Nonlinear effects in the electrodynamics of pure superconductors in a high frequency field

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The change of impedance of pure type-I superconductors in the presence of a constant magnetic field is investigated theoretically for $\omega \approx 2\Delta$. It is shown that the imaginary part of the impedance, found accurate to second order in the constant magnetic field increases as $\omega \to 2\Delta$. The effect is related to a singularity in the number density of states in pure superconductors.

It is well known that in pure superconductors the number density of states has a singularity at $\epsilon = \Delta$. Therefore, the derivative of the conductivity with respect to frequency diverges logarithmically at $\omega = 2\Delta$ even in the linear approximation. We shall show that in the nonlinear case the singularity in $N(\epsilon)$ leads to a logarithmic divergence for the correction $\Delta_1(q, \omega) \sim H_0 H_\omega$ to the superconducting order parameter for $\omega \to 2\Delta$ and $qv > \Delta$, where v is the Fermi velocity and $H_\omega(H_0)$ is the amplitude of the variable (constant) magnetic field at the surface of the superconductor. For small values of qv, $\Delta^{1/2}(\omega - 2\Delta)^{1/2} \ll qv \ll \Delta$, $\Delta_1(q, \omega)$ is also proportional to q^{-1}. The latter is associated with the fact that, a solution of the equation for the eigenvibrations of the superconducting order parameter exists[1] with $\omega = 2\Delta$ and $q = 0$ for pure superconductors when collisions are neglected. All this leads to the result that the imaginary part of the correction to the impedance, which is proportional to H_0^2, diverges like $\ln^3\alpha$ for Pippard superconductors with not very small Ginzburg-Landau parameters κ, and in the opposite case this correction contains $\ln^2\alpha$, where $\alpha = |\omega - 2\Delta|/\Delta$. It is obvious that both relaxation processes and the anisotropy of the energy gap cut off this divergence at $\omega = 2\Delta$. From the very beginning we shall neglect collisions; therefore, $|\omega - 2\Delta|$ cannot take a value smaller than the order of the reciprocal of the time τ between collisions; however, the parameter $\Delta\tau$ may be substantial for pure superconductors. Anisotropy of the energy gap leads to a value for α of the order of a (a denotes the gap anisotropy parameter, whose value may be 10^{-1} [2]). Taking account of the smallness of the parameter α, a theoretical investigation of a superconductor's impedance in a constant magnetic field for $\omega \sim 2\Delta$ and in the isotropic model is of definite interest. We note that the increase of the correction to the imaginary part of the impedance for $\omega \sim 2\Delta$ may be of additional interest in connection with investigations of the energy gap anisotropy in the case of weak anisotropy.

For pure superconductors the expression for $\Delta_1(q, \omega)$ may be found by various methods.[3-5] In particular, the following expression is obtained in[5] for the case of a superconducting half-space with specular reflection from the boundaries:

$$L_1(q,\omega)\Delta_1(q,\omega) = \frac{e^2\Delta}{2\pi m^2c^2}\int L_2(\mathbf{q}\omega;\mathbf{q}_1\omega_1;\mathbf{q}_2\omega_2) \times \delta(\mathbf{q}-\mathbf{q}_1-\mathbf{q}_2)\delta(\omega-\omega_1-\omega_2)\,d\mathbf{q}_1 d\mathbf{q}_2 d\omega_1 d\omega_2, \tag{1}$$

where

$$L_1(q,\omega) = \int d\mathbf{p}[\omega^2-4\Delta^2-(\mathbf{qv})^2][N(q,\omega)+N(q,-\omega)]\varepsilon^{-1}\,\mathrm{th}(\varepsilon/2T), \tag{2}$$

