Nonlinear Landau damping in an inhomogeneous plasma

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A theory of nonlinear damping (growth) of Langmuir waves in a collisionless inhomogeneous plasma is developed for the case when the degree of plasma inhomogeneity exceeds the critical value at which particle capture by the potential wells of the waves becomes impossible. An expression for the increment of the waves is derived and investigated as a function of the field amplitude and the inhomogeneity parameter in the entire regions of these parameters where there are no trapped particles.

1. INTRODUCTION

As is well known, the Landau-damping constant in a homogeneous plasma vanishes in time because of the phase mixing in the resonance region of velocity space[1-3]. In an inhomogeneous plasma, the phase mixing is never complete, since owing to the variation of the phase velocity of the wave, the resonance region is continuously renovated. As a result of this, the decrement (increment) of the wave never vanishes.

The characteristic parameter that determines the distinctive competition between the nonlinearity and inhomogeneity effects for Langmuir waves is the quantity $2|\alpha|\tau^2$, where α is the inhomogeneity parameter, which is proportional to the phase acceleration of the wave:

$$\alpha=-\frac{\omega^2}{2k^2}\frac{dk}{dx}, \tag{1}$$

while τ is the nonlinear phase-mixing time, which, for monochromatic waves, has the form

$$\tau=(m/eEk)^{1/2}, \tag{2}$$

E being the wave-filled amplitude:

$$\mathscr{E}(x,t)=E(x,t)\cos\left[\int_0^x k(x')\,dx'-\omega t+\varphi(x,t)\right]; \tag{3}$$

E, φ, and k are assumed here to be slowly varying functions, k(x) being determined from the standard linear dispersion relation for the homogeneous plasma, a relation which, for Langmuir waves, has the form $\omega^2 = \omega_p^2(1 + 3k^2r_D^2)$.

It is easy to verify that the parameter $2|\alpha|\tau^2$ is equal to the ratio of the phase acceleration of the wave

$$\frac{d}{dt}\left[\frac{\omega}{k(x)}\right]=\frac{d}{dx}\left[\frac{\omega}{k(x)}\right]\frac{\omega}{k}=-\frac{\omega^2}{k^3}\frac{dk}{dx}$$

to eE/m, the amplitude of the acceleration of the electrons in the wave field (3). For $2|\alpha|\tau^2 < 1$ the nonlinear effects predominate over the inhomogeneity effects. For $2|\alpha|\tau^2 > 1$ the reverse situation obtains. In this case particle capture by the potential wells of the wave is already impossible, and this radically changes the general picture of the nonlinear interaction of the resonance particles with the wave.

The theory of nonlinear effects in an inhomogeneous plasma has thus far been considered for the case $2|\alpha|\tau^2 \ll 1$, when the inhomogeneity effects can be considered to be weak[4-7]. In[7] an attempt was also made to consider the case of the highly inhomogeneous plasma ($2|\alpha|\tau^2 > 1$). However, the results of this paper have, in our opinion, a very limited region of applicability.

In the present paper we develop a consistent theory, valid for $2|\alpha|\tau^2 \geq 1$, of the interaction of resonance particles with a monochromatic wave. On the basis of this, we compute the increment (decrement) of the wave, this quantity being a generalization of the Landau increment in the case under consideration. We are then able to follow the behavior of the increment in the most interesting region where $2|\alpha|\tau^2 \to 1$, which is the border zone between the cases of strong and weak inhomogeneities.

2. THE KINETIC EQUATION AND THE EQUATIONS OF MOTION OF THE PARTICLES

We shall assume, as is usually done, that the amplitude of the wave is sufficiently small, so that $\omega\tau \gg 1$ and $1/\tau \ll kv_T$, where v_T is the thermal velocity of the plasma. Then the particles that are in resonance with the wave (3) and that, as is well known, have a velocity v in the interval

$$|v-\omega/k|\leqslant 1/k\tau\ll v_T\ll\omega/k, \tag{4}$$

fill a comparatively small volume of phase space. This enables us to significantly simplify the general kinetic plasma equation, which we write in the form

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}+\frac{\mathscr{F}}{m}\frac{\partial f}{\partial v}-\frac{e\mathscr{E}(x,t)}{m}\frac{\partial f}{\partial v}=0, \tag{5}$$

where $\mathscr{F}$ is the force due to the inhomogeneity of the medium and $\mathscr{E}(x, t)$ is the wave field (3), in which we shall drop the phase φ, since the latter turns out to be insignificant in the case when $2|\alpha|\tau^2 \geq 1$.

