The connection between various inclusive processes in quantum electrodynamics

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The connection between inclusive spectra $f(x, k_\perp) = d\sigma/\partial x \partial^2 k_\perp$ is considered. For the conjugate reactions $a + b \to c$ + anything and $a + \bar{c} \to \bar{b}$ + anything, as calculated by means of inelastic Born amplitudes, the relation $f_{b\to c}(x, k_\perp, s) = -x^{-1} f_{\bar{c}\to\bar{b}}(1/x, k_\perp/x, -xs)$ is obtained.

1. In the theory of hadron interactions at high energies there has been increasing interest in recent years in the study of inelastic processes with so-called inclusive experimental arrangements, in which one measures only the momenta of certain final particles, with arbitrary momenta of the other observed particles.[1]

In quantum electrodynamics, owing to the fact that the photon mass is zero, any observable process is necessarily inclusive, since it is accompanied by an infinite number of undetected photons. Recently a number of inclusive processes occurring in colliding beams have been investigated.[2-5]

In the present paper we show that inclusive cross sections calculated by means of inelastic amplitudes in the Born approximation and differing from each other by the interchange of an initial particle with one of the final particles are connected by simple relations, Eqs. (21) and (22). These relations considerably reduce the volume of the calculations, and are useful for securing consistency in calculations of various processes in quantum electrodynamics. Radiation corrections break these relations.

In describing hadron scattering processes at high energies, multiperipheral models are used with success. The relations (21) and (22) hold for these models, and are indeed simplified owing to scaling invariance. Reactions in which these relations are violated consequently cannot be described in the framework of simple multiperipheral models. Analogous relations for the matrix elements of processes in quantum electrodynamics have been widely used, in particular in papers by Bethe, Maximon, and Olsen.[6] They are first mentioned in the literature, under the name of the "substitution law," in the book of Jauch and Rohrlich.[7]

In Sec. 2 relations connecting the cross sections of conjugated processes are derived. In Sec. 3 these relations are illustrated with the examples of the following conjugated reactions:

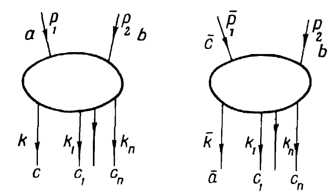
1) $e^+e^- \to e^+e^-\gamma$ and $e^+\gamma \to e^+e^+e^-$,

2) $e^+e^- \to e^+e^-\gamma\gamma$, $e^+\gamma \to e^+e^+e^-\gamma$ and $\gamma\gamma \to e^+e^+e^-e^-$,

and also in terms of the symmetry properties of the reaction $e^+e^- \to e^+e^+e^-e^-$.

In conclusion, in Sec. 4 we discuss the possibility of using the relations to analyze inclusive spectra involving hadrons, and also consider the causes of breaking the relations, by taking non-Born diagrams into account.

2. Let us consider the reaction $a + b \to c + c_1 + \ldots + c_n$, and the reaction $b + \bar{c} \to \bar{a} + c_1 + \ldots + c_n$:



(1)

We shall suppose that $s = (p_1 + p_2)^2 \gg m_1^2$, and also that $-s = -(-k + p_2)^2 \gg m_1^2$, where

$$\bar{p}_1 = -k, \quad \bar{k} = -p_1. \tag{1a}$$

Let $M_{a\to c}$ be the amplitude of the first reaction, and let $M_{\bar{c}\to\bar{a}}$ be that of the second. Then the matrix elements are connected by the crossing relation

$$M_{\bar{c}\to\bar{a}}(\bar{p}_1, \bar{k}) = M_{a\to c}(p_1, k). \tag{2}$$

We note that the relation (22) connects two amplitudes in the physical and in the nonphysical regions of the respective reactions. In the Born approximation the analytic continuation of Eq. (2) is trivial. We are interested in the behavior of the cross sections at large energies, so that it is convenient to work in the Sudakov variables[1)] α, β, $k_\perp$,[8] where[2)]

$$k = \alpha p_2' + \beta p_1' + k_\perp, \quad k_i = \alpha_i p_2' + \beta_i p_1' + k_{i\perp},$$
$$p_1' = p_1 - \frac{m^2}{s} p_2, \quad p_2' = p_2 - \frac{m^2}{s} p_1, \tag{3}$$
$$k_\perp p_1' = k_\perp p_2' = 0, \quad k_{\perp i} p_1' = k_{\perp i} p_2' = 0.$$

