Drift of particles in the field of a gravitational wave

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Photons, and also particles of nonzero rest mass, propagating in the field of a gravitational wave, undergo a periodic "bending of the ray." The deflection effect is small, but can be built up by repeated reflections of the photon between a pair of ideally reflecting mirrors. The drift length, i.e., the distance Δl between the point of departure of the photon and the point of its return to one of the mirrors, is given in order of magnitude by the formula $\Delta l/\lambda = hQ$, where h and λ are the amplitude and length of the gravitational wave, and Q is the number of reflections between the mirrors, placed at a distance $\lambda/2$ from each other. The direction of the drift depends on the phase of the gravitational wave at the instant the photon starts, and on its polarization. This effect can be used for the experimental observation of gravitational waves.

A gravitational wave (GW) can be detected by its action on test particles; a photon can play the role of test particle. The motion of particles in the field of a weak GW has much in common with the motion of electrons in the field of an electromagnetic wave,[1] if we neglect terms quadratic in the field strength of the electromagnetic wave. The moving particle undergoes periodic changes of its energy and periodic deflections from the mean direction of motion owing to the so-called "electric" and "magnetic" components of the gravitational field. The amplitude of these changes is small, but the effect can be systematically built up, if by means of reflecting mirrors we make the particle execute a motion of a definite kind. So that the effect may be entirely due to the field of the GW and have no contribution from the mirrors, these must be at rest in the chosen reference system and must be ideal mirrors. In other words, on reflection there must be no change of the energy of the particle and of its tangential components of momentum, and the normal component of the momentum must change sign.

We write the metric of a weak plane elliptically polarized GW in the form

$$ds^2=c^2dt^2-(dx^1)^2-(1-a)(dx^2)^2-(1+a)(dx^3)^2+2b\,dx^2dx^3,$$
$$a=h_+\sin[q(x^0-x^1)+\psi_+],\qquad b=h_\times\sin[q(x^0-x^1)+\psi_\times]. \tag{1}$$

The choices $h_\times = 0$ or $h_+ = 0$ correspond to two states of linear polarization, and for the two states of circular polarization the choices are $h_+ = h_\times$, $\psi_\times = \psi_+ \pm \pi/2$.

The world lines of free particles, geodesics of the space-time (1), are

$$x^0=\xi^0+\frac{\Omega}{c}s+\frac{1}{2}\frac{(u^3)^2-(u^2)^2}{c-u^1}A_+-\frac{u^2u^3}{c-u^1}A_\times,$$
$$x^1=\xi^1+\frac{\Omega u^1}{c^2}s+\frac{1}{2}\frac{(u^3)^2-(u^2)^2}{c-u^1}A_+-\frac{u^2u^3}{c-u^1}A_\times,$$
$$x^2=\xi^2+\frac{\Omega u^2}{c^2}s-u^2A_+-u^3A_\times,$$
$$x^3=\xi^3+\frac{\Omega u^3}{c^2}s+u^3A_+-u^2A_\times. \tag{2}$$

Here s is a parameter, ξ^α, Ω, u^i are arbitrary constants, and

$$A_+=\frac{h_+}{q(c-u^1)}\cos(\eta+\psi_+),\qquad A_\times=\frac{h_\times}{q(c-u^1)}\cos(\eta+\psi_\times),$$
$$\eta=\frac{\Omega}{c}\left(1-\frac{u^1}{c}\right)qs+q(\xi^0-\xi^1).$$

The components of the four-momentum of the particle are $k^\alpha = dx^\alpha/ds$. For light-cone geodesics the choice of the constants is such that $k^\alpha k^\beta g_{\alpha\beta} = 0$.

The equations (2) describe the motion of particles relative to a system of free test bodies $x^i = \text{const}$ which realize the synchronous reference system (1). The four-momentum component ck^0 is the energy of the particle. As we see from (2), it varies periodically along the trajectory. In real situations the amplitude of this change is small and could hardly be measured.[3] The change of the energy can be built up by sending the particle (or photon) along a closed "light guide."[4,5] In the geometric-optics approximation which we consider, the light guide can be a system of mirrors which make the particle move along a definite trajectory. Ya. B. Zel'dovich has pointed out that for a systematic change of the energy to be possible there is no necessity of a closed waveguide; a section of straight waveguide with mirrors at the ends is enough. Numerical estimates of the effect produced in this case are given in[6]. We shall here consider a particle reflected from two mirrors, in the space between which it moves freely. The forward and backward trajectories of the particle do not coincide, and differ by a quantity of the order of h, but the change of energy is exactly the same as if the particle moved along a waveguide whose position coincides with its "unperturbed" trajectory.