Let us now introduce in place of t and v the new independent variables[1)]

$$2\xi=\int_0^x k(x')\,dx'-\omega t+\pi,$$

$$2\dot{\xi}=k(x)[v-\omega/k(x)]. \tag{6}$$

Taking (4) into account, we obtain the kinetic equation in the form (cf.[4])

$$\frac{\omega}{k}\frac{\partial f}{\partial x}+\dot{\xi}\frac{\partial f}{\partial \dot{\xi}}+\left(\frac{\cos 2\xi}{2\tau^2}-\alpha_1\right)\frac{\partial f}{\partial \dot{\xi}}=0, \tag{7}$$

where $\alpha_1 = \alpha - k\mathscr{F}/2m$. Since the quantity

$$k\mathscr{F}/2m\sim v_T^4(k/\omega)^2dk/dx\ll\alpha,$$

we shall henceforth neglect the difference between α_1 and α. As will become apparent from the final results, we can assume, without loss of generality, that α = const. Furthermore, we shall, for definiteness, assume that $\alpha > 0$ (the changes that must be made in the following relations for the case of $\alpha < 0$

are obvious). Taking the foregoing remarks into account, we can write down the ordinary differential equations corresponding to (7):

$$\frac{d\xi}{d\theta}=u,\quad \frac{du}{d\theta}=\beta\cos 2\xi-1; \tag{8}$$

$$\theta=\frac{\alpha^{1/2}}{\omega}\int_0^x k(x')dx',\quad u=\frac{\xi}{\sqrt{\alpha}}; \tag{9}$$

$$\beta=1/2\alpha\tau^2\leqslant 1. \tag{10}$$

From (8) follows the "energy" conservation law

$$\varepsilon=u^2+2\xi-\beta\sin 2\xi. \tag{11}$$

The plots of the "potential energy" $P(\xi) = 2\xi - \beta\sin 2\xi$ for $\beta > 1$ and $\beta < 1$ are shown in Fig. 1. We see, in accord with the assertion made in the Introduction, that in the first case there exist potential wells in which there are particles trapped by the wave, while in the second case there are no trapped particles.

Let us write the solution to Eqs. (8) in the form

$$\int_{2\xi_0}^{2\xi}\frac{dz}{(\varepsilon-z+\beta\sin z)^{1/2}}=2\theta. \tag{12}$$

Under the condition (10), the integral (12) can be expressed in terms of known functions. For this purpose let us set

$$z-\beta\sin z=y. \tag{13}$$

For $\beta < 1$ the function $y(z)$ increases monotonically with increasing z, and, consequently, it has a single-valued inverse function. If we set

$$z=y+\sum_{n=1}^{\infty}a_n\sin ny \tag{14}$$

(the coefficients a_n will be found below), then Eq. (12) assumes the form

$$u_0-u+\sum_{n=1}^{\infty}\left(\frac{\pi n}{2}\right)^{1/2}a_n\{\cos n\varepsilon[C(n^{1/2}u_0)-C(n^{1/2}u)] + \sin n\varepsilon[S(n^{1/2}u_0)-S(n^{1/2}u)]\}=\theta, \tag{15}$$

where $C(w)$ and $S(w)$ are Fresnel integrals.

Let us now determine the coefficients a_n. Differentiating (14) with respect to y, we obtain

$$na_n=\frac{2}{\pi}\int_0^{\pi}\frac{dz}{dy}\cos ny\,dy=\frac{2}{\pi}\int_0^{\pi}\cos[n(z-\beta\sin z)]dz.$$

Thus,

$$a_n=\frac{2}{n}J_n(\beta n), \tag{16}$$

where $J_n(\beta n)$ is a Bessel function.