Let us also introduce Sudakov's parametrization for the momenta of the particles in the conjugated reaction:

$$\bar{k} = \bar{\alpha} \bar{p}_2' + \bar{\beta} \bar{p}_1' + \bar{k}_\perp, \quad k_i = \alpha_i' \bar{p}_2' + \beta_i' \bar{p}_1' + k_{i\perp}', \ \bar{p}_2 = p_2. \tag{4}$$

We now find the law of transformation of the Sudakov variables for the change to the conjugated reaction. Expressing the vector p_1 in terms of the Sudakov parameters of the conjugated reaction:

$$p_1 = -\bar{\alpha} \bar{p}_2' + \bar{\beta} k' - \bar{k}_\perp, \quad \bar{p}_2' \approx p_2,$$
$$k' = k - m^2 p_2'/2kp_2, \tag{5}$$

and then going back again to the Sudakov variables of the direct reaction:

$$p_1' - \frac{m^2}{s} p_2' = p_1 = -\bar{\alpha} p_2' - \bar{k}_\perp + \bar{\beta}\left(\alpha p_2' + \beta p_1' + k_\perp - \frac{m^2 p_2'}{2kp_2}\right) \tag{6}$$

and comparing the coefficients of p_1', p_2', and $k_\perp$, we get

$$\bar{\beta} = \frac{1}{\beta}, \quad \bar{k}_\perp = \bar{\beta} k_\perp = \frac{k_\perp}{\beta}, \tag{7}$$

$$\bar{\alpha} = \frac{\alpha}{\beta} + \frac{\beta^2 - 1}{\beta^2 s}, \quad \bar{s} = -s\beta. \tag{8}$$

Carrying out analogous operations for the vectors k_i, we get

$$k_{i\perp}'=k_{i\perp}-\frac{\beta_i}{\beta}k_\perp,\quad \beta_i'=-\frac{\beta_i}{\beta},\quad \alpha_i'=\alpha_i-\beta_i\bar{\alpha}. \tag{9}$$

Accordingly, in Sudakov variables the crossing relation (2) takes on the following appearance:

$$M_{a\to c}(k_\perp,\beta,k_{i\perp},\beta_i,s)=M_{\bar{c}\to\bar{a}}(\bar{k}_\perp,\bar{\beta},k_{i\perp}',\beta_i',-\beta s), \tag{10}$$

where the variables on the right-hand side are defined by Eqs. (7)—(9).

Let us now examine the connection between the cross sections of conjugated processes.[3)] To do so we write the expression for the cross section in Sudakov variables:

$$d\sigma_{a\to c}=\frac{1}{2s}|M_{a\to c}|^2\frac{s}{2}d\alpha\,d\beta\,d^2k_\perp\frac{\delta^+(k^2-m^2)}{(2\pi)^3}\prod_{i=1}^{n}\frac{s}{2}d\alpha_i d\beta_i$$

$$\times\frac{\delta^+(k_i^2-m^2)}{(2\pi)^3}d^2k_{i\perp}\frac{2}{s}(2\pi)^4\delta\Big(\sum_{i=1}^{n}k_{i\perp}+k_\perp\Big)\delta\Big(\sum_{i=1}^{n}\beta_i+\beta-1\Big)\delta\Big(\sum_{i=1}^{n}\alpha_i+\alpha-1\Big)$$

$$=\frac{1}{2s}|M_{a\to c}|^2d\beta\frac{\theta(\beta)}{2\beta(2\pi)^3}d^2k_\perp\prod_{i=1}^{n-1}d\beta_i\,d^2k_i\frac{\theta(\beta_i)}{2\beta_i(2\pi)^3}$$

$$\times\frac{2\cdot 2\pi}{s}\theta\Big(1-\beta-\sum_{i=1}^{n-1}\beta_i\Big)\Big[1-\beta-\sum_{i=1}^{n-1}\beta_i\Big]^{-1}\delta\Big(\frac{m^2-k_\perp^2}{s\beta}+\sum_{i=1}^{n-1}\frac{m^2-k_{i\perp}^2}{s\beta_i} \tag{11}$$

$$+\Big[m^2-\Big(\sum_{i=1}^{n-1}k_{i\perp}+k_\perp\Big)^2\Big]\Big[s\Big(1-\beta-\sum_{i=1}^{n-1}\beta_i\Big)\Big]^{-1}-1\Big)$$

$$\equiv f_{a\to c}(\beta,\beta_i,k_\perp,k_{i\perp},s)\,d\beta\,d^2k_\perp\theta(\beta)$$

$$\times\prod_{i=1}^{n-1}d\beta_i\,d^2k_{i\perp}\theta(\beta_i)\theta\Big(1-\beta-\sum_{i=1}^{n-1}\beta\Big).$$

