The pion spectrum in nuclear matter and pion condensation

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The spectrum of π^+, π^-, and π^0 mesons has been determined in nuclear matter with $Z=0$ (neutron stars) and $Z=N$ (nuclei), taking into account the nucleon correlations. It is shown that in neutron matter, at a density $n > n_c^+ \approx 0.4 n_0$, a negative energy branch, $\omega^{(+)} < -\epsilon_F^{(n)}$, appears in the spectrum of π^+ mesons. This leads to an instability of the protons in such a medium ($p \to n + \pi^+$), i.e., the neutron-star matter must consist of neutrons with a small admixture of positive pions, the charge of which is compensated by electrons. For neutron densities $n > n_c^{\pm} \approx 0.8 n_0$ there appears an instability with respect to the production of pairs of $\pi^+\pi^-$ mesons, and for approximately the same density the field of the π^0 mesons becomes unstable. This leads to phase transitions of the second kind in the system, with formation of pion condensates. The condensate of π^- mesons proposed in[4] does not appear, even for very large densities. In nuclear matter with $Z=N$ at densities $n = n_c \approx 0.6 n_0$ smaller than the nuclear density n_0 there appears an instability with respect to the simultaneous production of π^+, π^-, and π^0 mesons, leading to a phase transition of the second kind with the production of an electrically neutral condensate of π^+, π^-, and π^0 mesons. This should affect the characteristic properties of nuclei.

1. INTRODUCTION

In a sufficiently deep potential well, an energy level may descend to a depth at which particle creation from the vacuum becomes possible. In the case of fermions the stability of the vacuum is guaranteed by the Pauli principle: particle creation ceases when the "dangerous energy levels" are filled. In the case of bosons the process stops only when the repulsion between the particles makes further creation energetically unprofitable. The stability of the boson vacuum in an external field has been studied in [1]. It was shown that in a nucleon medium an instability may arise leading to the formation of a pion condensate, after which the system becomes stable.

In [2] one of the authors has developed a method for the determination of the spectrum of pions in nuclear matter, method which was based on the separation of the most important diagrams. It consisted in separating those diagrams which vary substantially for four-momenta of the order $m_\pi c$. The other less important diagrams are replaced by already known constants, determined from the comparison of theoretical and experimental results (such as, e.g., the constant f describing the pion-nucleon interaction, or the coupling constants g^{nn} and g^{pn} of the spin NN interaction in nuclei). We shall return below to these questions.

In [2] it was shown that the pion field becomes unstable for a nucleon density n_c smaller than the usual nuclear density n_0, leading to the conclusion that the pion condensate must exist in atomic nuclei. Since the square of the condensate field φ_0^2 turns out to be a periodic function of the coordinates, it follows that the density of nucleons in nuclei in the nucleus has a periodic structure (with a period of the same order as the average distance between the nucleons).

In the case of a medium with $N=Z$, the case considered in [2], a static electrically neutral condensate is obtained with the fields $\varphi_{\pi^+}^2 = \varphi_{\pi^-}^2 = \varphi_{\pi^0}^2$. For a medium with $Z \ll N$ the spectrum of π^0 mesons remains the same as in the case $Z=N$, whereas the spectrum of the charged mesons changes substantially [3]. For a density $n \approx 0.4 n_0$ together with the pion branch ($\omega \to 1$ for $k \to 0$), there appears a branch of excitations with the quantum numbers of the π^+ mesons but with energy $\omega^{(+)} < 0$. It turns out that $\omega^{(+)} + \epsilon_F^{(n)} < 0$. It is usually assumed that in a neutron star there is a small number of protons in addition to the neutrons, and that the charge of the former is compensated by electrons. Since for $n > 0.4 n_0$ one has $\omega^{(+)} + \epsilon_F^{(n)} < 0$, at such a density the protons will be replaced by π^+ mesons. The density of the π^+ mesons is determined by the equation $\omega^{(+)} + \epsilon_F^{(el)} = 0$. For a density $n_c^0 = 0.8 n_0$ there appears a condensate of π^0 mesons, and for approximately the same density there appears an electrically neutral condensate of π^- and π^+ mesons.

We note that the π^- condensate proposed in the paper of Sawyer and Scalalpino[4] does not appear, at least up to very large densities of nuclear matter. These papers make use of a simplified Hamiltonian, which takes into account only the interaction of the nucleons with the field of a π^--meson condensate having the form of a plane wave. In a correct calculation this simplified discussion also does not lead to the formation of a π^- condensate. The reason for the error is an incorrect use in [4] of the mean-field method, resulting in an incorrect expression for the energy density of the star in the presence of the π^- condensate. The incorrectness of the expression is visible already from the fact that for a density below the critical value, the energy given in [4] has a nonanalytic dependence on the coupling constant (it contains the absolute value of this constant linearly!). A detailed analysis of the method used in [4] is given in [5].

We use this occasion to make some remarks on the objections to the method of obtaining the pion spectrum developed in [2]. One of these objections consisted in the fact that the exchange part of the particle-hole ("pole") diagram in the polarization operator had to be left out, since it was putatively already taken into account in the observable amplitude for resonant scattering [6]. We shall stop below in detail on this, but we remark already here that the expression for the polarization operator proposed in [6] contradicts crossing symmetry and time-reversal invariance, since for $Z=N$ it yields a polarization operator which contains odd powers of the pion frequency.

Another objection is that taking into account the repulsion at small distances must diminish the pole part of the polarization operator. The interaction forces between nucleons lead indeed to a decrease of the polarization operator, however, this decrease is taken into account rigorously in the method of [2] by introducing the empirical spin-spin coupling constant between the nucleons, constant which was determined by comparing the theoretical and experimental values of the magnetic moments and the beta-decay rates. Thus, this objection is based on a misunderstanding. It is discussed in more detail in [7] (cf. also Sec. 2, Item 5).

Below we discuss in detail all important ingredients which determine the polarization operator, and we obtain the spectra of π^+, π^-, π^0 mesons in the absence of a condensate, both for the case $Z = N$ and for the case $Z \ll N$ (neutron star).

A condition for a phase transition of the second kind (the density of the condensate increases from its zero value) is the appearance of an instability in these spectra. In order to consider the possibility of a phase transition of the first kind it is necessary to solve a more complicated problem: find the energy of the system for an arbitrary density of the condensate and compare it with the energy of the system in which there is no Bose condensation, or (for $n > n_c$) with the energy of the system in which the above-mentioned phase transitions of the second kind have occurred. This problem was solved in [5] for the model involving nucleons plus the π^- condensate with a single propagation vector **k**. It turned out that in this model there is neither a phase transition of the first kind nor one of the second kind, with formation of a π^- condensate.

In order to determine the magnitude and coordinate dependence of the condensate field, as well as the energy of the system, it is necessary to determine the Lagrange function of the nucleons and mesons in the presence of the condensate. In [2] this problem was solved in the model $\mathcal{L}' = \lambda\varphi^4/4$. It turned out that the presence of the condensate stabilizes the system: the pion spectra become stable in the presence of the condensate (all excitation frequencies are positive). Simultaneously with the appearance of the condensate there appears a Goldstone branch of excitations of low frequencies. In order to solve a similar problem in a real system it is necessary to determine the variations of the polarization operator in the presence of a condensate, from where one can find an additional term in the Lagrange function which replaces the model expression used above. The determination of the Lagrange function in the presence of the condensate field allows one to answer the question whether phase transitions of the first kind are possible in the system. Finally, one must solve the problem of determining the condensate field and the energy in a system of finite dimensions, in view of applications to nuclear theory.