3. THE DISTRIBUTION FUNCTION

The formula (15) enables us to determine the distribution function in the resonance region. From the Liouville theorem we have

$$f(x,\xi,u)=f(0,\xi_0,u_0)=f_0\left(\frac{\omega}{k}\right)+\frac{2\sqrt{\alpha}}{k}(u_0-\theta)f_0'\left(\frac{\omega}{k}\right), \tag{17}$$

where ξ_0 and u_0 are the initial values of the generalized coordinates and velocities expressed with the aid of (15) in terms of ξ, u, and x, while $f_0(v)$ is the waveless-plasma distribution function, which we have expanded in the neighborhood of the resonance velocity $v = \omega/k$, retaining two terms. Into the expressions for the increment and the other quantities connected with the influence of the wave enters the quantity

$$\delta f=f(x,\xi,u)-F(x,\xi,u), \tag{18}$$

where $F(x, \xi, u)$ is the unperturbed-plasma distribution function that satisfies Eq. (7) for $\tau = \infty$ (i.e., for $\mathscr{E} = 0$). We can, to the same degree of accuracy as in (17), write

$$F(x,\xi,u)=f_0\left(\frac{\omega}{k}\right)+\frac{2\alpha^{1/2}u}{k}f_0'\left(\frac{\omega}{k}\right). \tag{19}$$

Substituting (17) and (19) into (18) and using (16), we obtain

$$\delta f=\frac{1}{k}f_0'\left(\frac{\omega}{k}\right)\sum_{n=1}^{\infty}(2\pi\alpha n)^{1/2}a_n\{\cos n\varepsilon[C(n^{1/2}u)-C(n^{1/2}u_0)] + \sin n\varepsilon[S(n^{1/2}u)-S(n^{1/2}u_0)]\}. \tag{20}$$

Let us show how the expression for the distribution function as given by ordinary perturbation theory is derived from this formula. For $|u| \gg 1$ and $|u_0| \gg 1$, we can, using the well-known asymptotic expansions for the Fresnel integrals, write

$$C(n^{1/2}u_0)=\frac{\operatorname{sign}u_0}{2}+\frac{\sin nu_0^2}{(2\pi n)^{1/2}u_0},\quad S(n^{1/2}u_0)=\frac{\operatorname{sign}u_0}{2}-\frac{\cos nu_0^2}{(2\pi n)^{1/2}u_0} \tag{21}$$

and similar expressions for the argument u. It must be borne in mind here that perturbation theory is applicable only for sufficiently small θ, when the dominant contribution is made by the region where $\operatorname{sign} u_0 = \operatorname{sign} u$.

Substituting (21) into (20), and expressing ϵ in terms of u, ξ and u_0, ξ_0 (in accordance with (11)) in the corresponding terms in (20), we obtain for δf, after simple transformations, the expression

$$\delta f=\frac{\sqrt{\alpha}}{k}f_0'\left(\frac{\omega}{k}\right)\sum_{n=1}^{\infty}a_n\left\{\frac{\sin[n(2\xi_0-\beta\sin 2\xi_0)]}{u_0}-\frac{\sin[n(2\xi-\beta\sin 2\xi)]}{u}\right\}. \tag{22}$$

Taking (13) and (14) into account, we can write

$$\sum_{n=1}^{\infty}a_n\sin[n(2\xi-\beta\sin 2\xi)]=\beta\sin 2\xi. \tag{23}$$

Thus, (22) assumes the form

$$\delta f=\frac{\alpha^{1/2}\beta}{k}f_0'\left(\frac{\sin 2\xi_0}{u_0}-\frac{\sin 2\xi}{u}\right), \tag{24}$$

which, when expressed in terms of the ordinary variables, coincides with the well-known expression that follows from perturbation theory.

