Vertex functions and Green functions in the (4 - ϵ)-dimensional theory of phase transitions

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The vertex functions, polarization operator, and Green functions in the $(4-\epsilon)$-dimensional theory of phase transitions with a φ^4 interaction are calculated for small values of ϵ by direct summation of perturbation-theory graphs. The expressions obtained are valid both in the region of applicability of perturbation theory and in the scaling region.

1. INTRODUCTION

As is well known[1], the Landau theory of phase transitions holds with logarithmic accuracy in four-dimensional space. It is natural to expect that in a $(4-\epsilon)$-dimensional space with $\epsilon \ll 1$ the deviation from the theory will be small. Wilson and Fisher therefore proposed using ϵ as a small parameter to calculate critical indices [2]. Using renormalization-group ideas, Wilson [3] has calculated the indices to order ϵ^3 and has obtained an expression for the four-point function in the scaling region. Tsuneto and Abrahams[4] have done the same using Ward identities. However, the field-theory equations were not solved in these papers (it is well known that the theory of phase transitions is formally equivalent to field theory). On the other hand, Larkin and Khmel'nitskiĭ [1] showed that in a four-dimensional field theory with a φ^4 interaction a logarithmic situation obtains and the principal contribution to the vertex functions is determined by the parquet graphs. It is natural that these same graphs should also give the main contribution in a $(4-\epsilon)$-dimensional theory with $\epsilon \ll 1$, although the logarithm is replaced by a power function with small exponent ϵ. This is connected with the fact that for small ϵ the power function, like the logarithm, is large and slowly-varying.

In the present paper, by summation of parquet graphs in $(4-\epsilon)$-dimensional space, explicit expressions are obtained for the vertex functions, polarization operator and Green function in the whole range of variation of the momenta, and the connection with perturbation theory is made. It is shown, in particular, that in the scaling region the coefficient of the corresponding power of the momentum in the vertex with two external points and one angle and in the polarization operator is a power function of ϵ with a non-integer exponent. Critical indices are also calculated. The expressions obtained for the indices coincide with the results of the papers [3,4].

2. CALCULATION OF THE VERTEX FUNCTIONS

We shall consider the theory of an n-component field φ_α, the Hamiltonian of which is equal to [3]

$$\frac{H}{T}=\int d^dx\left\{\frac{\varkappa_0^2}{2}\varphi^2(\mathbf{x})+\frac{1}{2}[\nabla\varphi(\mathbf{x})]^2+\frac{\lambda}{8}[\varphi^2(\mathbf{x})]^2\right\}, \quad (1)$$

$$\varphi^2=\sum_i\varphi_i^2,\qquad (\nabla\varphi)^2=\sum_i(\nabla\varphi_i)^2;$$

d is the number of dimensions. We shall define the Green function

$$G_{ij}(\mathbf{r})=\langle\varphi_i(\mathbf{r})\varphi_j(0)\rangle; \quad (2)$$

the averaging is performed with weight $e^{-H/T}$. It is perfectly obvious that $G_{ij}(\mathbf{r})=\delta_{ij}G(\mathbf{r})$. The zeroth-order Green function in the momentum representation is equal to

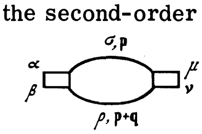
$$G_0(\mathbf{k})=1/(k^2+\varkappa_0^2). \quad (3)$$

We note that a factor $1/2!$ must be associated with each closed loop, as is shown easily in the derivation of the perturbation theory.