Let us compare $f_{a\to c}(\beta,\beta_i,k_\perp,k_{i\perp},s)$ and $f_{\bar{c}\to\bar{a}}(\bar{\beta},\beta_i',k_{i\perp}',\bar{k}_\perp,\bar{s})$. We have

$$f_{\bar{c}\to\bar{a}}(\bar{\beta},\beta_i',\bar{k}_\perp,k_i',\bar{s})=\frac{1}{2|\bar{s}|}|M_{\bar{c}\to\bar{a}}|^2 \tag{12}$$

$$\times\frac{1}{(2\pi)^3\cdot 2\bar{\beta}}\prod_{i=1}^{n-1}\frac{1}{2(2\pi)^3\beta_i'}\frac{4\pi}{|\bar{s}|}\Big[1-\bar{\beta}-\sum_{i=1}^{n-1}\bar{\beta}_i\Big]^{-1}.$$

$$\times\delta\Big(-1+\frac{m^2-\bar{k}_\perp^2}{\bar{s}\bar{\beta}}+\sum_{i=1}^{n-1}\frac{m^2-k_{i\perp}'^2}{\bar{s}\beta_i'}$$

$$+\Big[m^2-\Big(\sum_{i=1}^{n-1}k_{i\perp}'+k_\perp\Big)^2\Big]\Big[\bar{s}\Big(1-\bar{\beta}-\sum_{i=1}^{n-1}\bar{\beta}_i\Big)\Big]^{-1}\Big).$$

Substituting Eqs. (8) and (9) in this equation, we get

$$f_{a\to c}(\beta,\beta_i,k_\perp,k_{i\perp},s)=\Big(-\frac{1}{\beta}\Big)^n f_{\bar{c}\to\bar{a}}(\bar{\beta},\beta_i',\bar{k}_\perp,k_i',\bar{s}). \tag{13}$$

Let us denote the differential inclusive cross section $d\sigma/\partial\beta\partial\beta_i\partial^2k_\perp\partial^2k_{i\perp}$ for the process

$$a+b\to c+\sum_{i=1}^{m}c_i+\text{ anything}$$

by $\varphi_{a\to c+m}$. Using the fact that the dependence of $k_{i\perp}'$ on $k_{i\perp}$ and that of β_i' on β_i are linear [see (9)], and also that in Eqs. (11) and (12) the integration over $k_{i\perp}$ is taken over the entire plane, we have, integrating Eq. (13),

$$\varphi_{a\to c+m}(\beta,\beta_i,k_\perp,k_{i\perp},s)=\Big(-\frac{1}{\beta}\Big)^{m+1}\varphi_{\bar{c}\to\bar{a}+m}(\bar{\beta},\beta_i',\bar{k}_\perp,k_{i\perp}',\bar{s}). \tag{14}$$

In particular, for a one-particle inclusive process the relation will be

$$\varphi_{a\to c}(\beta,k_\perp,s)=-\frac{1}{\beta}\varphi_{\bar{c}\to\bar{a}}\Big(\frac{1}{\beta},\frac{k_\perp}{\beta}\;-\beta s\Big), \tag{15}$$

and after integration over $k_\perp$ we get

$$\frac{d\sigma_{a\to c}}{d\beta}\equiv\tilde{\varphi}_{a\to c}(\beta,s)=-\beta\tilde{\varphi}_{\bar{c}\to\bar{a}}\Big(\frac{1}{\beta},-\beta s\Big). \tag{16}$$

Let us denote by $\tilde{\varphi}_{a\to c+m}$ the inclusive distribution in fractions of the energy. Then

$$\tilde{\varphi}_{a\to c+m}(\beta,\beta_1,\beta_m,s)=(-1)^{m+1}\frac{1}{\beta^{m-1}}\tilde{\varphi}_{\bar{c}\to\bar{a}+m}(\bar{\beta},\beta_1',\beta_n',\bar{s}). \tag{17}$$

If the particles have spin, then one must average over spins of the initial particles and sum over spins of the final particles. This means that in the right-hand sides of Eqs. (13)—(17) one must insert a factor $\eta = s_a^{-1}s_c^{+1}$, where s_a and s_c are the numbers of spin states of the respective particles a and c. Besides this, if the particles a and c have different statistics we must multiply the right-hand sides of Eqs. (13)—(17) by −1, since the density matrices of a fermion and an antifermion differ in sign [cf., e.g., the connection further on between Eqs. (25a) and (25b)].

