar^2(m_1+m_2)}\left(\frac{N}{V}\right)^{1/3}\ll 1;$$

where N_1 and N_2 are the numbers of particles of the first and second components, $N=N_1+N_2$ is the total number of particles, and V is the volume of the system, are satisfied.

It should be noted with respect to the Hamiltonian (1.1) that it is invariant under two independent gauge transformations of the first kind: $a_{\mathbf{k}}\rightarrow a_{\mathbf{k}}e^{i\varphi_1}$, $b_{\mathbf{k}}\rightarrow b_{\mathbf{k}}e^{i\varphi_2}$, i.e., it commutes with the total particle number operator of each of the components

$$\hat{N}_1=\sum_{\mathbf{p}}a_{\mathbf{p}}^{+}a_{\mathbf{p}},\quad \hat{N}_2=\sum_{\mathbf{p}}b_{\mathbf{p}}^{+}b_{\mathbf{p}},$$

which allows us to introduce two independent condensate "wave functions": $a_0\sim N_{10}^{1/2}e^{i\varphi_1}$ and $b_0\sim N_{20}^{1/2}e^{i\varphi_2}$. For a moving solution the presence of the two independent phases φ_1 and φ_2 enables the two components of the solution to flow with different superfluid velocities (see the following section). We assume that φ_1 and φ_2 are equal to zero in the case of a stationary solution.

At temperatures close to absolute zero we can, following Bogolyubov[6], expand (1.1) in powers of the small quantities $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ ($p\neq 0$). We obtain

$$\hat{H}=\frac{N_1^2g_1}{2V}+\frac{N_2^2g_2}{2V}+\frac{N_1N_2g_3}{V}+\sum_{\mathbf{p}}\frac{\mathbf{p}^2}{2m_1}a_{\mathbf{p}}^{+}a_1$$

$$+\sum_{\mathbf{p}}\frac{\mathbf{p}^2}{2m_2}b_{\mathbf{p}}^+b_{\mathbf{p}}+\frac{N_1g_1}{2V}\sum_{\mathbf{p}\neq0}(a_{\mathbf{p}}a_{-\mathbf{p}}+a_{\mathbf{p}}^+a_{-\mathbf{p}}^++2a_{\mathbf{p}}^+a_{\mathbf{p}})$$

$$+\frac{N_2g_2}{2V}\sum_{\mathbf{p}\neq0}(b_{\mathbf{p}}b_{-\mathbf{p}}+b_{\mathbf{p}}^+b_{-\mathbf{p}}^++2b_{\mathbf{p}}^+b_{\mathbf{p}}) \tag{1.2}$$

$$+\frac{\sqrt{N_1N_2}\,g_3}{V}\sum_{\mathbf{p}\neq0}(b_{\mathbf{p}}a_{-\mathbf{p}}+a_{\mathbf{p}}^+b_{-\mathbf{p}}^++a_{\mathbf{p}}^+b_{\mathbf{p}}+b_{\mathbf{p}}^+a_{\mathbf{p}}).$$

In deriving (1.2), we used the relations

$$N_1=a_0^2+\sum_{\mathbf{p}\neq0}a_{\mathbf{p}}^+a_{\mathbf{p}},\qquad N_2=b_0^2+\sum_{\mathbf{p}\neq0}b_{\mathbf{p}}^+b_{\mathbf{p}}.$$

In order to diagonalize (1.2), we carry out the canonical transformation (see, for example,[7]):

$$a_{\mathbf{p}}=\sum_{\nu=1,2}u_{1\nu}(\mathbf{p})\xi_{\mathbf{p}\nu}+v_{1\nu}^*(\mathbf{p})\xi_{-\mathbf{p}\nu}^+, \tag{1.3}$$

$$b_{\mathbf{p}}=\sum_{\nu=1,2}u_{2\nu}(\mathbf{p})\xi_{\mathbf{p}\nu}+v_{2\nu}^*(\mathbf{p})\xi_{-\mathbf{p}\nu}^+.$$