We turn now to the calculation of the vertex function $\Gamma_{\alpha\beta\mu\nu}$. In the first approximation of perturbation theory, we have

$$\gamma_{\alpha\beta\mu\nu}=\lambda I_{\alpha\beta\mu\nu},$$
$$I_{\alpha\beta\mu\nu}=\delta_{\alpha\beta}\delta_{\mu\nu}+\delta_{\alpha\mu}\delta_{\beta\nu}+\delta_{\alpha\nu}\delta_{\beta\mu}. \quad (4)$$

We shall consider the second-order graph:



(5)

This equals

$$\Delta\Gamma_{\alpha\beta\mu\nu}=-\frac{\lambda^2}{2}J_{\alpha\beta\mu\nu}\int\frac{d^dp}{(2\pi)^d}G(\mathbf{p})G(\mathbf{p}+\mathbf{q}),$$
$$J_{\alpha\beta\mu\nu}=I_{\alpha\beta\sigma\rho}I_{\sigma\rho\mu\nu}. \quad (5')$$

As usual, we shall assume that for large momenta $G(p)=p^{-2+\eta}$. On the other hand, as shown in the papers of Wilson and Fisher [2,3], and as will be shown in Sec. 3 of this paper, η is small ($\eta=0.02$). Therefore, we shall assume in this Section that $\eta=0$.

We shall consider the region of large momenta $q \gg \kappa$, where κ is the renormalized value of κ_0 (at the phase-transition point, κ vanishes). We note that for $d=4$ the integral in (5) diverges logarithmically. For finite ϵ, it is not difficult to see that the integral is determined by the range of variation: $p\sim qe^{1/\epsilon}\gg q$. We can therefore write (Λ is the momentum cutoff)

$$\Delta\Gamma_{\alpha\beta\mu\nu}=-\frac{\lambda^2}{2}J_{\alpha\beta\mu\nu}\int_q^\Lambda\frac{d^dp}{(2\pi)^d}G^2(\mathbf{p}). \quad (6)$$

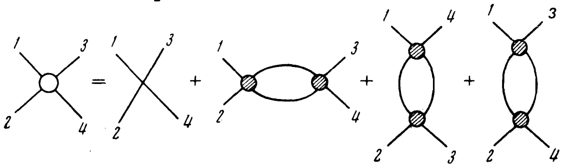
Putting [3]

$$\frac{d^dp}{(2\pi)^d}\to K_dp^{3-\varepsilon}dp,\qquad K_d=\left\{2^{d-1}\pi^{d/2}\Gamma\left(\frac{d}{2}\right)\right\}^{-1}, \quad (7)$$

we have ($\epsilon=4-d$)

$$\Delta\Gamma_{\alpha\beta\mu\nu}=-\frac{\lambda^2}{2}J_{\alpha\beta\mu\nu}\frac{K_d}{\varepsilon\Lambda^\varepsilon}\left[\left(\frac{\Lambda}{q}\right)^\varepsilon-1\right]. \quad (8)$$

From (8) it can be seen that for $\epsilon\ll 1$ the function obtained varies very slowly (when $(\Lambda/q)^\epsilon$ changes by a factor of e, q should change by a factor of $e^{1/\epsilon}$). In this case as in the logarithmic situation (for $\epsilon=0$), the principal graphs are the parquet graphs [1,5]. To calculate the vertex in the parquet approximation, it is convenient to use the method proposed by Polyakov[6]. As we have seen, the power divergences arise on account of the two-particle sub-units. Selecting in each graph the two-particle sub-unit in which the momentum of integration is smallest, we can obtain for the vertex the equation



(9)

Here the external momenta are equal to

$$\mathbf{k}_{1,2}=\mathbf{k}/2\pm\mathbf{q}_1,\quad \mathbf{k}_{3,4}=-\mathbf{k}/2\pm\mathbf{q}_2\quad \mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4=0,$$

and the external indices are equal to α, β, μ and ν respectively. The shaded circles are vertices in which all the virtual momenta $\kappa_i \gg p$, where p is the momentum of integration in the equation.