We shall now show that the relations (15) and (16) remain the same if there are identical particles, and see how this alters the other relations. We call attention to the fact that this is the main point of difference between inclusive and exclusive cross sections.

Let us consider the exclusive reaction $b+\bar{\mu}\to(n+1)N+m\mu$, where μ and N are particles of different kinds, and the conjugated reaction $b+\bar{N}\to nN+(m+1)\mu$. Here Eq. (13) is altered in the following way:

$$(n+1)f_{\bar{\mu}\to(n+1)N+m\mu}=(m+1)\Big(-\frac{1}{\beta}\Big)^{n+m+1}f_{\bar{N}\to nN+(m+1)\mu}, \tag{18}$$

since there is a factor $(k!)^{-1}$ in the definition of the cross section for the production of k identical particles.

We shall show that the inclusive reactions

$$b+\bar{\mu}\to(n+1)N+m\mu+\text{anything} \tag{19a}$$

$$b+\bar{N}\to nN+(m+1)\mu+\text{anything} \tag{19b}$$

are connected by a relation analogous to (18).

Consider the contributions of the exclusive cross sections

$$b+\bar{\mu}\to(n+1+r_1)N+(m+r_2)\mu \tag{20a}$$

to the inclusive process (19a), and of those of the process

$$b+\bar{N}\to(r_1+n)N+(m+1+r_2)\mu \tag{20b}$$

to the inclusive process (19b). It is obvious that in (20a) we must take each cross section with the weight $C_{n+1+r_1}^{n+1}$, $C_{m+r_2}^{m}$, and also those in (20b) with the weight $C_{r_1+n}^{n}C_{m+1+r_2}^{m+1}$. Using (18), we have after ingrating over all unobserved particles:

$$\varphi_{\bar{\mu}\to(n+1)N+m\mu}=\Big(-\frac{1}{\beta}\Big)^{n+m+1}\varphi_{\bar{N}\to nN+(m+1)\mu}\frac{C_{r_1+n}^{n}C_{m+1+r_2}^{m+1}(n+1+r_1)}{C_{n+1+r_1}^{n+1}(r_2+m+1)C_{m+r_2}^{m}}$$

$$=\Big(-\frac{1}{\beta}\Big)^{n+m+1}\varphi_{\bar{N}\to nN+(m+1)\mu}\frac{n+1}{m+1}.$$

Thus we get the following connection:

$$\begin{aligned}&(m+1)\varphi_{\bar{\mu}\to(n+1)N+m\mu}(\beta,k_\perp,\beta_i,k_{i\perp},s)\\&=\Big(-\frac{1}{\beta}\Big)^{n+m+1}(n+1)\varphi_{\bar{N}\to nN+(m+1)\mu}(\bar{\beta},\bar{k}_\perp,\beta_i',k_{i\perp}',\bar{s})\end{aligned} \tag{21}$$

and, in particular, for the case of an ordinary inclusive reaction $a + b \to c$ + anything (i.e., $m = n = 0$), Eqs. (15) and (16) remain valid. In the case when the particle c to be detected is an antiparticle relative to a, the functions $\tilde{\varphi}_{a\to c}$ and $\tilde{\varphi}_{\bar{c}\to\bar{a}}$ coincide. Denoting both of them by χ, we have:

$$\chi(x, s) = -x\chi(1/x, -xs). \tag{22}$$

These relations (13)–(17), (21), (22) determine the behavior of the cross section of the conjugated reaction in the nonphysical region. In particular, in the relation (22) the physical region of variation of x, $0 < x < 1$, is connected with the nonphysical region, $\bar{x} > 1$, and consequently it becomes necessary to continue through the point $x = 1$. Analogously, in Eq. (21) β_i must be continued from the region $\beta_i > 0$ into the region $\beta_i < 0$. In such an analytical continuation there are, first, changes of the signs of s and u, and also of their relative value, and second, (p_1, k_i) and (k, k_i) change their signs. Accordingly, there can be obstacles to the analytical continuation (in Regge-pole language) as follows: 1) singularities of the inelastic amplitudes, associated with signature phenomena in s, 2) singularities at $x = 1$, associated with the three-reggeon limit in the reaction $a + b \to c$ + anything, and corresponding signature factors for these reggeons, 3) low-energy singularities, associated with thresholds in the final state with respect to (k, k_i) and (p_i, k_i). In the Born approximation for the inelastic amplitudes all of these singularities are absent.