Suppose that a particle with the four-momentum $k^\alpha = \{\omega/c, \omega v^i/c^2\}$ starts out at time $x^0 = 0$ from the point $x^i = 0$, which lies on the first mirror. A systematic change of the energy is possible if after reflection from the second mirror the particle returns to the same point after exactly one period of the GW, with a different energy. Then, choosing the orientation of the first mirror so that the particle starts off again along the same trajectory, we can repeat the whole process. The increments of energy will then accumulate. Using Eqs. (2) and requiring that the stated conditions be satisfied, including small terms of order h, we can find the change of frequency $\Delta\omega/\omega$, the distance L between the points of reflection on the mirrors, and the orientation of the mirrors. It is clear that in the main approximation L must be equal to $n\pi V/qc$, where n is an integer and $V = [(v^1)^2 + (v^2)^2 + (v^3)^2]^{1/2}$ is the speed of the particle. For the increment $\Delta\omega/\omega$ which appears after one reflection and return of the particle to its starting point, we easily get the formula

$$\frac{\Delta\omega}{\omega}=2\frac{v^1}{c}\frac{1}{c^2-(v^1)^2}\sin\left[\left(1-\frac{v^1}{c}\right)\frac{n\pi}{2}\right]\Big\{h_+[(v^2)^2$$

$$-(v^3)^2]\cos\left[\left(1-\frac{v^1}{c}\right)\frac{n\pi}{2}+\psi_+\right]+2h_\times v^2v^3\cos\left[\left(1-\frac{v^1}{c}\right)\frac{n\pi}{2}+\psi_\times\right]\Bigg\}. \quad (3)$$

We see from this equation that there is no effect if $v^1 = 0$ or $v^2 = v^3 = 0$, and also in some other cases depending on the polarization of the GW, for example for $h_\times = 0$ and $v^2 = v^3$, or $\psi_+ = \pi/2 - (1 - v^1/c)n\pi/2$. For fixed momentum components the sign of the effect depends on the phase of the GW at the starting time of the particle. If we want to keep the trajectory of the particle unchanged to accuracy h, the orientation of the mirrors must be such that the directions of incidence and reflection of the particle do not coincide with the normal to the mirror, but make with it a small angle proportional to h. It is advantageous to place the mirrors at the smallest permissible distance, $n = 1$, so that the accumulation of the frequency shift will be faster. Using particles moving with a speed less than that of light, we can place the mirrors at a distance smaller than $\lambda/2$, but then $\Delta\omega/\omega$ is also smaller. After Q reflections from the second mirror a photon ($V = c$) receives a total increment $\Delta\omega/\omega$ larger than (3) by the factor Q, and given in order of magnitude by

$$\Delta\omega/\omega=hQ. \quad (4)$$

Let us now consider the effect of the systematic drift of the particle. It has much in common with the effect of systematic change of the energy. Let the coordinate surfaces $x^2 = 0$ and $x^2 = L$ be ideally reflecting mirrors; that is, on reflection the momentum components k^0, k^1, k^3 do not change, and k^2 changes sign. At time $x^0 = 0$ let the particle start from the point $x^i = 0$, moving exactly along the normal to the mirror. In other words, its four-momentum, not neglecting any terms of order h, is $k^\alpha = \{\omega/c, 0, \omega V/c^2, 0)$, whatever the phase of the GW. If we assign to the event of the particle's starting the parameter value $s = 0$, the corresponding geodesic is of the form

$$x^0=\frac{\omega}{c}s\left[1-\frac{1}{2}h_+\left(\frac{V}{c}\right)^2\sin\psi_+\right]+\frac{1}{2}\frac{V}{c}B_+,$$
$$x^1=-\frac{1}{2}\frac{\omega}{c}s\left(\frac{V}{c}\right)^2h_+\sin\psi_++\frac{1}{2}\frac{V}{c}B_+, \quad (5)$$
$$x^2=\omega\frac{V}{c}s(1-h_+\sin\psi_+)+B_+,\quad x^3=-\omega\frac{V}{c}sh_\times\sin\psi_\times+B_\times,$$