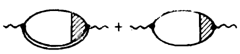
2. CALCULATION OF THE PION POLARIZATION OPERATOR

1. The Diagrams which Determine the Polarization Operator

The pion energy in nuclear matter is described in terms of the polarization operator $\Pi(k, \omega)$ ($\hbar = m_\pi = c = 1$):

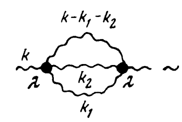
$$\omega^2 = 1 + k^2 + \Pi(k, \omega). \qquad (1)$$

We show that in the region of k and ω values which interests us ($\omega \lesssim 1$, $k \sim 1$) the polarization operator consists of the sum of two terms

$\Pi(k,\omega) =$  $\equiv \Pi_R + \Pi_p'$

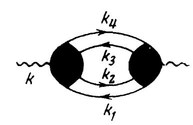
The first term, denoted by Π_R, corresponds to the production of a nucleon hole in the Fermi sea together with an $N_{33}^*(1232)$ isobar (the "resonant term"). The "pole" term Π_p' corresponds to an excitation of the type particle-hole in the medium. All other diagrams, not having parts connected by a particle and hole or a hole and isobar, are determined by large 4-momenta of the intermediate states, and either contribute little, or differ little from the corresponding vacuum diagrams, which are already taken into account in the observed pion mass, used in (1), or, finally are contained in the effective mass m^* of the nucleon, which will be used below ($m^* \approx 0.9$ m).

This is easily seen considering some characteristic diagrams of this type. For example, the diagram



$$\sim \int \lambda^2 \frac{d^4k_1\, d^4k_2}{k_1^2 k_2^2 (k-k_1-k_2)^2} \sim \int \lambda^2(k_1^2) \frac{k_1^7\, dk_1}{k_1^6}$$

is determined by values of k_1^2 which are important in the form factor $\lambda(k_1^2)$, i.e., by values of the order of the square of the mass of the nucleon, m^2, or of the corresponding resonance (i.e., also of the order of m^2). We now consider the diagram



in which the hole 4-momenta k_1 and k_2 are bounded by the conditions $k_{1,3} \le p_F$ and $E_{1,3} \le \epsilon_F$. The ratio of this diagram to the corresponding vacuum diagram, which in place of the hole contains an antiparticle, is easily seen to be $\sim p_F^6/m^6$, i.e., this diagram contributes little compared to the one taken into account in the pion mass. Similarly, all vertices which do not contain parts joined only by a particle-hole or isobar-hole have the "radius" $\sim m^{-1}$ and can be replaced by constants taken from experiment.

We now estimate the error in the pion mass appearing owing to the fact that the incoming pion lines in Π_{vac} are taken off the mass shell, namely for $k^2 - m_\pi^2 = \Pi$. Since the vacuum part of the polarization operator changes considerably for k^2 of the order of m^2 or of the order of the square of the mass of the corresponding resonance, we have

$$\delta m_\pi^2 \sim \frac{\delta \Pi_{vac}}{\delta k^2}(k^2 - m_\pi^2) \sim \left(\frac{m_\pi}{m}\right)^2 m_\pi^2.$$

As we can see, this error is small.

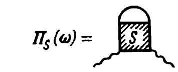
The vertex of the πNN interaction is selected of the form $-f(\bar\psi\gamma_\mu\gamma_5\tau\psi)\partial_\mu\boldsymbol{\varphi}$, where $f = g/2m = 1.0$, m is the mass of the nucleon ($m = 6.7$) and $\boldsymbol{\varphi} = \{\varphi_1, \varphi_2, \varphi_3\}$ is the meson field ($\varphi_{\pi^\pm} = 2^{-1/2}(\varphi_1 \pm i\varphi_2)$, $\varphi_0 = \varphi_3$). More precisely, we shall use the limit of this expression for nonrelativistic nucleons. In this case the vertex operator has the form:

$$\mathcal{T}_0^\alpha = if(\boldsymbol{\sigma}\mathbf{k})\tau^\alpha. \qquad (2)$$

It is clear that the vertex $\mathcal{T}_0^{\alpha}$ selected this way describes the interaction of nucleons with mesons which are only in the P state. Therefore it is necessary to add a part which is determined by the S-state πNN interaction:

$$\Pi_P' = \Pi_P + \Pi_S.$$

For the determination of Π_P we shall make use of the vertex (2) (i.e., $\Pi_P \to 0$ as $k \to 0$). The operator Π_S takes into account the interaction of the mesons with nucleons in the S state. In other words,



(3)

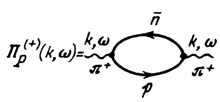
where the block S describes the amplitude for S scattering of the pion on a nucleon of the medium. Thus,

$$\Pi(k,\omega) = \Pi_P(k,\omega) + \Pi_R(k,\omega) + \Pi_S(\omega). \qquad (4)$$

We now study each of these terms in part.

2. The Pole Term of the Polarization Operator. Allowance for the Nucleon Correlations

We consider a medium consisting of particles of a single type, e.g., neutrons. The pole part of the polarization operator of the π^+ mesons, without taking into account the nucleon correlations, is easily calculated:



(5)

$$= 4f^2k^2 \int \frac{d^3p}{(2\pi)^3} \frac{n(p)}{\omega - \varepsilon(\mathbf{p}+\mathbf{k}) + \varepsilon(p)} = -4f^2k^2 \frac{mp_F}{2\pi^2} \Phi_1(k,\omega);$$

$\epsilon(p) = p^2/2m$; $n(p)$ are the neutron occupation numbers. Here and in the sequel m will denote the mass of the nucleon quasiparticles in nuclear matter. As was shown in [8], m is approximately 0.9 of the nucleon mass in vacuum. In our computations we have varied the value of m within limits of 20%. This has had no practical influence on the result.

The function $\Phi_1(k, \omega)$ is given by the expression

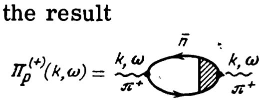
$$\Phi_1(k,\omega) = \frac{m^2}{2k^3p_F}\left\{\frac{a^2-b^2}{2}\ln\frac{a+b}{a-b} - ab\right\}, \qquad (6a)$$

$$a = \omega - k^2/2m, \quad b = kv_F;$$

$$\Phi_1(k,\omega) = -\frac{\pi^2 n}{mp_F}\frac{1}{\omega - k^2/2m}, \quad |a| \gg b. \qquad (6b)$$

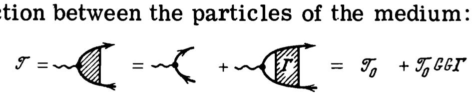
The density is $n = p_F^3/3\pi^2$.

Taking into account the interaction between the nucleons leads to the result



(7)

The shaded region of the graph takes into account the interaction between the particles of the medium:



where Γ is the scattering amplitude for the particle-hole channel, and G is the Green's function of the nucleon in the medium. This relation can be rewritten in the form of an equation for the exact vertex $\mathcal{T}$, which has been investigated in detail in [8] (pp. 178–181, pp. 161–163):

$$\mathcal{T} = \mathcal{T}_0 + \Gamma^{\omega} A \mathcal{T}. \qquad (8)$$

Here Γ^{ω} is the amplitude for the interaction between the quasiparticles near the Fermi surface, A is the product of the pole parts of the single-particle Green's functions. We have set equal to unity the charge of the quasiparticles of the form $\tau_\alpha \sigma_2$ with respect to the external field.

It follows from the spin structure of the vacuum vertex ($\mathcal{T}_0 \propto \sigma$) that in a medium which has no average spin only the terms corresponding to the spin-spin interaction of the nucleons ($\Gamma^{\omega} \propto \sigma_1 \cdot \sigma_2$) contribute to $\mathcal{T}$. Retaining only isospin symmetric terms, we write Γ^{ω} in the form

$$\frac{mp_0}{\pi^2}\Gamma^{\omega} = (g + g'\tau\tau')\sigma\sigma', \qquad (9)$$

where p_0 is the momentum of the Fermi limit of atomic nuclei (in mesonic units $p_0 = (1.5\pi^2 n_0)^{1/3} \approx 2$). The quantities g and g′ are functions of ω and k (the energy and momentum transferred in the particle-hole channel). They differ from the corresponding quantities introduced in [8] in that, by the definition of the polarization operator, they do not include diagrams having one meson in the particle-hole channel. This difference disappears as $k \to 0$.