We shall assume that $q_1 \gg q_2 \gg k$. We put

$$x=\left(\frac{\Delta}{q_1}\right)^\varepsilon,\quad y=\left(\frac{\Lambda}{q_2}\right)^\varepsilon,\quad z=\left(\frac{\Lambda}{k}\right)^\varepsilon,\quad t=\left(\frac{\Lambda}{p}\right)^\varepsilon, \tag{10}$$
$$z>y>x>1.$$

Then, as in Polyakov's paper [6], we obtain the following equation for the vertex $\Gamma_{\alpha\beta\mu\nu}(x, y, z)$:

$$\Gamma_{\alpha\beta\mu\nu}(x,y,z)=\gamma_{\alpha\beta\mu\nu}-\beta\int_1^x dt\{\Gamma_{\alpha\sigma\mu\rho}(t,t,t)\Gamma_{\sigma\beta\rho\nu}(t,t,t)$$
$$+\Gamma_{\alpha\sigma\nu\rho}(t,t,t)\Gamma_{\sigma\beta\rho\nu}(t,t,t)+\Gamma_{\alpha\beta\sigma\rho}(t,t,t)\Gamma_{\sigma\rho\mu\nu}(t,t,t)\} \tag{11}$$
$$-\beta\int_x^y dt\Gamma_{\alpha\beta\sigma\rho}(x,t,t)\Gamma_{\sigma\rho\mu\nu}(t,t,t)-\beta\int_y^z dt\Gamma_{\alpha\beta\sigma\rho}(x,t,t)\Gamma_{\sigma\rho\mu\nu}(y,t,t),$$

where

$$\beta=K_d/2\varepsilon\Lambda^\varepsilon. \tag{12}$$

We introduce the notation

$$\Gamma_{\alpha\beta\mu\nu}(x,y,y)=\Gamma_{1\alpha\beta\mu\nu}(x,y),$$
$$\Gamma_{\alpha\beta\mu\nu}(x,x,x)=\Gamma_{0\alpha\beta\mu\nu}(x),$$
$$\Gamma_{\alpha\beta\mu\nu}(x,y,z)=T_1(x,y,z)I_{\alpha\beta\mu\nu}+T_2(x,y,z)R_{\alpha\beta\mu\nu}, \tag{13}$$
$$\Gamma_{1\alpha\beta\mu\nu}(x,y)=P_1(x,y)I_{\alpha\beta\mu\nu}+P_2(x,y)R_{\alpha\beta\mu\nu},$$
$$\Gamma_{0\alpha\beta\mu\nu}(x)=P_0(x)I_{\alpha\beta\mu\nu},$$
$$R_{\alpha\beta\mu\nu}=\delta_{\alpha\beta}\delta_{\mu\nu}.$$

From (11), taking (4) and (13) into account, we obtain

$$T_1(x,y,z)=\lambda-\beta(n+8)\int_1^x dt\,P_0^2(t)$$
$$-2\beta\int_x^y dt\,P_1(x,t)P_0(t)-2\beta\int_y^z dt\,P_1(x,t)P_1(y,t),$$

$$T_2(x,y,z)=-\beta(n+2)\int_x^y dt[P_1(x,t)+P_2(x,t)]P_0(t) \tag{14}$$
$$-\beta\int_y dt\{(n+2)[P_1(x,t)P_1(y,t)+P_1(x,t)P_2(y,t)$$
$$+P_2(x,t)P_1(y,t)]+nP_2(x,t)P_2(y,t)\}.$$

It can be seen from (14) that T_1 and T_2 are expressed in terms of P_1, P_2 and P_0. It is therefore convenient to have equations for these quantities immediately. Putting y = z in (14), we obtain

$$P_1(x,y)=\lambda-\beta(n+8)\int_1^x dt\,P_0^2(t)-2\beta\int_x^y dt\,P_1(x,t)P_0(t),$$

$$P_2(x,y)=-\beta(n+8)\int_x dt[P_1(x,t)+P_2(x,t)]P_0(t). \tag{15}$$

We put x = y in (15); then,

$$P_0(x)=\lambda-\beta(n+8)\int_1^x dt\,P_0^2(t) \tag{16}$$

We begin by solving Eq. (16). We differentiate (16) with respect to x and solve the equation obtained:

$$P_0(x)=\lambda/t(x),\quad t(x)=1+\lambda\beta(n+8)(x-1). \tag{17}$$

Next, from (15) and (16) we have

$$P_1(x,y)=P_0(x)-2\beta\int_x^y dt\,P_1(x,t)P_0(t). \tag{18}$$

We differentiate (18) with respect to y and solve the equation:

$$P_1(x,y)=\lambda[t(x)]^{-(n+6)/(n+8)}[t(y)]^{-2/(n+8)}. \tag{19}$$

Solving the second Eq. (15), we have

$$P_2(x,y)=\lambda\frac{n+2}{n}\{[t(x)]^{-6/(n+8)}[t(y)]^{-(n+2)/(n+8)} \tag{20}$$
$$-[t(x)]^{-(n+6)/(n+8)}[t(y)]^{-2/(n+8)}\}.$$