Here $\xi_{\mathbf{p}\nu}$ and $\xi_{\mathbf{p}\nu}^+$ are the new Bose operators and the functions $u_{\alpha\nu}(\mathbf{p})$ and $v_{\alpha\nu}(\mathbf{p})$ satisfy the relations

$$\sum_{\alpha}[u_{\alpha\mu}(\mathbf{p})u_{\alpha\nu}^*(\mathbf{p})-v_{\alpha\mu}(\mathbf{p})v_{\alpha\nu}^*(\mathbf{p})]=\delta_{\mu\nu}, \tag{1.4a}$$

$$\sum_{\alpha}[u_{\alpha\mu}(\mathbf{p})v_{\alpha}\ (\mathbf{p})-u_{\alpha\nu}(\mathbf{p})v_{\alpha\mu}\ (\mathbf{p})]=0,$$

as well as the relations

$$\sum_{\nu}[u_{\alpha\nu}(\mathbf{p})u_{\beta\nu}^*(\mathbf{p})-v_{\beta\nu}(\mathbf{p})v_{\alpha\nu}^*(\mathbf{p})]=\delta_{\alpha\beta}, \tag{1.4b}$$

$$\sum_{\nu}[u_{\beta\nu}(\mathbf{p})v_{\alpha\nu}^*(\mathbf{p})-u_{\alpha\nu}(\mathbf{p})v_{\beta\nu}^*(\mathbf{p})]=0.$$

The Hamiltonian (1.2) is transformed into the form

$$\hat{H}=E_0+\sum_{\mathbf{p}\neq0}\sum_{\nu=1,2}E_\nu(\mathbf{p})\xi_{\mathbf{p}\nu}^+\xi_{\mathbf{p}\nu}. \tag{1.5}$$

Dropping in the expression for E_0 the terms of higher order than the first in the interaction constants, we can write

$$E_0=\frac{N_1^2g_1}{2V}+\frac{N_2^2g_2}{2V}+\frac{N_1N_2g_3}{V}. \tag{1.6}$$

The functions $u_{\alpha\nu}(\mathbf{p})$ and $v_{\alpha\nu}(\mathbf{p})$ and the energies $E_\nu(\mathbf{p})$ are determined from the system of equations

$$\sum_{\beta}\{[S_{\alpha\beta}(\mathbf{p})-E_\nu(\mathbf{p})\delta_{\alpha\beta}]u_{\beta\nu}(\mathbf{p})+R_{\alpha\beta}v_{\beta\nu}(\mathbf{p})\}=0, \tag{1.7}$$

$$\sum_{\beta}\{R_{\alpha\beta}u_{\beta\nu}(\mathbf{p})+[S_{\alpha\beta}(\mathbf{p})+E_\nu(\mathbf{p})\delta_{\alpha\beta}]v_{\beta\nu}(\mathbf{p})\}=0,$$

where

$$S_{\alpha\beta}(\mathbf{p})=\begin{pmatrix}\dfrac{\mathbf{p}^2}{2m_1}+\dfrac{N_1g_1}{2V} & \dfrac{\sqrt{N_1N_2}\,g_3}{V}\\ \dfrac{\sqrt{N_1N_2}\,g_3}{V} & \dfrac{\mathbf{p}^2}{2m_2}+\dfrac{N_2g_2}{2V}\end{pmatrix}$$

$$R_{\alpha\beta}=\begin{pmatrix}\dfrac{N_1g_1}{V} & \dfrac{\sqrt{N_1N_2}\,g_3}{V}\\ \dfrac{\sqrt{N_1N_2}\,g_3}{V} & \dfrac{N_2g_2}{V}\end{pmatrix}. \tag{1.8}$$

From the condition that the determinant of the system (1.7) should be equal to zero, we obtain the elementary-excitation energies$^{1)}$

$$E_{1,2}(\mathbf{p})=p\left\{\frac{1}{2}\left(\frac{\mathbf{p}^2}{4m_1^2}+\frac{\mathbf{p}^2}{4m_2^2}+\frac{N_1g_1}{m_1V}+\frac{N_2g_2}{m_2V}\right)\right.$$

$$\left.\pm\left[\frac{1}{4}\left(\frac{\mathbf{p}^2}{4m_1^2}-\frac{\mathbf{p}^2}{4m_2^2}+\frac{N_1g_1}{m_1V}-\frac{N_2g_2}{m_2V}\right)^2+\frac{g_3^2N_1N_2}{m_1m_2V^2}\right]^{1/2}\right\}^{1/2}. \tag{1.9}$$

which reduce at low momenta to

$$E_{1,2}(\mathbf{p})=c_{1,2}p, \tag{1.10}$$

$$c_{1,2}=\left\{\frac{1}{2}\left(\frac{N_1g_1}{m_1V}+\frac{N_2g_2}{m_2V}\right)\pm\left[\frac{1}{4}\left(\frac{N_1g_1}{m_1V}-\frac{N_2g_2}{m_2V}\right)^2+\frac{N_1N_2g_3^2}{m_1m_2V^2}\right]^{1/2}\right\}^{1/2}. \tag{1.11}$$

Thus, the spectrum of the single-particle excitations has two branches with a linear dispersion law at low momenta (as it should be—in accordance with the Goldstone theorem). In order to ascertain the physical meaning of these branches, it is necessary to find the physical quantities to whose oscillations they correspond.