The functions g and g′ are phenomenological parameters of the theory of Fermi-systems and are determined through a comparison of the theoretical and experimental data. From an analysis of the magnetic moments of nuclei and the rates of beta decay, the following values have been obtained for g and g′ [9]:

$$g = 0{,}5, \quad g' = 0{,}8.$$

For a medium with $Z \ll N$ the functions g and g′ are unknown, but we assume that they differ little from their values in ordinary nuclei. In the determination of the meson spectra the values of g and g′ have been varied within wide ranges and the behavior of the branches of the spectrum and the values of the parameters which are interesting us have varied little (cf. Sec. 3).

Substituting into (8) the expression for the vacuum vertex of the π^+ meson $\mathcal{T}_0^+ = 2^{1/2} if \tau_+ k\sigma_z$ (where $\tau^+ = \frac{1}{2}(\tau_1 + i\tau_2)$) and using (9), we obtain for the exact vertex of the π^+ meson, for an arbitrary relation between Z and N the expression

$$\mathcal{T}^+ = \frac{\mathcal{T}_0^+}{1 + g^- p_0^{-1}[p_F^{(n)}\Phi_1(k,\omega;p_F^{(n)}) + p_F^{(p)}\Phi_1(-k,-\omega;p_F^{(p)})]}, \qquad (10)$$

where $g^- = 2g'$; $p_F^{(n)}$, $p_F^{(p)}$ are the momenta of the Fermi limit of the protons and the neutrons, and $\Phi_1(k, \omega)$ is defined by the expression (6).

Below we shall need the exact vertex of the π^0 meson in the medium. Making use of the expression for the vacuum vertex function of the π^0 meson, $\mathcal{T}_0^0 = if\tau_3 k\sigma_z$ we obtain from (8)

$$\mathcal{T}^0 = ifk\sigma_z[(1 - g\Phi^{(+)})\tau_3 + Ig\Phi^{(-)}]$$

$$\times\left[1 - g^{nn}\Phi^{(+)} + 4gg'\frac{p_F^{(n)}p_F^{(p)}}{n^2}\Phi(k,\omega;p_F^{(n)})\Phi(k,\omega;p_F^{(p)})\right]^{-1} \qquad (11)$$

$$\Phi^{(\pm)} = \frac{p_F^{(n)}}{p_0}\Phi(k,\omega;p_F^{(n)}) \pm \frac{p_F^{(p)}}{p_0}\Phi(k,\omega;p_F^{(p)}).$$

here $g^{nn} = g + g'$ are the amplitudes of the spin-spin interaction of two identical particles at the Fermi limit, I is the isospin unit matrix, and

$$\Phi(k, \omega)=\Phi_1(k, \omega)+\Phi_1(-k, -\omega). \tag{12}$$

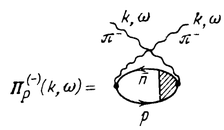
Thus, in a neutron medium ($p_F^{(p)} = 0$) it follows from Eq. (10) that the exact π^+-meson vertex is obtained from the vacuum vertex by multiplication with the factor

$$\left[1+g^- \frac{p_F^{(n)}}{p_0} \Phi_1(k, \omega)\right]^{-1}.$$

For the pole part of the polarization operator of the π^+ meson, defined by Eq. (7), we obtain in this case

$$\Pi_P^{(+)}(k, \omega)=-2f^2k^2 \frac{mp_F^{(n)}}{\pi^2} \frac{\Phi_1(k, \omega)}{1+g^-\Phi_1(k, \omega)p_F^{(n)}/p_0}. \tag{13}$$

The pole part of the polarization operator of the π^- meson in a neutron medium will have the following graphical representation

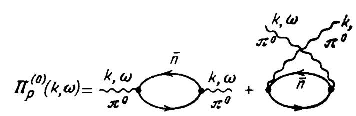


(14)

Comparing with the corresponding diagram for $\Pi_P^{(+)}$ it is obvious that

$$\Pi_P^{(-)}(k, \omega)=\Pi_P^{(+)}(-k, -\omega). \tag{15}$$

Finally, for the π^0 meson in a neutron medium, without taking into account the nucleon correlations, we have



(16)

$$=2f^2k^2 \int \frac{d^3p}{(2\pi)^3} \frac{n(\mathbf{p}+\mathbf{k})-n(p)}{\omega+\varepsilon(\mathbf{p}+\mathbf{k})-\varepsilon(p)}=-2f^2k^2 \frac{mp_F^{(n)}}{2\pi^2} \Phi(k, \omega).$$

The function $\Phi(k, \omega)$ is defined by the expression (12).

In analogy with the case of charged mesons, the interaction between the nucleons leads to the multiplication of (16) by the factor

$$\left[1+g^{nn} \frac{p_F^{(n)}}{p_0} \Phi(k, \omega)\right]^{-1}.$$

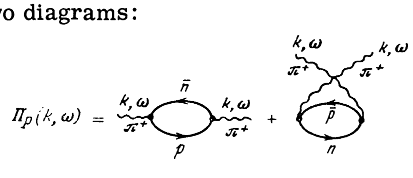
Thus, for $\Pi_P^{(0)}(k, \omega)$ we obtain finally

$$\Pi_P^{(0)}(k, \omega)=-f^2k^2 \frac{mp_F^{(n)}}{\pi^2} \Phi(k, \omega)\left[1+g^{nn} \frac{p_F^{(n)}}{p_0} \Phi(k, \omega)\right]^{-1}. \tag{17}$$

We now go over to a medium consisting of protons and neutrons; let in addition $p_F^{(p)} = p_F^{(n)} = p_F$. Since such a medium is isospin-invariant, the results will be the same for each of the components of the meson field φ_1, φ_2, φ_3 (and consequently also for $\varphi_{\pi^\pm}$ and φ_{π^0})

$$\Pi_P^{(+)}(k, \omega; N=Z)=\Pi_P^{(-)}(k, \omega; N=Z)=\Pi_P^{(0)}(k, \omega; N=Z)\equiv\Pi_P(k, \omega). \tag{18}$$

It suffices therefore to consider, e.g., only π^+ mesons. The pole part of the polarization operator of the π^+ meson, in distinction from (5), is determined in this case by two diagrams:



(19)

$$=4f^2k^2 \int \frac{d^3p}{(2\pi)^3} \frac{n^{(n)}(p)-n^{(p)}(\mathbf{p}+\mathbf{k})}{\omega+\varepsilon(p)-\varepsilon(\mathbf{p}+\mathbf{k})}=-4f^2k^2 \frac{mp_F}{2\pi^2} \Phi(k, \omega).$$

We have made use of the fact that $p_F^{(n)} = p_F^{(p)} = p_F$ as well as of the function $\Phi(k, \omega)$ defined by (12).