Substituting (17), (18) and (20) into (14), we obtain

$$T_1(x,y,z)=\lambda[t(x)]^{-(n+6)/(n+8)}\left\{\frac{n+6}{n+4}[t(y)]^{-2/(n+8)}\right.$$
$$\left.-\frac{2}{n+4}[t(y)]^{-(n+6)/(n+8)}[t(z)]^{(n+4)/(n+8)}\right\}, \tag{21}$$
$$T_2(x,y,z)=\frac{\lambda}{n}\left\{\frac{6(n+2)}{4-n}[t(x)]^{-6/(n+8)}[t(y)]^{-(n+2)/(n+8)}\right.$$
$$-\frac{(n+2)(n+6)}{n+4}[t(x)]^{-(n+6)/(n+8)}[t(y)]^{-2/(n+8)}$$
$$-\frac{(n+2)^2}{4-n}[t(x)t(y)]^{-6/(n+8)}[t(z)]^{(4-n)/(n+8)}$$
$$+\frac{2(n+2)}{n+4}[t(x)t(y)]^{-(n+6)/(n+8)}[t(z)]^{(n+4)/(n+8)}. \tag{22}$$

We now proceed to the analysis of the expressions obtained. First of all, we note that for a one-component system (n = 1) all the expressions are considerably simplified. For n = 1 we must replace the tensor $R_{\alpha\beta\mu\nu}$ by unity and replace $I_{\alpha\beta\mu\nu}$ by a factor equal to three. Then we have from (13), (17) and (19)–(21)

$$\Gamma(x,y,z)=3T_1(x,y,z)+T_2(x,y,z)=3\lambda\{2[t(x)]^{-2/3}[t(y)]^{-1/3}$$
$$-[t(x)t(y)]^{-2/3}[t(z)]^{1/3}\}, \tag{23}$$
$$\Gamma_1(x,y)=3P_1(x,y)+P_2(x,y)=3\lambda[t(x)]^{-2/3}[t(y)]^{-1/3}. \tag{24}$$

We now consider the region of momenta that are small compared with the momentum cutoff Λ. Then from (10), (17), (19) and (20) we have

$$\Gamma_{0\alpha\beta\mu\nu}(\mathbf{k})=ak^\varepsilon I_{\alpha\beta\mu\nu},$$
$$\Gamma_{1\alpha\beta\mu\nu}(\mathbf{q},\mathbf{k})=\alpha q^{\varepsilon(n+6)/(n+8)}k^{2\varepsilon/(n+8)}I_{\alpha\beta\mu\nu}+\alpha\frac{n+2}{n}\{q^{6\varepsilon/(n+8)}k^{(n+2)\varepsilon/(n+8)}$$
$$-q^{(n+6)\varepsilon/(n+8)}k^{2\varepsilon/(n+8)}\}R_{\alpha\beta\mu\nu}; \tag{25}$$

$$\alpha=\frac{2\varepsilon}{(n+8)K_d}\approx\frac{16\pi^2\varepsilon}{n+8}. \tag{26}$$

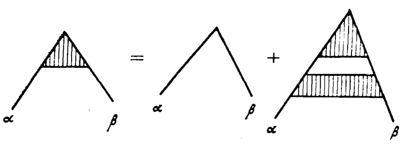
In the latter equality we have replaced K_d by K_4.