Let us introduce the concentration c and the density ρ through the formulas

$$\rho c=\frac{m_1N_1}{V},\qquad \rho(1-c)=\frac{m_2N_2}{V},\qquad \rho=\frac{m_1N_1+m_2N_2}{V}. \tag{1.12}$$

Then we have in the lowest nonvanishing order in the concentration (we assume here that $c \ll 1$) the expressions

$$c_1^2=\frac{\rho g_2}{m_2^2}, \tag{1.13a}$$

$$c_2^2=\frac{c\rho}{m_1^2g_2}(g_1g_2-g_3^2); \tag{1.13b}$$

c_1 coincides with the velocity of first sound in a mixture of two superfluid liquids (see[4]) and $u_1^2 = (\partial P/\partial\rho)_c$ if u_1^2 is computed in the zeroth approximation in the concentration$^{2)}$. Indeed, we have

$$u_1^2=\left(\frac{\partial P}{\partial\rho}\right)_c=-\frac{\partial}{\partial\rho}\left(\frac{\partial E_0}{\partial V}\right)\approx\frac{\rho g_2}{m_2^2}. \tag{1.14}$$

The quantity c_2 coincides with the velocity of third sound in a mixture of two superfluid liquids obtained in[4]. To first order in the concentration (see[4]), we have

$$u_3^2=c\left(\frac{\partial(\mu_1/m_1-\mu_2/m_2)}{\partial c}\right)_P. \tag{1.15}$$

Here μ_1 and μ_2 are the chemical potentials of the first and second components of the mixture:

$$\mu_1=\frac{\partial E_0}{\partial N_1}=\rho\left[\frac{cg_1}{m_1}+\frac{(1-c)g_3}{m_2}\right], \tag{1.16a}$$

$$\mu_2=\frac{\partial E_0}{\partial N_2}=\rho\left[\frac{(1-c)g_2}{m_2}+\frac{cg_3}{m_1}\right], \tag{1.16b}$$

ρ being given by the formula

$$P=-\frac{\partial E_0}{\partial V}=\rho^2\left[\frac{g_1c^2}{2m_1^2}+g_2\frac{(1-c)^2}{2m_2^2}+g_3\frac{c(1-c)}{m_1m_2}\right]. \tag{1.17}$$

Thus, using (1.16a), (1.16b), and (1.17), we find from the formula (1.15) to first order in the concentration that

$$u_3^2=\frac{c\rho}{m_1^2g_2}(g_1g_2-g_3^2), \tag{1.18}$$

which coincides with (1.13b). The velocities u_1 and u_3 found in[4] are the velocities of propagation of coupled oscillations in the pressure and concentration of a solution of two superfluid liquids. It is interesting that the purely imaginary diffusion branch of the spectrum of a solution of ordinary liquids is converted in a solution of two superfluid liquids into a real branch of the concentration fluctuations.

From (1.9) (see also (1.13a) and (1.13b) it is easy to derive the conditions for realness of the elementary-excitation energies:

$$g_2>0,\quad g_1g_2>g_3^2. \tag{1.19}$$

These conditions are sufficient for the fulfillment of the thermodynamic inequality $(\partial P/\partial V) < 0$, which can easily be verified with the aid of the formulas (1.17) and (1.12). The second condition in (1.19) also coincides with the condition for the absence of stratification of the phases in a solution of two Bose gases at T = 0°K. Indeed, in order that such a stratification will not occur, it is necessary that the chemical potential of, say, the first component in the mixture be less than the chemical potential of the same component in the pure form at the same pressure. In other words, for any c $(0 < c < 1)$ the inequality

$$\mu_1(c, \rho(c, P)) < \mu_1(1, \rho(1, P)). \tag{1.20}$$

should be satisfied. We can verify, using (1.16a) and (1.17), that (1.20) coincides with the second condition in (1.19).