In the higher approximations, in all of the diagrams we have checked, the singularities of the first and second types are only logarithmic, and the correct answer is evidently obtained if in the continuation these logarithms are understood in the arithmetic sense (cf.[9]). Singularities of the third type can seriously disrupt the entire procedure (see Sec. 4).

Let us examine some consequences of the relation (22) under the condition that for $s \to \infty$ the function χ becomes no longer dependent on s and that χ is analytic at the point $x = 1$. Then the general solution of the functional relation (22) can be written in the form

$$\chi(x) = (1-x)\varphi((1-x)^2/x), \tag{23}$$

where $\varphi(\gamma)$ is analytic at $\gamma = 0$ and is positive for $\gamma > 0$ (cf.[10]).

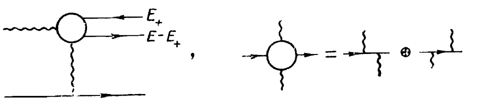
Let $\varphi(\gamma) = a + b\gamma$ for small γ; then to accuracy up to $(1-x)^4$ we have

$$\chi(x) = a(1-x) + b(1-x)^3 + b(1-x)^4.$$

In the general case the relation (23) requires that χ vanish as an odd power of $1 - x$.

3. We shall demonstrate the usefulness of the relations we have derived by using them to calculate some electrodynamic processes.[4)]

Let us consider the differential distribution in the energy fraction $x = E_+/E$ received by the positron from the photon in the process of photoproduction of a pair e^+e^- on an electron:

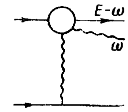


(24a)

This quantity has the form

$$d\sigma/dx = 2\alpha r_0^2[\ln(s^2x^2(1-x)^2) - 1] \cdot [1 - {}^4/_3 x(1-x)], \quad m_e = 1. \tag{25a}$$

By means of the relation (16) we shall find the spectrum in the fraction of the original electron energy which is carried away by the photon in the process of single bremsstrahlung in the collision of an electron with a nucleus[11]:

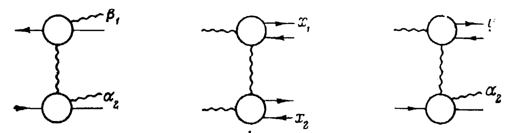


(24b)

(We note that in the right member of Eq. (16) we must change the sign because in this case the initial and final particles have different statistics.) We get

$$\frac{d\sigma}{dy} = 2\alpha r_0^2 \frac{1}{y}\left[\ln\left(s^2\frac{(1-y)^2}{y^2}\right) - 1\right]\left[y^2 + \frac{4}{3}(1-y)\right]. \tag{25b}$$

Let us now consider the process of double bremsstrahlung in opposite directions in the c.m.s. of colliding electrons:



(26)

The differential distribution in the fractions x_1 and x_2 of the initial energy of the e^+ and e^- that are carried off by the photons, for process a in (26) is of the form[12]

$$\frac{d^2\sigma}{d\beta_1\, d\alpha_2} = \frac{8\alpha^2 r_0^2}{\pi}\left[\eta_1\left(\frac{1}{\alpha_2} - 1\right)\left(\frac{1}{\beta_1} - 1\right) + \eta_2\alpha_2\beta_1 + \eta_3\left(\beta_1\left(\frac{1}{\alpha_2} - 1\right) + \alpha_2\left(\frac{1}{\beta_1} - 1\right)\right)\right], \tag{27a}$$

$$\eta_1 = \frac{5}{4} + \frac{7}{8}\xi(3), \quad \eta_2 = \frac{7}{8}\xi(3), \quad \eta_3 = \frac{1}{2} + \frac{7}{8}\xi(3).$$

Multiplying the right-hand side of Eq. (27a) by x_1x_2 and making the substitution $\alpha_2 \to x_1^{-1}$, $\beta_1 \to x_2^{-1}$, [see Eq. (16)], we obtain the differential distribution in the fractions of the energy carried off by the positrons of the final pairs in the process of production of two pairs by two photons [see diagram b of (26)]:

$$\frac{d^2\sigma}{dx_1\, dx_2} = \frac{8\alpha^2 r_0^2}{\pi}\left[\eta_1 x_1(1-x_1)x_2(1-x_2) + \eta_2 - \eta_3(x_1(1-x_1) + x_2(1-x_2))\right]. \tag{27b}$$