There is also an analogous, but more cumbersome, formula for $\Gamma_{\alpha\beta\mu\nu}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{k})$. We write it out for the case n = 1 only:

$$\Gamma(\mathbf{q}_1,\mathbf{q}_2,\mathbf{k})=3\alpha\{2q_1^{2\varepsilon/3}q_2^{2\varepsilon/3}-(q_1q_2)^{2\varepsilon/3}k^{-\varepsilon/3}\}, \tag{27}$$
$$\Gamma_1(\mathbf{q},\mathbf{k})=3\alpha q^{2\varepsilon/3}k^{\varepsilon/3}. \tag{27'}$$

The expressions (25) and (27), and also the analogous formulas for $\Gamma_{\alpha\beta\mu\nu}(q_1, q_2, k)$, define, in the general case, the correlation functions that depend on four coordinates. It is easily verified that the rule formulated by Polyakov [6] for the coalescence of the correlations is fulfilled for these functions. It may be hoped that the general structure of the vertex functions will also be conserved in the three-dimensional case, if we put $\epsilon = 1$, although, of course, the coefficients of the powers of the momenta can change.

We shall now calculate the vertex $F_{\alpha\beta}$, represented by graphs with two external points and one angle, and the polarization operator $\Pi(q)$. In the case when the two momenta defining $F_{\alpha\beta}$ are of the same order, we have for $F_{\alpha\beta}$ the following equation [1]:



(28)

or, analytically,

$$F_{\alpha\beta}(q) = \delta_{\alpha\beta} - \frac{1}{2}\int_q^{\Lambda} \frac{d^dp}{(2\pi)^d} F_{\mu\nu}(p) G^2(p) \Gamma_{0\mu\nu\alpha\beta}(p). \qquad (28')$$

The polarization operator $\Pi(q)$ equals [1]

$$\Pi(q) = \frac{1}{2}\int_q \frac{d^dp}{(2\pi)^d} F_{\alpha\beta}(p) F_{\alpha\beta}(p) G^2(p). \qquad (29)$$

Putting $F_{\alpha\beta}(p) = F\delta_{\alpha\beta}$ and solving (28) and (29) in the same way as we solved the equation for $P_0(q)$, we obtain

$$F(x) = [t(x)]^{-(n+2)/(n+8)},$$
$$\Pi(x) = \frac{n}{4-n}\frac{1}{\lambda}\{[t(x)]^{(4-n)/(n+8)} - 1\}. \qquad (30)$$

for $q \ll \Lambda$, we have

$$F(q) = \left(\frac{\alpha}{\lambda} q^\epsilon\right)^{(n+2)/(n+8)},$$
$$\Pi(q) = \frac{n}{(4-n)\lambda}\left(\frac{\alpha}{\lambda} q^\epsilon\right)^{-(4-n)/(n+8)} \qquad (31)$$

Since $\alpha \sim \epsilon$, it can be seen from (31) that the coefficients of the powers of q in F and Π are proportional to $\epsilon^{(n+2)/(n+8)}$ and $\epsilon^{-(4-n)/(n+8)}$ and cannot be expanded in a series in ϵ.

3. CALCULATION OF THE CRITICAL INDICES

To determine the index γ, we shall use the Ward identity [1]

$$\frac{dr}{d\varkappa_0^2} = F(0), \quad r = G^{-1}(0). \qquad (32)$$

Since for $q \to 0$, if we neglect the index η, we must put $\kappa \approx r^{1/2}$ in place of q in F(q), we have

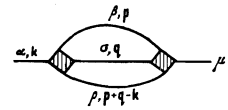
$$r \sim \varepsilon^{(n+2)/(n+8)} (\varkappa_0^2 - \varkappa_{0c}^2)^\gamma,$$
$$\gamma = 1 + \varepsilon(n+2)/2(n+8), \qquad (33)$$

where κ_{0c} is the critical value of κ_0. The expression (33) for γ was obtained in the papers [3,4].