2. A SOLUTION OF TWO NEAR-IDEAL BOSE GASES WITH INDEPENDENTLY MOVING CONDENSATES

In order to study the case of a solution with moving condensates, we write the Hamiltonian (1.1) in the coordinate representation:

$$\begin{aligned}\hat{H} = &\int d^3\mathbf{r}\, \hat{\psi}_1^+(\mathbf{r}) \left(-\frac{\hbar^2\nabla^2}{2m_1} - \mu_1\right) \hat{\psi}_1(\mathbf{r}) + \int d^3\mathbf{r}\, \hat{\psi}_2^+(\mathbf{r}) \left(-\frac{\hbar^2\nabla^2}{2m_2} - \mu_2\right) \hat{\psi}_2(\mathbf{r}) \\ &+ \frac{g_1}{2}\int d^3\mathbf{r}\, \hat{\psi}_1^+(\mathbf{r})\hat{\psi}_1^+(\mathbf{r})\hat{\psi}_1(\mathbf{r})\hat{\psi}_1(\mathbf{r}) + \frac{g_2}{2}\int d^3\mathbf{r}\, \hat{\psi}_2^+(\mathbf{r})\hat{\psi}_2^+(\mathbf{r})\hat{\psi}_2(\mathbf{r})\hat{\psi}_2(\mathbf{r}) \\ &+ \frac{g_3}{2}\int d^3\mathbf{r}[\hat{\psi}_2^+(\mathbf{r})\hat{\psi}_1^+(\mathbf{r})\hat{\psi}_1(\mathbf{r})\hat{\psi}_2(\mathbf{r}) + \hat{\psi}_1^+(\mathbf{r})\hat{\psi}_2^+(\mathbf{r})\hat{\psi}_2(\mathbf{r})\hat{\psi}_1(\mathbf{r})].\end{aligned} \tag{2.1}$$

In this Hamiltonian we have, using the usual procedure, also made a change of prescribed quantities from the numbers N_1 and N_2 of particles of the components of the solution to the chemical potentials μ_1 and μ_2, and $\hat{\psi}_1$ and $\hat{\psi}_2$ are the Bose operators for the particles of the first and second components. Let us, as usual, represent each of them as a sum consisting of a condensate part and an operator correction:

$$\hat{\psi}_\alpha(\mathbf{r}) = \psi_\alpha(\mathbf{r}) + \hat{\Phi}_\alpha(\mathbf{r}), \quad \alpha = 1, 2. \tag{2.2}$$

Notice that the epicondensate corrections $\hat{\Phi}_\alpha$ will now not exactly satisfy the Bose commutation relations (see [8]):

$$[\hat{\Phi}_\alpha(\mathbf{r});\ \hat{\Phi}_\beta^+(\mathbf{r}')] = \left(\delta(\mathbf{r}-\mathbf{r}') - \frac{\psi_\alpha(\mathbf{r})\psi_\alpha^+(\mathbf{r}')}{(N_{\alpha 0}N_{\beta 0})^{1/2}}\right)\delta_{\alpha\beta}. \tag{2.3}$$

Here $N_{\alpha 0}$ and $N_{\beta 0}$ are the numbers of condensate particles in the first and second components, $\alpha = 1, 2$; $\beta = 1, 2$.

Let us substitute (2.2) into (2.1). The terms linear in $\hat{\Phi}_\alpha(\mathbf{r})$ vanish if the equations

$$\begin{aligned}&\left\{-\frac{\hbar^2\nabla^2}{2m_1} - \mu_1 + g_1|\psi_1(\mathbf{r})|^2 + g_3|\psi_2(\mathbf{r})|^2\right\}\psi_1(\mathbf{r}) = 0, \\ &\left\{-\frac{\hbar^2\nabla^2}{2m_2} - \mu_2 + g_2|\psi_2(\mathbf{r})|^2 + g_3|\psi_1(\mathbf{r})|^2\right\}\psi_2(\mathbf{r}) = 0.\end{aligned} \tag{2.4}$$

are satisfied. These equations for the condensate functions of the mixture of Bose gases are analogous to the equation obtained for the Bose gas by Gross [9] and Pitaevskiĭ [10]. As in the case of ordinary superfluidity, from the nonsteady-state analogs of Eqs. (2.4) we can derive equations for two superfluid velocities $\mathbf{v}_{s1}$ and $\mathbf{v}_{s2}$ that coincide with the velocities obtained in [3] in a phenomenological manner and in [5] from the equations of motion for the anomalous mean quantities:

$$\begin{aligned}&m_1\frac{\partial \mathbf{v}_{s1}}{\partial t} + \nabla\left(\mu_1' + \frac{m_1\mathbf{v}_{s1}^2}{2}\right) = 0, \\ &m_2\frac{\partial \mathbf{v}_{s2}}{\partial t} + \nabla\left(\mu_2' + \frac{m_2\mathbf{v}_{s2}^2}{2}\right) = 0;\end{aligned}$$

here $\mathbf{v}_{s\alpha} = (\hbar/m_\alpha)\partial S_\alpha/\partial\mathbf{r}$, S_α is the phase of the condensate wave function defined in (2.2), and

$$\mu_1' = \mu_1 - |\psi_1|^{-1}(g_1|\psi_1|^2 + g_3|\psi_2|^2); \quad \mu_2' = \mu_2 - |\psi_2|^{-1}(g_2|\psi_2|^2 + g_3|\psi_1|^2).$$

Thus, we emphasize again that two independent superfluid motions with velocities $\mathbf{v}_{s1}$ and $\mathbf{v}_{s2}$ can exist in the solution under consideration.