4. COMPUTATION OF THE INCREMENT

We determine the increment of the wave from the formula

$$\gamma=-4\pi E^{-2}\overline{(j\mathscr{E})}=-\frac{16e\alpha^{1/2}}{k^2\partial\varepsilon/\partial\omega E}\int_{-\pi/2}^{\pi/2}d\xi\int_{-\infty}^{\infty}du\,\delta f\cos 2\xi, \tag{25}$$

where $\partial\epsilon/\partial\omega = 2/\omega$ in our case. Substituting δf into (25), we obtain

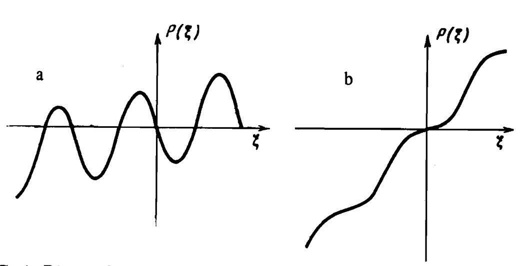

FIG. 1. Plots of the effective potential $P(\xi)$: a) $\beta > 1$, b) $\beta < 1$.

$$\gamma=\frac{16e\omega\alpha}{k^3E}\left(\frac{\pi}{2}\right)^{1/2}f_0'\left(\frac{\omega}{k}\right)\sum_{n=1}^{\infty}n^{1/2}a_n+$$

$$\cdot\int_{-\infty}^{\infty}du\int_{-\pi/2}^{\pi/2}d\xi\cos 2\xi\,[C(n^{1/2}u_0)\cos n\varepsilon+S(n^{1/2}u_0)\sin n\varepsilon], \qquad (26)$$

where we have dropped the u-odd terms, which do not contribute to the integral (26). It is implied here that the function $u_0 = u_0(\xi, u, \theta)$ is determined from the equations of motion. Let us now consider the two most interesting limiting cases.

1) $\theta \ll 1$. In this case the integral (26) is most easily evaluated in the following fashion. Using the fact that $du\,d\xi = du_0 d\xi_0$ and that $\xi = \xi_0 + u_0\theta$ for small values of θ, we can write this integral in the form

$$-\int du_0\,d\xi_0\sin 2\xi_0\sin(2u_0\theta)\,[C(n^{1/2}u_0)\cos n\varepsilon+S(n^{1/2}u_0)\sin n\varepsilon], \qquad (27)$$

where we have dropped the u_0-odd terms, which make no contribution to the integral. We note further that for $\theta \ll 1$ the dominant contribution is made by the region where $u_0 \gg 1$, so that we can use the asymptotic expressions (21). As a result, the integral (27) assumes the form

$$2\int_0^{\infty}du_0\sin(2\theta u_0)\,(\sin nu_0^2-\cos nu_0^2)\int_0^{\pi/2}d\xi_0\sin 2\xi_0\sin[n(2\xi_0-\beta\sin 2\xi_0)]$$

$$+2\left(\frac{2}{\pi n}\right)^{1/2}\int_0^{\infty}du_0\frac{\sin(2\theta u_0)}{u_0}\int_0^{\pi/2}\sin 2\xi_0\sin[n(2\xi_0-\beta\sin 2\xi_0)]. \qquad (28)$$

The u_0 integral in the first term is equal to

$$\left(\frac{\pi}{2n}\right)^{1/2}\left\{\sin\frac{\theta^2}{n}\left[S\left(\frac{\theta}{n^{1/2}}\right)-C\left(\frac{\theta}{n^{1/2}}\right)\right]+\cos\frac{\theta^2}{n}\left[C\left(\frac{\theta}{n^{1/2}}\right)+S\left(\frac{\theta}{n^{1/2}}\right)\right]\right\}.$$

For $\theta \ll 1$ this integral is of the order of θ, and therefore it can be neglected. The first integral in the second term in (28) is equal to $\pi/2$. Now substituting (28) instead of the integral into (26), taking into account the foregoing remarks, and noting that according to (13) and (14)

$$\sum_{n=1}^{\infty}a_n\sin[n(2\xi_0-\beta\sin 2\xi_0)]=\beta\sin 2\xi_0 \qquad (29)$$

we find that in the case under consideration

$$\gamma=\gamma_L, \qquad (30)$$

where γ_L is the increment of the linear theory:

$$\gamma_L=\frac{2\pi^2e^2\omega}{mk^2}f_0'\left(\frac{\omega}{k}\right). \qquad (31)$$