$$L_2(q\omega;q_1\omega_1;q_2\omega_2) = \int d\mathbf{p}(\mathbf{A}(q_1\omega_1)\mathbf{p})(\mathbf{A}(q_2\omega_2)\mathbf{p})\{[4\varepsilon^2-2\varepsilon(\omega+\omega_1) +\omega\omega_1+2\eta(\mathbf{q}_1\mathbf{v})+(\mathbf{q}_1\mathbf{v})(\mathbf{qv})]N(q_1,-\omega_1)N(q,-\omega)+(\omega_1\to-\omega_1,\omega\to-\omega) +[4\varepsilon^2-2\varepsilon(\omega_1-\omega_2)-\omega_1\omega_2+(\mathbf{q}_1\mathbf{v})(\mathbf{q}_2\mathbf{v})]N(q_1,-\omega_1)N(-q_2,\omega_2)\}\varepsilon^{-1}\,\mathrm{th}(\varepsilon/2T), \tag{3}$$

$$N^{-1}(q,\omega) = (\varepsilon+\omega+i\alpha')^2-\varepsilon^2-2\eta(\mathbf{qv})-(\mathbf{qv})^2,$$
$$\varepsilon^2=\eta^2+\Delta^2,\ \eta=p^2/2m-\mu,\ \alpha'\to+0,$$

A is the vector potential of the electromagnetic field, and div **A** = 0. If the frequencies ω_1 and ω_2 vanish simultaneously, expression (1) goes over into the appropriate formulas for the static limit, which was investigated in[6]. However, as was first noticed in[4], approaching the limit by letting $\omega_1 \to 0$, $\omega_2 \to 0$, $\omega_1/\omega_2 \to \infty$ no longer gives the static limit. This fact is related to the neglect of collisions,[4] and in our formulas it formally manifests itself in the appearance of terms proportional to ω_1^{-1} in Eq. (3). For $\omega_1 \neq 0$ but $\omega_2 \to 0$ these terms tend to zero, which cannot be said in the case when $\omega_1 \to 0$. Therefore, the limit of Eq. (1) as $\omega_1 \to 0$, $\omega_2 \to 0$, and $\omega_1/\omega_2 \to 1$ does not coincide with the limit obtained by letting $\omega_1 \to 0$, $\omega_2 \to 0$, and $\omega_1/\omega_2 \to \infty$. Assuming that $\omega_2 = 0$ and $\omega_1 \neq 0$ in what follows, we confine ourselves to the behavior of Δ_1 in a static magnetic field at alternating-field frequencies $\omega_1 > \tau^{-1}$.

Let $\omega_2 = 0$, i.e., let $\mathbf{A}(q_2 0)$ denote the vector potential of the constant magnetic field in the superconductor, and let $\mathbf{A}(q_1\omega)$ describe the high frequency field, where we assume q_1 and q_2 to be directed along the z axis. We shall investigate type-I superconductors. Let us change to a spherical coordinate system in the integrals with respect to dp in Eqs. (2) and (3); then, as usual, the integration with respect to the polar angle θ can be reduced to an integral with respect to $t = \cos\theta$ with infinite limits. In this approximation the integrals with respect to t are easily evaluated, and as a result $L_2(q\omega; q_1\omega_1; q_2\omega_2)$ takes the form

$$\int\limits_{\max(\Delta,\omega-\Delta)}^{\omega+\Delta}\frac{f_1(\varepsilon,\omega,q,q_1,q_2)\,d\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}(\Delta^2-(\omega-\varepsilon)^2)^{1/2}} + i\int\limits_{\Delta}^{\omega-\Delta}\frac{f_2(\varepsilon,\omega,q,q_1,q_2)\Theta(\omega-2\Delta)\,d\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}(\Delta^2-(\omega-\varepsilon)^2)^{1/2}}, \tag{4}$$

where f_1 and f_2 are real functions, and $\Theta(x)$ is the step function. We see that in the limit as $\omega \to 2\Delta$ the integrand has a singularity at $\epsilon = \Delta$, which leads to a logarithmic singularity in Re $L_2(q\omega; q_1\omega; q_2 0)$, whereas Im $L_2(q\omega; q_1\omega; q_2 0)$ [Eq. (4) contains that part of the expression for $L_2(q\omega; q_1\omega; q_2 0)$ which has an essential singularity in the integrand] remains finite because the range of integration over ϵ vanishes at $\omega = 2\Delta$. Performing the relatively simple calculations, we find