The specific heat c is proportional to $\Pi(0)$, and therefore

$$c \sim \varepsilon^{-(4-n)/(n+8)} (\varkappa_0^2 - \varkappa_{0c}^2)^{-\varepsilon(4-n)/2(n+8)}. \qquad (34)$$

We turn now to the calculation of the index η. For this we must calculate the Green function. It is easy to show that in the parquet approximation Σ is defined by the graph



(35)

where p and q are the smallest of the virtual momenta and $q \gg p \gg k$. We then obtain

$$\Sigma_{\alpha\mu}(k) = \Sigma(k)\delta_{\alpha\mu} = -\frac{1}{3!}\int_k^{\Lambda} \frac{d^dp}{(2\pi)^d} G(p) \int_p^{\Lambda} \frac{d^dq}{(2\pi)^d} \Gamma_{0\alpha\beta\sigma\rho}(q) \Gamma_{0\sigma\rho\mu\beta}(q) G^2(q). \qquad (36)$$

Taking into account that $\Gamma_{0\alpha\beta\mu\nu}(q) = P_0(q) I_{\alpha\beta\mu\nu}$, where $P_0(q)$ satisfies Eq. (16), which we rewrite in the form

$$P_0(k) = \lambda - \frac{n+8}{2}\int_k^{\Lambda} \frac{d^dp}{(2\pi)^d} G^2(p) P_0^2(p),$$

and that $I_{\alpha\beta\sigma\rho} I_{\sigma\rho\mu\beta} = 3(n+2)\delta_{\alpha\mu}$, we obtain

$$\Sigma(k) = -\frac{n+2}{n+8}\int_k^{\Lambda} \frac{d^dp}{(2\pi)^d} G(p)[P_0(p) - \lambda]$$
$$= -\frac{n+2}{n+8}\int \frac{d^dp}{(2\pi)^d} G(k-p)[P_0(p) - \lambda]. \qquad (37)$$

It can be seen from (37) that $\Sigma(k)$ diverges quadratically at large momenta, and it is therefore convenient to make a subtraction. We introduce

$$\Sigma_1(k) = \Sigma(k) - \Sigma(0)$$
$$= -\frac{n+2}{n+8}\int \frac{d^dp}{(2\pi)^d} [G(k-p) - G(p)] P_0(p). \qquad (38)$$

Since the renormalized quantity κ^2 is determined by the condition

$$\varkappa_0^2 - \Sigma(0, \varkappa^2) = \varkappa^2, \qquad (39)$$

the Green function equals

$$G = [k^2 + \varkappa^2 - \Sigma_1(k)]^{-1}. \qquad (40)$$

To determine η, we must consider the region of momenta $k \gg \kappa$. On the other hand, since $\eta > 0$, in this region

$$\Sigma_1(k) \sim k^{2-\eta} \gg k^2.$$

Therefore, in the region of interest,

$$G = -\Sigma_1^{-1}(k) = Ak^{-(2-\eta)} \qquad (41)$$

On the other hand, from (25) and (26) we have for $\Lambda \gg k \gg \kappa$:

$$P_0(k) = \frac{2\varepsilon}{(n+8)K_d A^2} k^{\varepsilon - 2\eta}. \qquad (42)$$

The expression (42) differs from (25) because in the derivation of (25) we assumed that $G(k) = k^{-2}$, and in the derivation of (42) we took for G(k) the formula (41).

From (38), (41) and (42) we obtain

$$k^{2-\eta} = \frac{2(n+2)\varepsilon}{(n+8)^2 K_d}\int \frac{d^dp}{(2\pi)^d} p^{\varepsilon - 2\eta}\left\{\frac{1}{|k-p|^{2-\eta}} - \frac{1}{p^{2-\eta}}\right\}. \qquad (43)$$

It can be seen from (43) that the integral is determined by the region of momenta $ke^{1/\eta} \gg k$, and therefore the integrand can be expanded in k. Taking into account that [3]

$$\int \frac{d^dp}{(2\pi)^d} = \frac{K_{d-1}}{2\pi}\int_0^\infty p^{3-\varepsilon} dp \int_0^\pi (\sin v)^{2-\varepsilon} dv, \qquad (44)$$

we obtain after simple computations the following expression for η:

$$\eta = \frac{n+2}{2(n+8)^2}\varepsilon^2,$$

which coincides with the corresponding expression in Wilson's paper [3].

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