Retaining in (2.1) only the terms quadratic in $\hat{\Phi}_\alpha$, we obtain

$$\begin{aligned}\hat{H} = E_0 &+ \int d^3r\, \hat{\Phi}_1^+(\mathbf{r})\left[-\frac{\hbar^2}{2m_1}\nabla^2 - \mu_1 + 2g_1|\psi_1(\mathbf{r})|^2 + g_3|\psi_2(\mathbf{r})|^2\right]\hat{\Phi}_1(\mathbf{r}) \\ &+ \int d^3\mathbf{r}\, \hat{\Phi}_2^+(\mathbf{r})\left[-\frac{\hbar^2}{2m_2}\nabla^2 - \mu_2 + 2g_2|\psi_2(\mathbf{r})|^2 + g_3|\psi_1(\mathbf{r})|^2\right]\hat{\Phi}_2(\mathbf{r}) \\ &+ \frac{g_1}{2}\int d^3\mathbf{r}[(\psi_1^*(\mathbf{r}))^2\hat{\Phi}_1(\mathbf{r})\hat{\Phi}_1(\mathbf{r}) + \hat{\Phi}_1^+(\mathbf{r})\hat{\Phi}_1^+(\mathbf{r})(\psi_1(\mathbf{r}))^2] \\ &+ \frac{g_2}{2}\int d^3\mathbf{r}[(\psi_2^*(\mathbf{r}))^2\hat{\Phi}_2(\mathbf{r})\hat{\Phi}_2(\mathbf{r}) + \hat{\Phi}_2^+(\mathbf{r})\hat{\Phi}_2^+(\mathbf{r})(\psi_2(\mathbf{r}))^2] \\ &+ g_3\int d^3\mathbf{r}[\hat{\Phi}_2^+(\mathbf{r})\hat{\Phi}_1^+(\mathbf{r})\psi_1(\mathbf{r})\psi_2(\mathbf{r}) + \hat{\Phi}_2^+(\mathbf{r})\hat{\Phi}_1(\mathbf{r})\psi_1^*(\mathbf{r})\psi_2(\mathbf{r}) \\ &\quad + \psi_2^*(\mathbf{r})\psi_1^*(\mathbf{r})\hat{\Phi}_1(\mathbf{r})\hat{\Phi}_2(\mathbf{r}) + \hat{\Phi}_1^+(\mathbf{r})\hat{\Phi}_2(\mathbf{r})\psi_2^*(\mathbf{r})\psi_1(\mathbf{r})].\end{aligned} \tag{2.5}$$

Here E_0 is given by the expression (1.6).

Let us separate out the variable phases in (2.2) in explicit form:

$$\psi_\alpha(\mathbf{r}) = e^{iS_\alpha(\mathbf{r})}[|\psi_\alpha(\mathbf{r})| + \hat{\eta}_\alpha(\mathbf{r})], \quad \alpha = 1, 2, \tag{2.6}$$

where

$$\hat{\Phi}_\alpha(\mathbf{r}) = e^{iS_\alpha(\mathbf{r})}\hat{\eta}_\alpha(\mathbf{r}).$$

Let us rewrite (2.5) with allowance for (2.6):