Integration of this expression over x_1 and x_2 gives the total cross section for production of two pairs by two photons[13]:

$$\sigma_{\gamma\gamma\to e^+e^+e^-e^-} = \frac{\alpha^4}{\pi}\left(\frac{175}{36}\xi(3) - \frac{19}{18}\right). \tag{28}$$

From the differential cross section for double bremsstrahlung, Eq. (27a), we can also obtain the differential cross section for production of a pair by a photon on an electron with the emission of an additional photon [diagram c of (26)]. To do so we must multiply the right-hand side of Eq. (27a) by y and make the substitution $\beta_1 \to y^{-1}$; the result is

$$\frac{d\sigma_{\gamma e^+\to\gamma e^+e^+e^-}}{dy\, d\alpha_2} = -\frac{8\alpha^4}{\pi}\left[\eta_1\left(\frac{1}{\alpha_2} - 1\right)y(1-y) - \eta_2\alpha_2 - \eta_3\left(\frac{1}{\alpha_2} - 1 - \alpha_2 y(1-y)\right)\right]. \tag{27c}$$

We note that the cross section (25b) remains unchanged if we multiply the right-hand side by $-(1-y)$ and everywhere replace $(1-y)$ with $(1-y)^{-1}$. The cross section (27b) is unchanged if we perform the same operation with $(1-x)$, and the cross sections (27a) and (27c) remain unchanged under the same operation with $(1-\beta_1)$ and $(1-y)$, respectively. This is a consequence of the relation (22) and the fact that we are considering the amplitude in the Born approximation, in which the cross sections for interaction of particle a with particle b and for interaction of the antiparticle a with the particle b are equal (in the limit as $s \to \infty$).

With increase of the number of particles in the final state the number of symmetry relations for the differential cross sections becomes larger. As an example let us consider the process $e^+e^- \to e^+e^+e^-e^-$ under kinematic conditions with the new pair moving in the direction of the electron. In the Weizsäcker-Williams approximation the differential cross section in the energy fractions $x_1 = E_1/E$, $x_2 = E_2/E$ of the final electrons in the c.m.s. system can be written in the following form:

$$d\sigma/dx_1dx_2 = f(x_1, x_2)\ln s, \tag{29}$$

where $f(x_1, x_2)$ satisfies the following symmetry relations, owing to (17):

$$f(x_1,x_2)=f(x_2,x_1)=f\left(\frac{x_1}{\Delta},\frac{x_2}{\Delta}\right) = f\left(\frac{1}{x_1},\frac{\Delta}{x_1}\right)=f\left(\frac{1}{x_2},\frac{\Delta}{x_2}\right), \tag{30}$$

where $\Delta = x_1 + x_2 - 1$.

There are contributions to $f(x_1, x_2)$ from three types of diagrams: Bethe-Heitler diagrams,[5,14] deceleration diagrams,[15] and interference diagrams[4] which arise from the fact that the final electrons are identical:

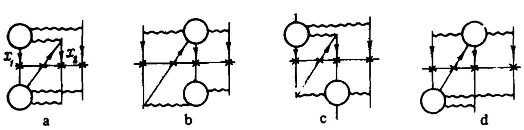
$$f(x_1, x_2) = f_{\rm BH}(x_1, x_2) + f_{\rm int}(x_1, x_2) + f_{\rm dec}(x_1, x_2). \tag{31}$$

Under the transformation $x_1 \to x_1/\Delta$, $x_2 \to x_2/\Delta$ the Bethe-Heitler diagrams go over into deceleration diagrams, so that

$$f_{\rm BH}(x_1, x_2) = f_\tau\ (x_1/\Delta, x_2/\Delta). \tag{32}$$

Consequently, to find the contribution of the deceleration mechanism it suffices to know that of the Bethe-Heitler mechanism.[14]

The quantity $f_{\rm int}(x_1, x_2)$ is the sum of contributions from four gauge-invariant sets of diagrams:



(33)

or

$$f_{\rm int}(x_1, x_2) = f_a(x_1, x_2) + f_b(x_1, x_2) + f_c(x_1, x_2) + f_d(x_1, x_2).$$

The expression for $f_d(x_1, x_2)$ is given in Appendix 2 of of[4], which also gives a connection between f_a, f_b, f_c, and f_d derived by means of a relation completely analogous to Eq. (17). Thus in this case the relation (17) allows us to shorten the work of calculation by a factor four. We shall not give the expression for $f_{\rm int}(x_1, x_2)$ because of its cumbersomeness; we remark only that it satisfies a relation identical with the relation (30) for f.