$$f_1(\Delta,2\Delta,q,q_1,q_2) = -4\pi^2(\mathbf{A}(q_1\omega)\mathbf{A}(q_2 0))m^4v^2 \times\left(\frac{q^2q_1}{|q|q_2}-\frac{q_1q}{|q_1|}-\frac{q_1^3}{|q_1|q_2}\right)\frac{1}{(q^2+q_1^2)}\,\mathrm{th}\frac{\Delta}{2T}. \tag{5}$$

Within logarithmic accuracy, we obtain the following result for $L_2(q\omega; q_1\omega; q_2 0)$ in the limit as $\omega \to 2\Delta$:

$$L_2(q\omega;q_1\omega;q_2 0) = f_1(\Delta,2\Delta,q,q_1,q_2)(2\Delta)^{-1}\ln\alpha. \tag{6}$$

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Hence it is clear that $L_2(q\omega; q_1\omega; q_20)$ diverges as $\omega \to 2\Delta$; however, as already mentioned, our investigation is valid for frequencies $|\omega - 2\Delta| > \tau^{-1}$. For frequencies $|\omega - 2\Delta| < \tau^{-1}$ it is necessary to take collisions explicitly into account; therefore, in our formulas α cannot be larger than $\Delta\tau$ at $\Delta\tau \gg 1$.

The vector potential of the electromagnetic field in (5) is determined in the linear approximation, i.e., for Pippard superconductors

$$|\mathbf{A}(q\omega)| = \frac{2H_\omega |q|}{|q|^3 + q_\omega^3} = 2H_\omega Y(q, \omega), \tag{7}$$

where q_ω is the characteristic Pippard momentum[6], which generally depends on the frequency. Substituting (7) in (5) we find

$$L_1(q,\omega)\Delta_1(q,\omega) = -\frac{4\pi^2 e^2 m^2 v^2}{c^2 q_0^4}(\mathbf{H}_0\mathbf{H}_\omega)\,\mathrm{th}\frac{\Delta}{2T}\ln\alpha F_1\left(\frac{q}{q_0}\right), \tag{8}$$

where at $z > 0$ we have

$$F_1(z) = \int_0^\infty \frac{x\,dx}{(x^3+1)[(x+z)^3+\beta^3]} - \int_0^\infty \frac{(z-x)^2 x\,dx}{(z^2+x^2)(x^3+1)[|x-z|^3+\beta^3]}. \tag{9}$$

Here $F_1(-z) = F_1(z)$ and $\beta = q_\omega/q_0$; the quantity $F_1(z)$ can be expressed in terms of elementary functions; however, we confine ourselves to the following limiting values:

$$F_1(z) = \begin{cases} 4\pi/3\sqrt{3}\,\beta(1+\beta+\beta^2), & z=0 \\ 2/z, & z \gg \beta \end{cases}. \tag{10}$$

For large wave vectors $qv > \Delta$, T the quantity $L_1(q, \omega)$ essentially does not depend on ω or T and is determined by its static value:[6]

$$L_1(q, \omega) = 8\pi m^2 v \ln(q\xi_0) = 2\pi m^2 v \Phi(q, \omega),$$

where ξ_0 is the coherence length at $T = 0$. The dependence on the wave vector is slight. The situation becomes complicated at small $qv < \Delta$, and it is no longer possible to calculate $L_1(q, \omega)$ at arbitrary temperatures.