2) Let us now consider the opposite limiting case when

$$\theta\gg1. \qquad (32)$$

Since $|u| \sim 1$ in the resonance region, we can, as can be seen from (15), set in this case

$$u_0\approx\theta. \qquad (33)$$

Substituting this into (26) and using the relations

$$\cos n\varepsilon=\cos nu^2\cos(2n\xi-\beta n\sin 2\xi)-\sin nu^2\sin(2n\xi-\beta n\sin 2\xi),$$
$$\sin n\varepsilon=\sin nu^2\cos(2n\xi-\beta n\sin 2\xi)+\cos nu^2\sin(2n\xi-\beta n\sin 2\xi), \qquad (34)$$

$$\int_{-\pi/2}^{\pi/2}d\xi\cos 2\xi\cos[n(2\xi-\beta\sin 2\xi)]=\frac{\pi}{\beta}J_n(\beta n)=\frac{\pi}{2\beta}na_n,$$

we obtain

$$\gamma=\gamma_L\beta^{-2}\sum_{n=1}^{\infty}na_n^2[C(n^{1/2}\theta)+S(n^{1/2}\theta)]. \qquad (35)$$

On account of the condition (32), we can replace C and S in (35) by their asymptotic forms (21), which yields

$$\gamma=\gamma_L\beta^{-2}\sum_{n=1}^{\infty}na_n^2(\beta). \qquad (36)$$

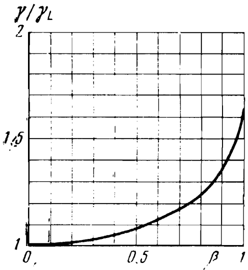

FIG. 2. Plot of the increment for $0 \leqslant \beta \leqslant 1$.

Let us now investigate the expression for γ. To estimate a_n for $n \gg 1$ (and any β), we can use the relation (see, for example,[8])

$$J_n(\beta n)\approx\frac{1}{\sqrt{\pi}}\left(\frac{2}{\beta n}\right)^{1/3}\mathrm{Ai}\left[\left(\frac{2n^2}{\beta}\right)^{1/3}(1-\beta)\right], \qquad (37)$$

where Ai(z) is the Airy function, which has the asymptotic form

$$\mathrm{Ai}(z)=\begin{cases}2\pi^{-1/2}z^{-1/4}\exp(-{}^2/_3z^{3/2}), & z>0\\ \pi^{-1/2}|z|^{-1/4}\cos({}^2/_3|z|^{3/2}-\pi/4), & z<0\end{cases} \qquad (38)$$

It follows from (16), (37), and (38) that the series Σna_n^2 converges for all β, but that for $\beta \sim 1$ this convergence is very slow:

$$na_n^2\sim\begin{cases}n^{-2}\exp[-{}^4/_3(2n^2/\beta)^{1/2}(1-\beta)^{3/2}], & \beta<1\\ n^{-5/3}, & \beta=1\end{cases} \qquad (39)$$

It follows from this, in particular, that for $\beta \ll 1$, it is sufficient in computing γ to take only a few terms of the series into account. As a result, for sufficiently low values of β we obtain

$$\gamma=\gamma_L(1+{}^1/_4\beta^2+{}^{11}/_{96}\beta^4) \qquad (40)$$

(for $\beta = 0.5$ the error in (40) does not exceed 5%). On the other hand, it follows from (39) that

$$\left.\frac{d\gamma(\beta)}{d\beta}\right|_{\beta=1}=\infty. \qquad (41)$$

The graph of $\gamma(\beta)$ is shown in Fig. 2.

In conclusion, let us discuss the results of the paper[7], where the expression (in our notation)

$$\gamma=\gamma_L\left[1-\frac{3\sqrt{\pi}}{4}\beta^2\theta\left(\sin\frac{\theta^2}{2}-\cos\frac{\theta^2}{2}\right)\right] \qquad (42)$$

is obtained under the condition (32) after very tedious calculations (the physical meaning of the last term in the square brackets is not discussed by the authors). Comparison with the foregoing shows clearly that the expression (42) is valid only for very small values of β (in this case it is obvious that we can neglect the term containing θ). But for $\beta \ll 1$ the result $\gamma \approx \gamma_L$ can be obtained by a considerably simpler method than was used in[7].

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[1])Such a change of variable is convenient in the "spatial Landau damping" problem, which is the problem we are considering here.

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