$$\begin{aligned}\hat{H} = E_0 &+ \int d^3\mathbf{r}\, \hat{\eta}_1^+(\mathbf{r})\left[-\frac{\hbar^2}{2m_1}(\nabla + i\nabla S_1(\mathbf{r}))^2 - \mu_1 + 2g_1|\psi_1(\mathbf{r})|^2\right. \\ &\left.+ g_3|\psi_2(\mathbf{r})|^2\right]\hat{\eta}_1(\mathbf{r}) + \int d^3\mathbf{r}\, \hat{\eta}_2^+(\mathbf{r})\left[-\frac{\hbar^2}{2m_2}(\nabla + i\nabla S_2(\mathbf{r}))^2 - \mu_2 + 2g_2|\psi_2(\mathbf{r})|^2\right. \\ &\left.+ g_3|\psi_1(\mathbf{r})|^2\right]\hat{\eta}_2(\mathbf{r}) + \frac{g_1}{2}\int d^3\mathbf{r}|\psi_1(\mathbf{r})|^2[\hat{\eta}_1(\mathbf{r})\hat{\eta}_1(\mathbf{r}) + \hat{\eta}_1^+(\mathbf{r})\hat{\eta}_1^+(\mathbf{r})] \\ &+ \frac{g_2}{2}\int |\psi_2(\mathbf{r})|^2[\hat{\eta}_2(\mathbf{r})\hat{\eta}_2(\mathbf{r}) + \hat{\eta}_2^+(\mathbf{r})\hat{\eta}_2^+(\mathbf{r})] + g_3\int d^3\mathbf{r}|\psi_1(\mathbf{r})||\psi_2(\mathbf{r})|. \\ &\times[\hat{\eta}_2^+(\mathbf{r})\hat{\eta}_1^+(\mathbf{r}) + \hat{\eta}_2^+(\mathbf{r})\hat{\eta}_1(\mathbf{r}) + \hat{\eta}_1(\mathbf{r})\hat{\eta}_2(\mathbf{r}) + \hat{\eta}_1^+(\mathbf{r})\hat{\eta}_2(\mathbf{r})].\end{aligned} \tag{2.7}$$

We diagonalize (2.7) by carrying out the canonical transformation

$$\hat{\eta}_\alpha(\mathbf{r}) = \sum_{\nu=1,2}\sum_l [u_{\alpha\nu}(\mathbf{r}, l)\xi_\nu(l) + v_{\alpha\nu}^*(\mathbf{r}, l)\xi_\nu^+(l)]. \tag{2.8}$$

The l sum in (2.8) is over all the particles that are not in the condensate. The functions u and v satisfy the relations

$$\sum_l\sum_{\alpha=1,2}[u_{\alpha\nu}(\mathbf{r}, l)u_{\alpha\mu}^*(\mathbf{r}', l) - v_{\alpha\nu}(\mathbf{r}', l)v_{\alpha\mu}^*(\mathbf{r}, l)] = \bar{\delta}(\mathbf{r}, \mathbf{r}')\delta_{\mu\nu},$$

$$\sum_l\sum_{\alpha=1,2}[u_{\alpha\nu}(\mathbf{r}, l)v_{\alpha\mu}(\mathbf{r}', l) - v_{\alpha\nu}(\mathbf{r}, l)u_{\alpha\mu}(\mathbf{r}'l)] = 0$$

(here $\bar{\delta}(\mathbf{r}, \mathbf{r}')$ denotes the expression in the round brackets on the right-hand side of the formula (2.3)), as well as the relations

$$\sum_{\nu=1,2}\int d^3\mathbf{r}[u_{\alpha\nu}(\mathbf{r}, l)u_{\beta\nu}^*(\mathbf{r}, m) - v_{\beta\nu}(\mathbf{r}, m)v_{\alpha\nu}^*(\mathbf{r}, l)] = \delta_{lm}\delta_{\alpha\beta},$$

$$\sum_{\nu=1,2}\int d^3\mathbf{r}[u_{\beta\nu}(\mathbf{r}, l)v_{\alpha\nu}^*(\mathbf{r}, m) - u_{\alpha\nu}(\mathbf{r}, m)v_{\beta\nu}^*(\mathbf{r}, l)] = \delta_{lm}\delta_{\alpha\beta}.$$

After substituting (2.8) into (2.7), we obtain

$$\hat{H} = E_0 + \sum_{\nu=1,2}\sum_l E_\nu(l)\xi_\nu^+(l)\xi_\nu(l). \tag{2.9}$$

without allowance for the terms of order higher than the first in the interaction constants, E_0 is given by the expression (1.6).

The system of differential equations for the determination of the functions u and v and the excitation energies $E_\nu(l)$ is entirely similar to the system of algebraic equations (1.7):

$$\sum_{\beta}\{[\hat{S}_{\alpha\beta}(\mathbf{r})-E_{\nu}(l)\delta_{\alpha\beta}]u_{\beta\nu}(\mathbf{r},l)+R_{\alpha\beta}(\mathbf{r})v_{\beta\nu}(\mathbf{r},l)\}=0,$$
$$\sum_{\beta}\{R_{\alpha\beta}(\mathbf{r})u_{\beta\nu}(\mathbf{r},l)+[\hat{S}_{\alpha\beta}^{*}(\mathbf{r})+E_{\nu}(l)\delta_{\alpha\beta}]v_{\beta\nu}(\mathbf{r},l)\}=0, \qquad (2.10)$$