4. In this section we shall demonstrate with one example how the relation given above are broken when the radiation corrections to the inelastic amplitudes are taken into account. Let us consider the two conjugated reactions:

$$e^-e^- \to e^- + \text{ anything}; \quad e^+e^- \to e^+ + \text{ anything} \tag{34}$$

The distribution in the fraction Δ of the energy of the initial positron which is carried away by the final positron in the second of the reactions (34) is the same as the analogous distribution for the first reaction, up to very high orders of perturbation theory, in which the inelastic amplitudes with one-photon and with two-photon exchange begin to interfere. Therefore, if we do not go to such high orders, we expect that the simple relation (16) will be valid for each of these reactions:

$$f(\Delta, s) = -\Delta f(1/\Delta, -\Delta s). \tag{16a}$$

We shall show, however, that this relation is violated as soon as we study the radiation corrections to the simplest exclusive process, $e^-e^- \to e^-e^-\gamma$, which leads to a nontrivial dependence of the inclusive cross section on Δ. [We note that the relation (16a) is satisfied for the process $e^-e^- \to e^-e^-\gamma\gamma$ in the Born approximation[15] (see also (27a)).] From the relation (16a) it follows in particular that for small $1 - \Delta$ and large s the cross section should be of the form

$$f(\Delta) = c_1\Delta/(1-\Delta) + c_2(1-\Delta) + O((1-\Delta)^3). \tag{35}$$

This is precisely the form of the cross section for this reaction as calculated in the Born approximation for $s \to \infty$ [cf. Eq. (25b)]. We shall show, however, that the behavior of the first radiation correction $f_1(\Delta)$ for $\Delta \to 1$ is as follows [see Eq. (41) below]:

$$f_1(\Delta)\,|_{\Delta\to1} \to \text{const},$$

and this contradicts the relation (35), which is a consequence of the substitution law, Eq. (16a).

To calculate $f_1(\Delta)$ we use the Weizsäcker-Williams method:

$$d\sigma_{ee\to ee\gamma} = \frac{2\alpha}{\pi}\ln s\int_0^\infty \frac{d\varkappa}{\varkappa}\, d\sigma_{e\gamma\to e\gamma}, \tag{36}$$

where $d\sigma_{e\gamma\to e\gamma}$ is the radiation correction to the differential cross section for elastic γe scattering, the momenta of the electron and photon in the initial state being respectively p and k. The invariant $\kappa = 2(k, p)$ is connected with the energy of the photon in the laboratory system: $\omega = \kappa/2m$. The differential cross section can be expressed in terms of the radiation correction $u_1(\kappa_1, \kappa_2)$ to the amplitude for the Compton effect in the following way:

$$\begin{gathered} d\sigma_{e\gamma\to e\gamma} = -2\alpha r_0^2 \frac{dt}{m^2\varkappa^2} u_1(\varkappa_1, \varkappa_2), \\ u_1(\varkappa, \varkappa_2) = P_1(\varkappa_1, \varkappa_2) + P_1(\varkappa_2, \varkappa_1), \\ \varkappa_1 \equiv \varkappa = 2(k, p), \quad \varkappa_2 = -2(k', p), \\ t = (k-k')^2 = \varkappa_1 + \varkappa_2, \end{gathered} \tag{37}$$

where k' is the momentum of the final photon. The expression for $P_1(\kappa_1, \kappa_2)$ has been calculated by Brown and Feynman.[16] We need only the expression for $d\sigma_{ee\to ee\gamma}$ for $\Delta \to 1$. The condition for the final photon to be real, expressed in Sudakov variables, can be written:

$$1-\Delta = (k_\perp'^2 - \lambda^2)/\varkappa_2, \tag{38}$$

where λ is the photon mass.