First let us consider the case $T \ll T_c$. Assuming $\tanh(\epsilon/2T) \approx 1$ in Eq. (2), let us first evaluate the integral with respect to η:

$$\Phi(q,\omega) = \frac{2}{\Delta}\int_0^1 dt\left\{-\frac{q^2v^2t^2}{2\Delta} + \pi\sqrt{q^2v^2t^2+4\Delta^2-\omega^2}\,[\Theta(q^2v^2t^2+4\Delta^2-\omega^2) - i\Theta(\omega^2-4\Delta^2-q^2v^2t^2)]\right\} \tag{11}$$

for $qv < \Delta$ and $|\omega - 2\Delta| < \Delta$. Integrating with respect to t in Eq. (11) we find

$$\Phi(q,\omega) = 2\alpha^{1/2}\left\{-\frac{2}{3}\alpha^{1/2}x^2 + \pi[x^2+\mathrm{sign}(2\Delta-\omega)]^{1/2} + \frac{\pi}{x}\mathrm{sign}(2\Delta-\omega)\ln|x^2 + [x^2+\mathrm{sign}(2\Delta-\omega)]^{1/2}| - \frac{i\pi^2}{2x}\Theta(\omega-2\Delta)\begin{bmatrix}1, & x>1\\ 2\pi^{-1}\arcsin x, & x<1\end{bmatrix}\right\}, \tag{12}$$

where $x = qv|\omega^2 - 4\Delta^2|^{-1/2}$. For negative values of the expression under the radical sign in (12), it is necessary to set the corresponding terms equal to zero.

In the other limiting case, $T \sim T_c$ and $\Delta < T_c$, we divide the entire range of integration over η into two parts: $|\eta| < 2\Delta$ and $|\eta| > 2\Delta$. In the first region we set $\tanh(\epsilon/2T) \sim \epsilon/2T$ and integrate over η. Making use of the condition $\Delta > qv$, we first integrate with respect to the polar angle and then with respect to η in the second region, and as a result we find for $\Phi(q, \omega)$ an expression that coincides with (12) after multiplying the entire expression (12) by $\Delta/2T$, and after additionally multiplying the first term in the square brackets by

$$\gamma = -4T\Delta\int_{2\Delta}^\infty \frac{d\eta}{\varepsilon^3}\,\mathrm{th}\frac{\varepsilon}{2T}. \tag{13}$$

It is clear that the quantity $L_1(q, \omega)$ becomes small at small values of qv and $|\omega - 2\Delta| < \Delta$, which leads to an increase of $\Delta_1(q, \omega)$ as $q \to 0$. This is due to the existence of a solution with $\omega = 2\Delta$ and $q = 0$ satisfying the dispersion equation $L_1(q, \omega) = 0$ for the natural vibrations of the superconducting order parameter in pure superconductors.[1]

Expressing Δ in terms of the critical magnetic field and utilizing the relationship (see[6])

$$q_0^3 = \pi m^2 v^2 c^{-2}\Delta\,\mathrm{th}(\Delta/2T),$$

we find

$$\frac{\Delta_1(q,\omega)}{\Delta} = -\frac{4}{\pi^2}\frac{(H_0H_\omega)}{H_c^2 q_0}\Phi^{-1}(q,\omega)\ln\alpha F_1\left(\frac{q}{q_0}\right) \tag{14}$$

for $T \ll T_c$. An additional factor $1 - (T/T_c)$ appears near the critical temperature in the Pippard region ($\kappa^2 < 1 - T/T \ll 1$). Comparison with the results of[6] indicates that the principal difference in the high-frequency case at large wave vectors lies in the appearance of the logarithmic singularity. However, at small q the difference from the static case is more substantial; in particular, $\Delta_1(q, \omega)$ becomes a complex quantity and at $qv > \Delta^{1/2}|\omega - 2\Delta|^{1/2}$ we have $\Delta_1(q, \omega) \sim q^{-1}$, which leads to an anomalously large variation of $\Delta_1(x)$.[5]