where

$$\hat{S}_{\alpha\beta}(\mathbf{r}) = \begin{pmatrix} -\frac{\hbar^2}{2m_1}(\nabla+i\nabla S_1(\mathbf{r}))^2-\mu_1+2g_1|\psi_1(\mathbf{r})|+g_3|\psi_2(\mathbf{r})|^2, & g_3|\psi_1(\mathbf{r})||\psi_2(\mathbf{r})| \\ g_3|\psi_1(\mathbf{r})||\psi_2(\mathbf{r})|, & -\frac{\hbar^2}{2m_2}(\nabla+i\nabla S_2(\mathbf{r}))^2-\mu_2+2g_2|\psi_2(\mathbf{r})|^2+g_3|\psi_1(\mathbf{r})|^2 \end{pmatrix}$$
$$R_{\alpha\beta}(\mathbf{r})=\begin{pmatrix} g_1|\psi_1(\mathbf{r})|^2, & g_3|\psi_1(\mathbf{r})||\psi_2(\mathbf{r})| \\ g_3|\psi_1(\mathbf{r})||\psi_2(\mathbf{r})|, & g_2|\psi_2(\mathbf{r})|^2 \end{pmatrix} \qquad (2.11)$$

Let us consider in greater detail the case when both condensates move uniformly with different velocities $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\psi_\alpha(\mathbf{r})=(n_{\alpha0})^{1/2}\exp[im_\alpha\mathbf{v}_\alpha\mathbf{r}/\hbar], \quad \alpha=1, 2. \qquad (2.12)$$

Here $n_{\alpha0} = N_{\alpha0}/V$ ($\alpha = 1, 2$) are the densities of the first and second condensates. The condensate densities coincide up to terms of order higher than the first in the interaction constants with the densities of the corresponding Bose gases.

Further, from the Eqs. (2.4) we have:

$$\mu_1=n_{10}g_1+n_{20}g_3+{}^1/_2m_1\mathbf{v}_1^2,$$
$$\mu_2=n_{20}g_2+n_{10}g_3+{}^1/_2m_2\mathbf{v}_2^2. \qquad (2.13)$$

In the case of uniform motion of the condensates the role of the variable l in the formulas (2.8)–(2.11) is played by the momentum $\mathbf{p}$.

Setting

$$u_{\alpha\nu}(\mathbf{r},\mathbf{p})=V^{-1/2}u_{\alpha\nu}(\mathbf{p})e^{i\mathbf{pr}/\hbar},$$
$$v_{\alpha\nu}(\mathbf{r},\mathbf{p})=V^{-1/2}v_{\alpha\nu}(\mathbf{p})e^{i\mathbf{pr}/\hbar}, \qquad (2.14)$$

Substituting (2.14) into (2.10), and taking (2.13) into consideration, we obtain, instead of a system of differential equations, a system of algebraic equations of the form (1.7):

$$\sum_{\beta}\{[S_{\alpha\beta}(\mathbf{p})-E_{\nu}(\mathbf{p})\delta_{\alpha\beta}]u_{\beta\nu}(\mathbf{p})+R_{\alpha\beta}v_{\beta\nu}(\mathbf{p})\}=0,$$
$$\sum_{\beta}\{R_{\alpha\beta}u_{\beta\nu}(\mathbf{p})+[S_{\alpha\beta}^{*}(\mathbf{p})+E_{\nu}(\mathbf{p})\delta_{\alpha\beta}]v_{\beta\nu}(\mathbf{p})\}=0 \qquad (2.15)$$

with the matrices (see (2.11))

$$S_{\alpha\beta}(\mathbf{p})=\begin{pmatrix} \frac{\mathbf{p}^2}{2m_1}+\mathbf{pv}_1+g_1n_{10}, & g_3\sqrt{n_{10}n_{20}} \\ g_3\sqrt{n_{10}n_{20}}, & \frac{\mathbf{p}^2}{2m_2}+\mathbf{pv}_2+g_2n_{20} \end{pmatrix},$$
$$S_{\alpha\beta}^{*}(\mathbf{p})=\begin{pmatrix} \frac{\mathbf{p}^2}{2m_1}-\mathbf{pv}_1+g_1n_{10}, & g_3\sqrt{n_{10}n_{20}} \\ g_3\sqrt{n_{10}n_{20}}, & \frac{\mathbf{p}^2}{2m_2}-\mathbf{pv}_2+g_2n_{20} \end{pmatrix}$$
$$R_{\alpha\beta}=\begin{pmatrix} g_1n_{10}, & g_3\sqrt{n_{10}n_{20}} \\ g_3\sqrt{n_{10}n_{20}}, & g_2n_{20} \end{pmatrix}.$$