It is obvious from Eq. (38) that for $\Delta \to 1$ the square of the momentum transferred from the initial photon to the final photon is much smaller than their energy in the c.m.s. We must consider two regions in the integration over κ:

1) The region $-t \sim m^2 \ll \kappa$ corresponds to the ordinary three-reggeon limit; if the photon has mass $\lambda \sim m$ for $\Delta \to 1$ there is a contribution only from this region, and this contribution is small:

$$\frac{d\sigma}{d\Delta} \sim \int_{m^2/(1-\Delta)}^{\infty} \frac{d\varkappa}{\varkappa}\frac{1}{\varkappa} \sim (1-\Delta)$$

and does not contradict Eq. (35).

2) The region of integration $-t \ll \kappa \sim m^2$, for $\Delta \to 1$, comes in only for $\lambda^2 = 0$, according to Eq. (38). In this region $\kappa_2 \approx -\kappa_1 \equiv -\kappa$, and the expression for the cross section of the Compton effect, Eq. (37), can be written in the following form

$$d\sigma_{e\gamma\to e\gamma} = -2r_0^2\alpha\frac{d\Delta}{\varkappa}u_1(\varkappa, -\varkappa), \quad \Delta\to 1, \tag{39}$$

where in this special case the expression for $u_1(\kappa_1, \kappa_2)$ can be greatly simplified[17]:

$$u_1(\varkappa, -\varkappa) = \varkappa^2\left\{\frac{3-\varkappa^2}{(1-\varkappa^2)^2}\ln\varkappa + \frac{3}{\varkappa^2} + \frac{1}{2\varkappa^2(\varkappa+1)}\right.$$
$$+\frac{1}{2\varkappa^2(1-\varkappa)} + \frac{2}{\varkappa^3}\int_0^{-\varkappa}\frac{d\gamma}{1-\gamma}\ln(-\gamma)\left[-\frac{4}{\varkappa} - 2 + \frac{\varkappa}{2}\right] \tag{40}$$
$$\left. -\frac{2}{\varkappa^3}\int_0^{\varkappa}\frac{d\gamma}{1-\gamma}\ln\gamma\left[\frac{4}{\varkappa} - 2 - \frac{\varkappa}{2}\right]\right\}.$$

Using Eqs. (39) and (40), we can rewrite the expression (36) in the form

$$d\sigma = f_1(\Delta, s)\,d\Delta,$$
$$f_1(\Delta, s)|_{\Delta\to 1} = -\frac{4\alpha^2 r_0^2}{\pi}\ln s \tag{41}$$
$$\times\int_0^\infty \frac{d\varkappa}{\varkappa^3}u_1(\varkappa, -\varkappa) = \frac{8}{3}\pi\alpha^2 r_0^2 \ln s.$$

Accordingly we have found that for $\Delta \to 1$ the quantity $f_1(\Delta, s)$ approaches a constant, which contradicts the relation (35), and consequently the relation (16a). This means that when there are radiation corrections to the inelastic amplitudes the simple relation (16) for the inclusive cross section does not hold, generally speaking. It is not known whether there exists a generalization of this relation which takes in cases in which the inelastic amplitudes are not treated by perturbation theory. In this connection, whether or not Eq. (16) holds can serve as a criterion of whether a given process is described by a sum of Born inelastic amplitudes. This is especially interesting for strong interactions, where there is no completely consistent theory.

Processes of multiple production at high energies have been successfully described by the use of multiperipheral models, in which all processes are described by sums of Born inelastic amplitudes. Consequently, it would seem that the relations of Sec. 2 would apply to them. Usually, however, in multipheripheral models one introduces propagators of the Regge type with singular multipliers, and also form-factors. This changes the analytic structure of the expressions and makes it impossible to continue the relation (22) through the point $x = 1$. At the same time it is not clear how large the singularity at $x = 1$ is, nor whether the function violating Eq. (22) is a small correction. Therefore it is of particular interest to test the main predictions following from the relations of Sec. 2, namely:

a) In reactions of the type $p\pi^+ \to \pi^-$ + anything, for x close to unity the differential cross section goes to zero as an odd power of $(1 - x)$.

b) The distributions for the reactions $pK^+ \to \pi^+$ + anything and $p\pi^- \to K^-$ + anything match each other for x close to unity. The experimental data now available are insufficient for such testing.

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1)For simplicity we suppose that the masses of all the particles are equal.
2)We recall that the variable β coincides with the Feynman variable X.
3)We assume that all the particles are scalar and nonidentical. The complications that arise when we do not use these assumptions will be discussed later.
4)In this section we shall consider cross sections integrated over the perpendicular components of the momenta of all final particles.

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