As was to be expected, the taking into account of the nucleon correlations in this case leads to the same results for π^+, π^-, and π^0 mesons. Indeed, the isospin structure of the vacuum vertices in an isospin-symmetric medium does not change (for $p_F^{(n)} = p_F^{(p)}$ the second term in (11) vanishes) and the taking into account of the nucleon correlations reduces to multiplication of the vacuum vertices by the same factor

$$\left[1+g^- \frac{p_F}{p_0} \Phi(k, \omega)\right]^{-1},$$

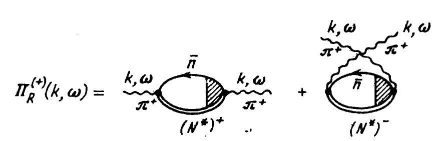
so that, with the interactions between the nucleons of the medium taken into account, the polarization operator takes the form:

$$\Pi_P(k, \omega)=-2f^2k^2 \frac{mp_F}{\pi^2} \Phi(k, \omega)\left[1+g^- \frac{p_F}{p_0} \Phi(k, \omega)\right]^{-1}. \tag{20}$$

The expressions (13), (17), and (20) for the pole part of the polarization operator, with the indicated values of g and g′ are valid for $\omega < \epsilon_F$ [8]. In the case of a neutron ($Z \ll N$) medium, together with $\omega \approx 0$ are important frequencies $\omega \approx 1 > \epsilon_F$. For such large ω one may assume that the function g^- in (13), describing the spin-spin interaction of the proton and the neutron approaches its vacuum value $g^-_{vac} = 0.8$ (cf. [8], p. 314).

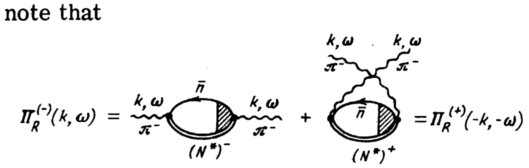
3. The Resonant Part of the Polarization Operator

We start with the case of a medium consisting of neutrons ($Z = 0$). The part of the polarization operator which is due to the production of a neutron hole and the isobar $N_{33}^*(1232)$ has the following graphic representation



(21)

We note that

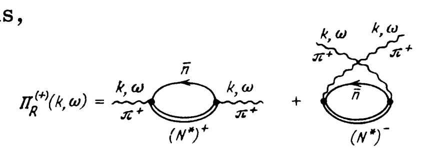


(22)

The shaded vertex in (21) describes the πNN^* interaction in the medium. By the definition of the resonant part of the polarization operator, this vertex must not contain in a section a particle-hole pair (such diagrams are taken into account in the pole term).

The expression for the πNN^* vertex in the medium can be obtained similarly to the way we have derived the πNN vertex. But the results of Ericson and Hüfner[10], who have obtained good agreement with experiment in their description of pion scattering on nuclei in the region of the N_{33}^* resonance by making use of vacuum vertices, suggest that the corrections due to the presence of the medium are not large in the πNN^* vertex. In the same paper it was shown that the change of the isobar mass in the medium is insignificant.

Thus,



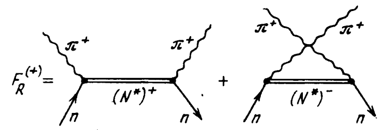
(21a)

and the πNN^*-interaction vertex is equal to its vacuum value.

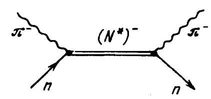
Making use of the usual methods of computation for (21a), it is easy to see that owing to the large resonance energy $\omega_R = 2.4$ one may neglect the difference between the kinetic energies of the isobar and the nucleon, and thus obtain for Π_R:

$$\Pi_R^{(\pm)} = -4\pi n F_R^{(\pm)}, \qquad (23)$$

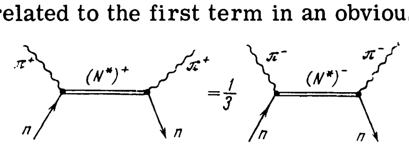
where $n = p_F^3/3\pi^2$ is the density of the medium, $F_R^{(\pm)}$ is the resonant part of the vacuum amplitude for forward scattering of a $\pi^\pm$-meson on a neutron in the laboratory system (l.s.). Graphically $F_R^{(\pm)}$ has the form



The second diagram (on the mass shell and in the resonance region it is 1% of the first diagram) represents the u channel for the diagram



which is related to the first term in an obvious way:



Thus,

$$F_R^{(+)} = a(k)k^2\left[\frac{1}{\omega_R - \omega - i\gamma(k)k^3} + \frac{3}{\omega_R + \omega + i\gamma(k)k^3}\right], \qquad (24)$$

$$F_R^{(-)}(k, \omega) = F_R^{(+)}(-k, -\omega),$$

where $a(k)$ and $\gamma(k)$ are chosen so that on the mass shell ($\omega^2 = 1 + k^2$) one obtains the usual resonance amplitude in the l.s., where

$$\gamma_0(k) \approx a(k) = 0.08/(1 + 0.23k^2), \qquad (25)$$

$\omega_R = 2.36$ is the resonance energy in the l.s. (the resonance mass has been taken to be 1232 MeV). We have taken all the parameters of the N_{33}^* resonance from the paper of Carter et al.[11]

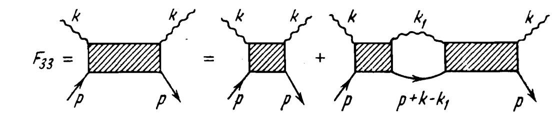
If one goes off the mass shell ($\omega^2 < 1 + k^2$) one must take into account the fact that the damping of the isobar is basically determined by the decay into a nucleon and one pion, and therefore, according to the unitarity condition the damping term must vanish for $\omega < 1$, and for $\omega > 1$ it must coincide with (25), i.e.,

$$\gamma = \gamma_0 \Theta(\omega - 1).$$

In a medium this condition changes on account of the appearance of low-lying excitations. However, since we shall be interested in values of $k \sim 1$, when $\gamma k^2 \ll 1$, then independently of the vanishing of the imaginary part one may neglect in our expressions, γ, which will be done in the sequel.

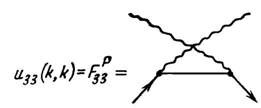
Recently [6], some confusion has arisen in relation to the definition of the concept "resonant amplitude." In this connection we make several remarks.

We write down the Bethe-Salpeter equation for the πN-scattering amplitude $F_{33}(k, p; k', p')$ (k and p are the 4-momenta of the pion and nucleon, respectively):



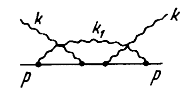
$$= u_{33}(k, k) + u_{33}(k, k_1) D(k_1) G(p + k - k_1) F_{33}(k_1, p + k - k_1; k, p). \qquad (26)$$

Here u_{33} by definition does not contain parts connected only by a pion and nucleon line. For small ω the principal role in $u(k, k)$ is played by the part of the pole amplitude corresponding to isospin $T = \frac{1}{2}$:



$$= -\frac{4}{3}\frac{f^2k^2}{\omega + k^2/2m}.$$

This cannot be asserted about the block $u_{33}(k, k_1)$, which enters into the second term of (26). It is easy to check that in the second term large intermediate 4-momenta $k_1 \sim m$ are important, as follows, e.g., from the diagram



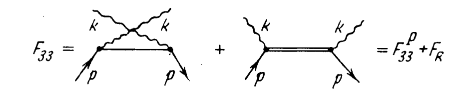
which is typical for the structure of the isobar. This means that for $\omega < 1$ the second term represents a function which depends weakly on ω (and has, of course, a maximum for $\omega \approx m_{N^*} - m$, corresponding to the N_{33}^* resonance). The fact that the intermediate momenta $k_1 \sim m$ allows one to talk of a small "radius" of the isobar, of the order of m^{-1}.

The expression for F_{33} obtained, e.g., in [12] and used with insignificant changes in [13],

$$F_{33} = -\frac{4}{3}\frac{f^2k^2}{\omega}\left(1 - \frac{\omega}{\omega_R} + i\frac{4}{3}\frac{f^2k^3}{\omega}\right)^{-1},$$

is incorrect, particularly off the mass shell, since it involves the assumption that $u_{33}(k, k_1)$ in the second term of (26) is proportional to F_{33}^P. Moreover, for $\omega < 1$, this expression contradicts the unitarity condition ($\mathrm{Im}F_{33}$ must vanish for $\omega < 1$).