By equating to zero the determinant of the system (2.15) we can find the energies of the excitations of a solution with two uniformly moving condensates. Equating the determinant to zero, we obtain the following equation:

$$\varepsilon^4+(a-2\delta^2)\varepsilon^2+2\delta d\varepsilon+b+\delta^2a+\delta^4=0. \qquad (2.16)$$

Here

$$\varepsilon=E-(\mathbf{pv}_1+\mathbf{pv}_2)/2, \quad \delta=(\mathbf{pv}_1-\mathbf{pv}_2)/2,$$
$$a=-A_1^2-A_2^2+B_1^2+B_2^2, \; b=(A_1A_2-2C^2)^2+(B_1B_2-2C^2)^2-(B_1A_2-2C^2)^2-(A_1B_2-2C^2)^2, \; d=-A_1^2+A_2^2+B_1^2-B_2^2,$$
$$A_\alpha=\mathbf{p}^2/2m_\alpha+g_\alpha n_{\alpha0}, \; B_\alpha=g_\alpha n_{\alpha0} \quad (\alpha=1, 2),$$
$$C=g_3\sqrt{n_{10}n_{20}}.$$

In the particular case of motion with equal velocities $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}$, $\delta = 0$, Eq. (2.16) is converted into the equation

$$\varepsilon^4+a\varepsilon^2+b=0, \qquad (2.17)$$

whose roots are given by the formula

$$E_\nu^m(\mathbf{p})=E_\nu^s(\mathbf{p})+\mathbf{pv}, \quad \nu=1, 2. \qquad (2.18)$$

Here $E_\nu^m(\mathbf{p})$ are the energies of the excitations of the solution with moving condensates, while the quantities $E_\nu^s(\mathbf{p})$, which are determined by the formula (1.9), are the corresponding energies for the case of stationary condensates. But if the condensates move with different velocities, then, as can be seen from (2.16), the excitation energies will depend on the half-sum and half-difference of these velocities, which will lead, at temperatures different from zero, to similar dependences of the currents, densities, and all the other thermodynamic quantities.

The exact expressions for the roots of Eq. (2.16) are too unwieldy, and therefore in solving this equation we can use the fact that one of the positive roots of Eq. (2.17) is much larger than the other at concentrations $c \ll 1$, i.e., when $c_1 \gg c_2$ (see (1.13a) and (1.13b)). If we assume that $v_1 \ll c_1$ and $v_2 \ll c_1$, but that v_1 and v_2 are entirely arbitrary in relation to c_2, then Eq. (2.16) will possess an analogous property.

We have

$$E_1^m(\mathbf{p})=\frac{\mathbf{pv}_1+\mathbf{pv}_2}{2}+(-a+2\delta^2)^{1/2},$$
$$E_2^m(\mathbf{p})=\frac{\mathbf{pv}_1+\mathbf{pv}_2}{2}+\frac{-\delta d-[(\delta d)^2-(a-2\delta^2)(b+\delta^2a+\delta^4)]^{1/2}}{a-2\delta^2}. \qquad (2.19)$$

The Landau condition for the existence of superfluidity in this case will be the condition that the smaller root $E_2^m(\mathbf{p}) > 0$. A simple analysis of Eq. (2.19) leads us to the conclusion that the Landau condition is violated when $|\mathbf{v}_1| \gtrsim c_2$ and $|\mathbf{v}_2| \gtrsim c_2$. Thus, upon the attainment by any of the components of the solution of a critical velocity of the order of c_2 (we abstract ourselves here from the possible existence of vortices), the superfluidity of this component vanishes. It should be emphasized that in the considered model for the solution, for $c \ll 1$, the Landau condition is violated at considerably lower velocities than in a pure Bose gas of the same density. This is due to the appearance of a second Goldstone excitation branch in a solution of two superfluid liquids.

In conclusion, the author considers it to be his pleasant duty to thank I. M. Khalatnikov, without whose stimulating influence the present work would never have been published. I am also grateful to V. L. Pokrovskiĭ for his benevolent criticism.

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[1] We consider only the two positive roots of the fourth-order equation. The two negative roots correspond to normalization conditions that differ from the conditions (1.4a) (see, for example, [7]).

[2] The retention in $(\partial P/\partial\rho)_c$ of terms of order c is senseless, since these terms were neglected in the derivation of the expression for u_1^2 itself (see [4]).

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Translated by A. K. Agyei
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