where

$$B_+=2\frac{h_+}{q}\frac{V}{c}\sin\left(\frac{\omega q}{2c}s+\psi_+\right)\sin\frac{\omega q}{2c}s,$$

and $B_\times$ is obtained from B_+ by replacing h_+ with $h_\times$ and ψ_+ with $\psi_\times$. The quantity L must be chosen so that the return of the particle to the first mirror will occur for $x^0 - x^1 = n2\pi/q$, i.e., at the same phase of the GW.[1)] If the point where it returns is shifted relative to the point where it started, then this shift can be increased by repeated reflections.

Let us set $L = n\pi V/qc$. It is not hard to show that when it returns after one reflection the particle's coordinates, including quantities of order h, will be

$$x^0=n\frac{2\pi}{q}(1+h_+\sin\psi_+)-\frac{n\pi}{q}\left(\frac{V}{c}\right)^2h_+\sin\psi_+, \quad (6)$$
$$x^1=-\frac{n\pi}{q}\left(\frac{V}{c}\right)^2h_+\sin\psi_+,\quad x^2=0,$$
$$x^3=2\frac{h_\times}{q}\frac{V}{c}[1-(-1)^n](\cos\psi_\times-n\pi\sin\psi_\times)$$

and its four-momentum will be

$$k^0=\frac{\omega}{c},\quad k^1=0,\quad k^2=-\frac{\omega V}{c^2},$$
$$k^3=-2\frac{\omega V}{c^2}[1-(-1)^n]h_\times\sin\psi_\times.$$

The shift along the x^3 axis and the appearance of the component k^3 are due to the presence of the x polarization of the GW. They can be avoided if we choose the number n to be even.

Let us confine ourselves to the shift along the x^1 axis. After one reflection it is $\Delta x^1 = -L(V/c)h_+\sin\psi_+$. The shift along the x^1 axis is absent only for $\psi_+ = 0$, π, i.e., in cases when the reflection of the particle occurs at a time when the coordinate and true distances between the mirrors are equal. The extreme values of the shift appear for $\psi_+ = \pi/2$, $3\pi/2$. For definiteness let us take $h_+ > 0$. Then a particle starting out with the value $\psi_+ = \pi/2$, which corresponds to the time of least true distance between the mirrors, is displaced toward negative x^1, and one starting out when $\psi_+ = 3\pi/2$, at the time of greatest true distance between the mirrors, is displaced toward positive x^1. Thus a system of two mirrors in the field of a GW can "sort" particles according to their starting times, shifting them in different directions. This effect is of course not connected with a change of the original orientation of the device projecting the particles. It is easy to verify that a vector which has no x^1 component at the initial time will not acquire any in the process of parallel transport along the timelike geodesic $x^i = 0$.

In the case of photons the smallest permissible distance between the mirrors is $L = \lambda/2$. After Q reflections the point to which the photon returns is shifted relative to the starting point by a distance Δl given in order of magnitude by the formula

$$\Delta l/\lambda=hQ. \quad (7)$$

As can be seen from Eqs. (5) and (6), there is no drift effect along the x^1 axis in those cases in which the mean trajectory of a particle projected along the normal to the mirror is also normal to it. This occurs for $\psi_+ = 0$, π. To obtain an effect in this case one must place the mirrors so that the x^1 component of their normal is not zero.

In conclusion we point out that formulas of the type of Eqs. (4) and (7) appear not only in the geometric-optics approximation, when we can speak of individual particles, but also in the wave theory. Such formulas, for example, describe the change of energy and phase of a standing electromagnetic wave in a resonator when acted upon by an incident GW.[6] In this case the factor Q is the figure of merit of the resonator. All of these effects assume the use of sufficiently short monochromatic GW with definite phase. In principle such GW can be generated under laboratory conditions.

[1)]It suffices to require that this equation be satisfied in the main approximation only, since the difference $x^0 - x^1$ appears in the metric in the arguments of harmonic functions which already have small factors h_+ or $h_\times$.

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