Thus in the region of ω and k in which we are interested ($\omega < 1$, $k \sim 1$), the amplitude F_{33} is represented by the sum of a pole part, corresponding to isospin $T = \frac{3}{2}$ and the resonant amplitude:



We are situated near the singularity of the first term F_{33}^P, therefore it is necessary to separate this term out from F_{33}, as we have done, including this diagram into the pole term of the polarization operator. It would be completely incorrect to consider this diagram as included in the resonant amplitude, and on that basis not to take it into account in the pole part of the polarization operator. Furthermore we note that in nuclear matter for $\omega < 1$ the pole part F_{33}^P strongly "feels" the Pauli principle, whereas F_R does not.

For the determination of the parameters of F_R it is necessary to subtract from the observed amplitude F_{33} the pole part F_{33}^P. This leads to a difference of about 5–10% in the parameters of F_R and F_{33}. In the sequel we shall neglect this difference.

Thus,

$$\Pi_R^{(+)}(k,\omega)=\Pi_R^{(-)}(-k,-\omega)=-4\pi nF_R^{(+)}(k,\omega)$$
$$=-4\pi na(k)k^2\left\{\frac{1}{\omega_R-\omega}+\frac{3}{\omega_R+\omega}\right\}. \qquad (27)$$

The polarization operator of the π^0 mesons in a neutron medium is easily obtained from an equation similar to the relation between the amplitudes for the scattering of π^0 and $\pi^\pm$ mesons:

$$\Pi_R^{(0)}(k,\omega)={}^1/_2[\Pi_R^{(+)}(k,\omega)+\Pi_R^{(-)}(k,\omega)]. \qquad (28)$$

Finally, we consider an isosymmetric medium, consisting of protons and neutrons ($p_F^p = p_F^n = p_F$). As already remarked above, the isospin invariance of the medium leads to the fact that the result is the same for π^+, π^-, and π^0 mesons, i.e.,

$$\Pi_R^{(+)}(k,\omega;N=Z)=\Pi_R^{(-)}(k,\omega;N=Z)=\Pi_R^{(0)}(k,\omega;N=Z)\equiv\Pi_R(k,\omega).$$

But for the π^0 mesons the polarization operator Π_R for a medium of given density n does not depend on its isospin composition (i.e., the relation between N and Z), i.e., has the form (28); consequently, for $N=Z$

$$\Pi_R(k,\omega)=-4\pi na(k)k^2\frac{4\omega_R}{\omega_R^2-\omega^2}, \qquad (29)$$

where n is the density of the medium, consisting this time of two types of particles, $n=2p_F^3/3\pi^2$.

4. The Contribution of the S-Wave πN Interaction to the Polarization Operator

Going over to a calculation of the part of the polarization operator which takes into account the S-wave πN interaction, Π_S, we note that the S amplitude is determined by large 4-momenta of the intermediate states and can therefore be taken into account in the gas approximation, i.e.,

$$\Pi_S^{(\pm)}=-4\pi nF_S^{(\pm)},$$

where $F_S^{(\pm)}$ denotes the vacuum amplitude for S-wave scattering of π^+ mesons by neutrons in the l.s.

The scattering amplitude for $\omega<1$ can be represented, following [14,15] as a series in powers of ω (one should add to it the nearest singularity–the S wave of the pole part of the amplitude, but in the whole region of interest (of ω and k values) one may neglect this contribution, within an accuracy of m_π/m). Making use of the crossing-symmetry conditions ($F_S^{(+)}(\omega) = F_S^{(-)}(-\omega)$), we obtain

$$F_S^{(+)}(\omega)-F_S^{(-)}(\omega)=c_1\omega+c_3\omega^3+\dots,$$
$$F_S^{(+)}(\omega)+F_S^{(-)}(\omega)=c_0+c_2\omega^2+\dots.$$

As shown by PCAC theory [14], we have for the l.s. amplitude

$$\frac{c_3}{c_1}\sim\frac{c_2}{c_0}\sim\frac{c_0}{c_1}\sim\frac{m_\pi}{m},\quad c_1\approx\frac{f^2}{g_A{}^2\pi}\approx0.23\quad(g_A=1.18). \qquad (30)$$

Thus, for $\omega\lesssim1$

$$F_S^{(+)}(\omega)-F_S^{(-)}(\omega)=c_1\omega,\quad F_S^{(+)}(\omega)+F_S^{(-)}(\omega)=0 \qquad (31)$$

or

$$F_S^{(\pm)}(\omega)=\pm{}^1/_2c_1\omega. \qquad (31a)$$

This implies a simple expression for c_1 in terms of observable amplitudes:

$$c_1=\frac{F_S^{(+)}(m_\pi)-F_S^{(-)}(m_\pi)}{m_\pi}=0.21.$$

This agrees beautifully with the theoretical value (30).

The relation between the scattering amplitude of the π^0 meson by a neutron, $F_S^{(0)}$, and the amplitudes $F_S^{(\pm)}$ is obvious:

$$F_S^{(0)}(\omega)={}^1/_2[F_S^{(+)}(\omega)+F_S^{(-)}(\omega)]=0.$$

Returning to the polarization operator, we obtain in the case $N=Z$

$$\Pi_S(\omega)=-4\pi nF_S^{(0)}=0. \qquad (32)$$

In a neutron medium ($Z=0$) we have

$$\Pi_S^{(+)}(\omega)=\Pi_S^{(-)}(-\omega)=-4\pi n\frac{c_1}{2}\omega=-1{,}4n\omega,$$
$$\Pi_S^{(0)}(\omega)=0. \qquad (33)$$

5. Sign Reversal of the Spin-Spin Interchange at $k^2\sim1$

Our equations (1) for the meson frequencies can also be obtained by considering the poles of the correlation function (the NN amplitude in the particle-hole channel).

The poles of the correlation function, corresponding to excitations with the quantum numbers of the pions, are associated in the case of a medium with $N=Z$ with the condition

$$1+g_t^-\frac{p_F}{p_0}\Phi(\omega,k)=0. \qquad (34)$$

Here the function $g_t^-(k,\omega)$ describes the complete spin-spin interaction between the nucleons, i.e., in addition to the quantity $g^-(k,\omega)$ introduced above, it contains a term produced by one-pion exchange:

$$g_t^-(k,\omega)=g^-(k,\omega)+\frac{2mp_0}{\pi^2}\frac{f^2k^2}{\omega^2-[1+k^2+\Pi^{(1)}(k,\omega)]}. \qquad (35)$$

By definition, $\Pi^{(1)}(k,\omega)$ contains all the parts of the polarization operator, with the exception of Π_P: $\Pi^{(1)}=\Pi-\Pi_P$.

Taking into account (35), the relation (34) coincides identically with the above listed dispersion equation (1). In a medium with $N=Z$ an instability arises if the meson spectrum contains points with $\omega=0$. It follows from (34) and from the fact that $0<\Phi(0,k)\le1$, it is necessary for the appearance of such points that $g_t^-(0,k)\le-p_0/p_F$ for some interval of k values.

Considering the diagrams which contribute to the spin-spin interaction of the nucleons one can see that only the one-meson diagram depends essentially on ω and k (for $\omega<\epsilon_F$ and $k\sim1$). The characteristic 4-momenta in all other diagrams, in particular in those which are responsible for the repulsion of nucleons at small distances, are of the order $2p_F$, $1/r_c$, or m ($r_c\approx0.3\hbar/m_\pi c$ is the range of the repulsive potential). Thus, the relation (35) for $\omega=0$ can be rewritten in the form

$$g_t^-(k)=g^-(0)-\frac{2mp_0}{\pi^2}\frac{f^2k^2}{1+k^2+\Pi^{(1)}(0,k)}+O\left(\frac{k^2}{4p_F{}^2},k^2r_c{}^2\right),$$
$$g_t^-(0)=g^-(0)=1{,}6. \qquad (36)$$

From this equation it follows that for densities of the order of the nuclear density the function $g_t^-(k)$ necessarily changes sign already at $k\sim1$ ($g_t^-=0$ for $k\approx0.8$, for $n=n_0$). From the expression (36) we easily obtain an estimate for the lower bound of the critical density from which an instability of nuclear matter may arise. Considering that $k\sim p_F$ we find that $g_t^-<-p_0/p_F$, starting with $p_F\approx1.6$ ($n\gtrsim0.25$).