We note that $\Delta_1(q, 0) \approx H_\omega H_{-\omega}$ also diverges logarithmically as $\omega \to 2\Delta$. The expression for $\Delta_1(q, 0)$ is obtained from (8) if we set $q_0 = q_\omega$ in Eq. (8), and the function $F_1(z)$ is replaced by

$$F_1'(z) = \int_0^\infty \frac{(z+x)(z+2x)x\,dx}{[x^2+(x+z)^2](1+x^3)(1+(x+z)^3)} - \frac{1}{z}\int_0^z \frac{x(z-x)\,dx}{(1+x^3)(1+(z-x)^3)} + \int_z^\infty \frac{(x-z)(2x-z)\,dx}{(1+x^3)(1+(x-z)^3)(x^2+(x-z)^2)}. \tag{9'}$$

Now let us consider the surface impedance of a superconductor in a constant magnetic field. To determine the correction proportional to H_0^2 to the impedance is it necessary to determine the first nonlinear term $j_3(q, \omega)$ in the expansion of the current density in powers of $\mathbf{A}$. The complete expression for $j_3(q, \omega)$ can be divided into two parts,[6] one of which does not depend explicitly on $\Delta_1(q, \omega)$ and the other is proportional to $\Delta_1(q, \omega)$. Following[4], we shall write out only the latter part, which turns out to be decisive in the limit as $\omega \to 2\Delta$:

$$j_3(q,\omega) = \frac{1}{2\pi}\int L_3(q\omega; q_1\omega_1; q_2\omega_2)\Delta_1(q_1,\omega_1)\delta(q-q_1-q_2)\times\delta(\omega-\omega_1-\omega_2)\,dq_1 dq_2 d\omega_1 d\omega_2, \tag{15}$$

$$\begin{aligned} L_3(q\omega; q_1\omega_1; q_2\omega_2) = \int \frac{d^3p}{(2\pi)^3}\frac{ie^2p_x\Delta}{3m^2c}(\mathbf{A}(q_2\omega_2)\mathbf{p})\{[4\varepsilon^2 - 2\varepsilon(\omega+\omega_1) \\ +2\eta(\mathbf{qv}) + (\mathbf{qv})(\mathbf{q_1v})]N(q_1,-\omega_1)N(q,-\omega) + [4\varepsilon^2 - 2\varepsilon(\omega+\omega_2) + \omega\omega_2 \\ -(\mathbf{qv})(\mathbf{q_2v})]N(q_2,-\omega_2)N(q,\omega) + [4\varepsilon^2 + 2\varepsilon(\omega_1-\omega_2) - \omega_1\omega_2 - 2\eta(\mathbf{q_2v}) \\ -(\mathbf{q_2v})(\mathbf{qv})]N(q_1,\omega_1)N(-q_2,-\omega_2) + \\ +(\omega\to-\omega, \omega_1\to-\omega_1, \omega_2\to-\omega_2)\}\varepsilon^{-1}\,\mathrm{th}(\varepsilon/2T). \end{aligned} \tag{16}$$

We shall show that $L_3(q\omega; q_1\omega; q_20)$ contains $\ln\alpha$ as $\omega \to 2\Delta$; hence it follows that $j_3(q, \omega)$ is proportional to the square of a large logarithm, whereas that term in the formula for $j_3(q, \omega)$ which does not explicitly depend on Δ_1 gives only the first power of the logarithm, as can be verified by direct calculation. In view of the cumbersome nature of these calculations, we shall not present them here. Setting $\omega_2 = 0$ in Eq. (16), i.e., assuming the field $\mathbf{A}(q_2\omega_2)$ to be static, we find

$$L_3(q\omega;\, q_1\omega;\, q_20) = -\frac{ie^2m^2v^2H_0}{12\pi c}\,\mathrm{th}\frac{\Delta}{2T}\ln\alpha Y(q_20)$$
$$\times\left\{\frac{q}{|q|}\left(\frac{1}{q_2}+\frac{q_2}{q^2+q_1^2}\right)-\frac{q_1}{|q_1|}\left(\frac{1}{q_2}+\frac{q+q_1}{q^2+q_1^2}\right)\right\}.$$