Such a behavior of $g_t^-(k)$ does not contradict the experimental facts. From an analysis of the magnetic moments and spectra of nuclei [9,16] the following values have been obtained for g_t^- at two points:

$$g_t^-(0)\approx 1{,}6, \quad g_t^-(2p_0)\approx -1.$$

Substituting the first of these values into (36) and taking into account that $|\Pi^{(1)}(0, 2p_0)|/4p_0^2 \ll 1$, we obtain

$$g_t^-(2p_0)\approx -1{,}2+O(k^2/4p_0^2, k^2r_c^2).$$

This means that even for $k^2 \sim 4p_0^2 \approx 16$ the correction terms in (36) do not give a large contribution.

3. THE PION SPECTRUM IN THE ABSENCE OF CONDENSATE AND THE INSTABILITY OF THE PION FIELD

1. Quantization of the Meson Field in the Medium

In the preceding sections we have investigated in detail the properties of the polarization operator $\Pi(k, \omega)$ of the mesons in the nucleon medium. We now discuss the basic properties of the solutions of the meson field equations. We start from the case of charged mesons, for which

$$[\omega^2-1-k^2-\Pi^{(\pm)}(k,\omega)]\varphi_{k,\omega}^{(\pm)}=0.$$

In place of studying separately the fields $\varphi^{(+)}$ and $\varphi^{(-)}$ for the π^+ and π^- mesons, we introduce the complex field

$$\Psi=\sum_k \{c_k^{(+)}a_k \exp(i\omega_k^{(+)}t-i\mathbf{kr})+c_k^{(-)}b_k^+\exp(-i\omega_k^{(-)}t+i\mathbf{kr})\},$$

where a_k and b_k are the annihilation operators of the π^+ and π^- mesons. The coefficients $c^{(+)}$ and $c^{(-)}$ are defined in the following manner.

The Lagrangian of the field Ψ has the form

$$\mathscr{L}=\sum_k \Psi_k^+[\omega_k^2-1-k^2-\Pi^{(+)}(k,\omega_k)]\Psi_k, \qquad (37)$$

where ω_k is an independent variable, not to be confused with the solution $\omega(k)$ of the equation of motion. Making use of the usual method of discussing a Lagrangian involving time derivatives of arbitrary order, it is easy to derive the following formula for the component T_{44} of the energy-momentum tensor:

$$T_{44}=\mathscr{H}=\sum_k\left(\omega_k\frac{\partial\mathscr{L}}{\partial\omega_k}\right)_{k=\omega(k)}-\mathscr{L}. \qquad (38)$$

Let us illustrate this relation on the example of the electromagnetic field in a medium with permittivity $\epsilon(\omega)$ and magnetic susceptibility $\mu(\omega)$. The time average of the Lagrange function, expressed in terms of the vector potential $\mathbf{A}$ ($\mathbf{B}=\text{curl}\ \mathbf{A}$, $\mathbf{E}=-\dot{\mathbf{A}}$ $|\mathbf{A}|=A_0\sin\omega t$) is of the form

$$\overline{\mathscr{L}}=\sum_k\left(\varepsilon\omega^2-\frac{k^2}{\mu}\right)\frac{A_0^2}{16\pi}.$$

From here we obtain with the help of (38) the following expression for the average field energy

$$\overline{U}=\frac{d(\varepsilon\omega)}{d\omega}\frac{E_0^2}{16\pi}+\frac{d(\mu\omega)}{d\omega}\frac{\mathscr{H}_0^2}{16\pi}$$

in agreement with [17].

From Eqs. (37) and (38) it is easy to obtain an expression for the Hamiltonian of the pion field

$$\mathscr{H}=\sum_k\Psi_k^+\left[\omega_k\left(2\omega_k-\frac{\partial\Pi^{(+)}}{\partial\omega_k}\right)\right]_{\omega_k=\omega(k)}\Psi_k$$

$$=\sum_k\{(c_k^{(+)})^2\omega^{(+)}(k)\Omega^{(+)}(k)a_k^+a_k+(c_k^{(-)})^2\omega^{(-)}(k)\Omega^{(-)}(k)b_k^+b_k\},$$

$$\Omega^{(\pm)}(k)=\left(2\omega_k-\frac{\partial\Pi^{(\pm)}(k,\omega)}{\partial\omega_k}\right)_{\omega_k=\omega^{(\pm)}(k)} \qquad (39)$$

(we have used the equality $\Pi^{(+)}(k,\omega)=\Pi^{(-)}(-k,-\omega)$). We require the Hamiltonian to have the form

$$\mathscr{H}=\sum_k\{\omega^{(+)}(k)a_k^+a_k+\omega^{(-)}(k)b_k^+b_k\},$$

where $\omega^{(\pm)}$ are the energies of the $\pi^\pm$ mesons. This implies that

$$(c_k^{(+)})^{-2}=\Omega^{(+)}(k), \quad (c_k^{(-)})^{-2}=\Omega^{(-)}(k) \qquad (40)$$

Thus,

$$\Psi(\mathbf{r},t)=\sum_k\left\{\frac{a_k\exp[i\omega^{(+)}(k)t-i\mathbf{kr}]}{[\Omega^{(+)}(k)]^{1/2}}+\frac{b_k^+\exp[-i\omega^{(-)}(k)t+i\mathbf{kr}]}{[\Omega^{(-)}(k)]^{1/2}}\right\}. \qquad (41)$$

The same result can also be obtained in another way. By the same method used for the computation of T_{44} one can determine the current 4-vector. From (37) we obtain the following expression for the charge density

$$j_0=e\frac{\partial\mathscr{L}}{\partial\omega_k}=e\sum_k\Psi_k^+\left(2\omega_k-\frac{\partial\Pi^{(+)}}{\partial\omega_k}\right)_{\omega_k=\omega(k)}\Psi_k. \qquad (42)$$

Requiring that j_0 be of the form

$$j_0=e\sum_k(a_k^+a_k-b_k^+b_k), \qquad (43)$$

we again obtain (40). The factor $2\omega-\partial\Pi/\partial\omega$ in (42) arises also from the Ward theorem, according to which the electromagnetic vertex (i.e., its 4-component) has the form

$$\Gamma_4^{el}=\frac{\partial D^{-1}}{\partial\omega}=2\omega-\frac{\partial\Pi}{\partial\omega} \quad D^{-1}=\omega^2-1-k^2-\Pi(k,\omega).$$

As follows from (41), the dependence of $\omega^{(+)}(k)$ for the π^+ mesons must be such that

$$\left[2\omega-\frac{\partial\Pi^{(+)}(k,\omega)}{\partial\omega}\right]_{\omega=\omega^{(+)}(k)}>0.$$

for the π mesons

$$\left[2\omega-\frac{\partial\Pi^{(-)}(k,\omega)}{\partial\omega}\right]_{\omega=\omega^{(-)}(k)}>0$$

or

$$\left[2\omega-\frac{\partial\Pi^{(+)}(k,\omega)}{\partial\omega}\right]_{\omega=-\omega^{(-)}(k)}<0.$$

We thus obtain the following rules for selecting solutions.

Assume we know the solutions $\omega(k)$ of the equation

$$\omega^2=1+k^2+\Pi^{(+)}(k,\omega).$$

The solutions lying in the region

$$2\omega-\partial\Pi^{(+)}(k,\omega)/\partial\omega>0,$$

correspond to π^+ mesons. The solutions situated in the region

$$2\omega-\partial\Pi^{(+)}(k,\omega)/\partial\omega<0,$$

become, after the substitution $\omega\to-\omega$ the dispersion law for the π^- mesons.

By analogy with the method by which we have obtained the representation (41) for the field of the charged mesons, it is easy to find the corresponding expression for the π^0 mesons. The density of π^0 mesons is

$$\rho_3=\sum_k\left(2\omega_k-\frac{\partial\Pi^{(0)}}{\partial\omega_k}\right)_{\omega_k=\omega(k)}\varphi_3^2. \qquad (44)$$

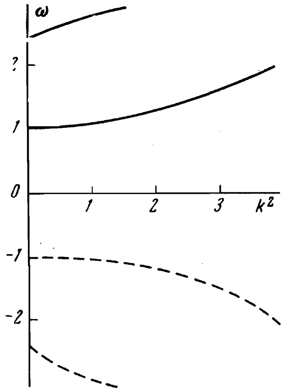


FIG. 1

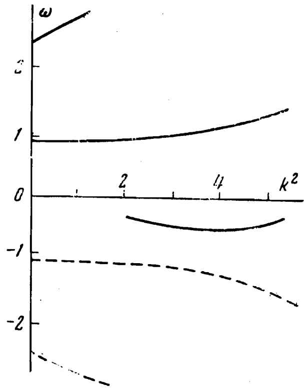


FIG. 2

FIG. 1. The spectrum of charged mesons in a neutron medium with density $n = 0.1 < n_c^+$ ($g^- = 0.8$). The solid line represents the energy of the π^+ mesons, $\omega^{(+)}$, the dotted line represents the energy of the π^- mesons with sign reversed ($-\omega^{(-)}$). For all k we have $\omega^{(+)} + \omega^{(-)} > 0$ and $\omega^{(+)} > 0$.

FIG. 2. The spectrum of charged mesons in a neutron medium with density $n = 0.3$ ($n_c^+ < n < n_c^{\pm}$, $g^- = 0.8$). For $2 \lesssim k^2 \lesssim 5.5$ the energy $\omega^{(+)}(k) < \epsilon_F^{(n)}$, which leads to an instability of the protons in such a medium with respect to the process $p \to n + \pi^+$. Everywhere $\omega^{(+)} + \omega^{(-)} > 0$.

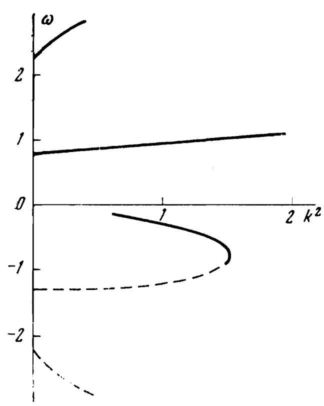


FIG. 3. The spectrum of charged mesons in a neutron medium with density $n = 0.5 > n_c^{\pm}$ ($g^- = 0.8$). For $k = k_c^{\pm} = 1.4$ we have $\omega^{(+)}(k) + \omega^{(-)}(k) = 0$. A system with such a density is unstable with respect to the production of $\pi^+\pi^-$-meson pairs.

Here $\Pi^{(0)}$ is an even function of ω, $\partial\Pi^{(0)}/\partial\omega = 2\omega\,\partial\Pi^{(0)}/\partial\omega^2$. One can see from the spectral decomposition that $\partial\Pi^{(0)}/\partial\omega^2 < 0$ (this can also be seen from the example of the expressions (20) and (29)). Consequently, the physical solutions for the π^0 meson correspond to the condition $\omega > 0$.

2. The Pion Spectra in the Case Z = 0 and Z = N

We first consider the case $Z = 0$. Collecting the results of Sec. 2, we obtain the following equation for the determination of the energy of the π^+ meson, $\omega(k)$:

$$\omega^2(k)=1+k^2+\Pi^{(+)}(k,\omega)=1+k^2-\frac{2mp_F}{\pi^2}f^2k^2\Phi_1(k,\omega)\left[1+g^-\frac{p_F}{p_0}\Phi_1(k,\omega)\right]^{-1}-1{,}4n\omega-4\pi na(k)k^2\left(\frac{1}{\omega_R-\omega}+\frac{3}{\omega_R+\omega}\right), \qquad (45)$$

where $n = p_F^3/3\pi^2$ is the density of nucleons.

Figure 1 shows the result of a numerical solution of Eq. (45) for $n < n_c^+$. The solid line represents the branch of the spectrum where $2\omega - \partial\Pi^{(+)}/\partial\omega > 0$, i.e., the branch corresponding to π^+ mesons, the dotted line represents the portion where $2\omega - \partial\Pi^{(+)}/\partial\omega < 0$, i.e., corresponds to π^- mesons (we recall that for π^- mesons the sign of ω changes to the opposite one). Starting with a density $n_c^+ = 0.2$ in the π^+-meson spectrum there appears a solution with $\omega < -\epsilon_F^{(n)}$ (cf. Fig. 2). The presence of such solutions leads to an instability of the proton in a neutron medium ($p \to n + \pi^+$). We note that for $n < n_c^{\pm}$ the meson energies are such that $(\omega^{(+)} + \omega^{(-)}) > 0$.

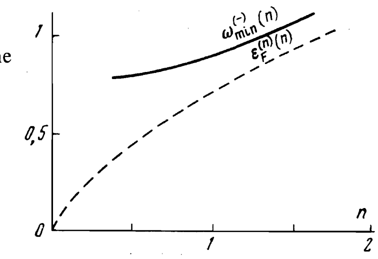


FIG. 4. The minimal energy $\omega_{min}^{(-)}(n)$ of the π^- mesons in the neutron medium as a function of the density n – represented by the solid line and the Fermi energy of the neutrons $\epsilon_F^{(n)}(n)$ (broken line); $\omega_{min}^{(-)}(n) > \epsilon_F^{(n)}$, therefore the process $n \to p + \pi^-$ is impossible.

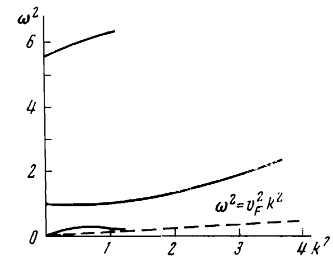


FIG. 5. The spectrum of π^0 mesons in a neutron medium of density $n = 0.3 > n_c^0$ ($g^{nn} = 1$). The three branches of the spectrum correspond to the three types of excitations: the isobar-hole type ("resonance branch", $\omega(k = 0) \approx \omega_R$), the mesonic branch ($\omega(k = 0) = 1$), and the particle-hole type ("spin-acoustic" branch). For all k, $\omega^2 > 0$.

The spectrum of charged mesons in a medium with a density $n \geq n_c^{\pm} = 0.4$ (Fig. 3) is distinguished by a characteristic peculiarity: the presence of a point where $\omega^{(+)} + \omega^{(-)} = 0$ (at this point $2\omega - \partial\Pi^{(+)}/\partial\omega = 0$, $d\omega/dk = \infty$), i.e., a system with such a density is unstable with respect to the production of pairs of $\pi^+\pi^-$ mesons, similar to what happened in a strong electric field.[1]

Figure 4 represents the minimal energy of the π^- meson as a function of the neutron density, $\omega_{min}^{(-)}(n)$ for $n > n_c^{\pm}$, from where it can be seen that even without taking into account the stabilizing action of the condensate, $\omega_{min}^{(-)} - \epsilon_F^{(n)} > 0$. This implies the impossibility of a phase transition of the second kind with the formation of a π^- condensate ($n \to p + \pi^-$).