For $T \ll T_c$, the following expression is obtained for the current density:

$$j_3(q,\omega) = \frac{icH_0(\mathbf{H}_0\mathbf{H}_\omega)}{6\pi^5H_c^2}\ln^2\alpha F_3\left(\frac{q}{q_0}\right), \qquad (18)$$

where

$$F_3(z) = \int dxdy\,\delta(x+y-z)\left\{\frac{z}{|z|}\left(\frac{1}{y}+\frac{y}{z^2+x^2}\right)\right.$$
$$\left.-\frac{x}{|x|}\left(\frac{1}{y}+\frac{z+x}{z^2+x^2}\right)\right\}Y(y,0)F_1(x)\Phi^{-1}(q_0x,\omega). \qquad (19)$$

For $x \to 0$, $\Phi^{-1}(q_0x, \omega)$ diverges as $\omega \to 2\Delta$, and within logarithmic accuracy the contribution of the point $x = 0$ to the value of the function $F_3(z)$ is given by

$$\frac{4F_1(0)\Delta}{\pi(z^2+1)q_0v}\{\ln\alpha+3i\}. \qquad (20)$$

At $T \approx T_c$ a factor $1 - (T/T_c)$ appears in (18), expression (20) is multiplied by $2T/\Delta$, and the α in expression (20) is replaced by $|\alpha\gamma|$. The imaginary part in (20) is small in comparison with the real part; however, we shall not neglect these small terms since they lead to a change in the real part of the impedance. The values $x \approx 1$ in the integral (19) will give the following contribution to the value of the function $F_3(z)$ for $z < 1$:

$$F_1(0)z(1-\ln z)\ln^{-1}(q_0\xi_0). \qquad (20')$$

It is seen that, for clearly expressed type-I superconductors at not too small values of α, expression (20′) will have larger values than those given by (20).

It is convenient to express the impedance change $Z(H_0) - Z(0)$ in the presence of a magnetic field in terms of the complex quantity[4]

$$\delta_1(\omega) = -\frac{c}{3i\omega}[Z(H_0)-Z(0)] = \frac{4}{3icH_\omega}\int dqY(q,\omega)j_3(q,\omega). \qquad (21)$$

At $\omega = 0$ the quantity $\delta_1(0)$ is purely real and has the meaning of the correction to the penetration depth of a static field into the superconductor. Substituting expression (18) in (21), we find

$$\frac{\delta_1(\omega)}{\delta_0} = \frac{H_0^2}{\sqrt{3}\,\pi^5H_c^2}\ln^2\alpha\int_0^\infty\frac{xF_3(x)\,dx}{x^3+\beta^3} \qquad (22)$$

at $T \ll T_c$, where $\delta = 4/3\sqrt{3}q_0$ is the penetration depth of a static magnetic field. If $F_3(x)$ is determined by expression (20), then $\delta_1(\omega)$ is proportional to $\ln^3\alpha$; in the case when expression (20′) is largest, $\delta_1(\omega) \sim \ln^2\alpha$ and the complete effect is entirely connected with the singularity in the number density of states at $\epsilon = \Delta$.

Comparing our results with the static case,[6] we see that the principal difference as $\omega \to 2\Delta$ is the appearance of $\ln^2\alpha$ in expression (22) for clearly pronounced Pippard superconductors with $\kappa \ll 1$. We note that similar expressions, which we shall not present here, are obtained for the dependence of the impedance on H_ω^2 in the absence of a constant magnetic field as $\omega \to 2\Delta$. The variation of the impedance in the presence of a constant magnetic field has been investigated experimentally in a number of articles (see, for example,[7]); however, as a rule these investigations were for $\omega \ll 2\Delta$. The case $\omega \approx 2\Delta$ was studied in[8]; there, however, only the variation of the absorption in the presence of a field H_0 was investigated, whereas a noticeable increase of the correction to the impedance can be expected only for the imaginary part, i.e., only for the variation of the penetration depth.

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