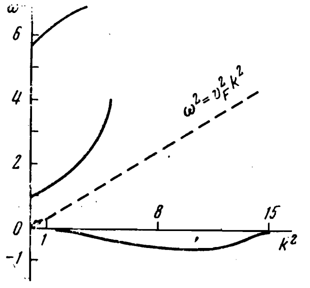
The dispersion law for the π^0 meson has the following form:

$$\omega^2=1+k^2+\Pi^{(0)}(k,\omega)=1+k^2-\frac{mp_F}{\pi^2}f^2k^2\Phi(k,\omega)\left[1+g^{nn}\frac{p_F}{p_0}\Phi(k,\omega)\right]^{-1}-\frac{16\pi a(k)k^2\omega_R n}{\omega_R^2-\omega^2}. \qquad (46)$$

A numerical solution of this equation yields the spectrum $\omega^2(k^2)$ represented in Fig. 5 for $n > n_c^0 = 0.4$ For a density $n > n_c^0$ (Fig. 6) there appears a region $\omega^2 < 0$, which signifies an instability with respect to the production of neutral mesons.

We now treat the case $Z = N$. As we already said, in such a medium isospin invariance implies that the results for all mesons are the same. The energy $\omega(k)$ of the meson is determined by the equation

$$\omega^2=1+k^2+\Pi(k,\omega)=1+k^2-\frac{2mp_F}{\pi^2}f^2k^2\Phi(k,\omega)\left[1+g^-\frac{p_F}{p_0}\Phi(k,\omega)\right]^{-1}-\frac{16\pi na(k)k^2\omega_R}{\omega_R^2-\omega^2} \qquad (47)$$



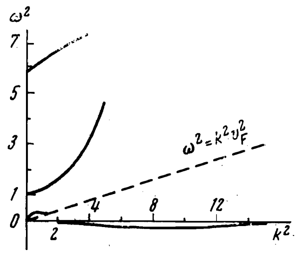


FIG. 6 FIG. 7

FIG. 6. The spectrum of π^0 mesons in a neutron medium with density $n = 0.9 > n_c^0$ ($g^{nn} = 1$). For $1.5 \lesssim k^2 \lesssim 15$ there exists a branch with $\omega^2 < 0$. A system with such a density is unstable with respect to the production of π^0 mesons.

FIG. 7. The meson spectrum in a medium with Z = N at a density $n = n_0 = 0.5 > n_c$ (n_0 is the nuclear density, $g^- = 1.6$). For $2 \lesssim k^2 \lesssim 15$ the quantity $\omega^2 < 0$. The system is unstable with respect to the production of $\pi^+\pi^-$, and π^0 mesons.

$Z=0$					$Z=0$			$Z=N$		
g^-	$n_c^{\pm}$	$k_c^{\pm}$	n_c^+	k_c^+	g^{nn}	n_c^0	k_c^0	g^-	n_c	k_c
0	0.37	1.4	0.2	p_F	0	0.2	2.4	0	0.1	1.6
0.3	0.39	1.6			1	0.4	2.5	0.8	0.2	2.2
0.6	0.41	1.6			1.6	0.6	2.5	1.6	0.3	2.4
0.8	0.43	1.6								

This equation has the same form as the one for neutral mesons in the case $Z = 0$.

For $n < n_c$ the spectrum is analogous to Fig. 5; for $n > n_c = 0.3$, as before, there appears a region $\omega^2 < 0$ (cf. Fig. 7), but for $N = Z$ this means already an instability with respect to the simultaneous production of π^+, π^-, and π^0 mesons.

The table lists the characteristic parameters for different values of g^- and g^{nn}. The quantities k_c^0 and k_c are the values of k for which $\omega^2 = 0$, and accordingly $n = n_c^0$ ($Z = 0$) and $n = n_c$ ($Z = 0$); $k_c^{\pm}$ is the value of k for which $\omega^{(+)} + \omega^{(-)} = 0$.

3. Instability of the Pion Field and Pion Condensation

As we see from Figs. 1–7, there are three branches of the pion spectrum corresponding to the three types of possible excitations: the meson branch ($\omega(k=0) \approx 1$) the resonance branch (excitations of the isobar-hole type) ($\omega(k=0) \approx \omega_R$) and the particle-hole branch, which we shall call the spin-acoustic branch. The resonant branch is of interest in questions related to pion-nucleus scattering in the region of the N_{33}^* resonance. The two others are essential in studying questions of stability.

In a symmetric medium, as already noted, the polarization operator is an even function of the frequency and the physical solutions correspond to $\omega > 0$. Starting from $n = n_c = 0.3$ there appears a solution with $\omega^2 < 0$ for all mesons (an interval of k^2 in which $\omega^2 < 0$ increases from zero as n increases). It follows that a pion condensate must exist in the nucleus, and the presence of this condensate will influence the computations of the different characteristics of the nucleus, e.g., its binding energy, etc.

Near the transition point the condensate field has the form $\varphi_0 \propto \sin k_0 x$, where k_0 corresponds to the minimum of the energy and approximately corresponds to a minimum of ω^2 on the spin-acoustic branch ($k_0 \sim p_F$) [2]. Since φ_0^2 turns out to be a periodic function of the coordinates, the nucleon density has a modulation with the same period. Thus, apparently, the nucleus has a stratified structure, the direction of which is coupled to the spin of the nucleus. This should lead to peculiar phenomena of the type of Bragg reflections in the scattering of particles on oriented nuclei.

As a result of the phase transition there appears a low-lying (Goldstone) branch. The existence of such oscillations can be an argument in favor of the existence of a condensate.

For $Z \ll N$ (a neutron star) at a density $n = n_c^+$ the π^+-mesons acquire a branch with energy $\omega^{(+)} < 0$. At the instant of appearance of this branch $\omega^{(+)} + \epsilon_F^{(n)} = 0$ ($k = k_c^+ = p_F$). As the density increases further $|\omega^{(+)}|$ grows faster than $\epsilon_F^{(n)}$. This has an important implication. Usually one assumes that in a neutron star there exists an admixture of protons, the charge of which is compensated by electrons. Since $\omega^{(+)} + \epsilon_F^{(n)} < 0$ the protons will convert into neutrons and π^+-mesons ($p \to n + \pi^+$). The equilibrium number of π^+ mesons and electrons is determined by the equation $\omega^{(+)} + \epsilon_F^{(l)} = 0$.

At a density $n = n_c^{\pm} = 0.4$ the energies of the pair of $\pi^+\pi^-$ mesons $\omega^{(+)} + \omega^{(-)}$ vanishes for $k_c^{\pm} = 1.6$, leading to the formation of a $\pi^+\pi^-$ condensate. Approximately for the same density there appears a condensate of π^0 mesons. As was shown in [1] and then in [2,3] the appearance of the condensate stabilizes the system. The formation of a condensate corresponds to a phase transition of the second kind.

The minimal energy of the π^- meson before the appearance of the $\pi^+\pi^-$ and π^0 condensates is larger than $\epsilon_F^{(n)}$ and thus the phase transition of the second kind with formation of a π^- condensate is impossible. When the condensates appear the minimal energy of the π^- mesons increases, as was shown in [3], so that $\omega^{(-)} - \epsilon_F^{(n)} > 0$, at least up to very large densities, which proves the assertion that the π^- condensate which was conjectured in [4] does not in fact exist.

In ref. [2,3] the problem of coordinate-dependence and energy of the condensate field was solved in the $\mathscr{L}' = \lambda\varphi^4/4$ model. This model describes the picture only from a qualitative point of view. In order to solve a realistic problem it is necessary to determine the variation of the Lagrange function of the system in the presence of the condensate. This problem, as well as the problem of the possibility of phase transitions of the first kind will be discussed by the authors in the sequel.

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46