Damping law for the external fields of a collapsing body

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The laws of variation of the electromagnetic and gravitational fields in the space surrounding a collapsing star are obtained. At large distances from the star, the wave part of the field following the exponential front falls off according to a power law of the type $1/r\tau^{l+2}$, where τ is the time reckoned at the given point from the instant the exponential front passes the point. The period during which this power law remains valid is determined by the distance R over which the gravitational field of the collapsar is comparable to the fields of the surrounding bodies. In the idealized case when the collapsing star is the only gravitational-field source, the period during which our law remains valid is, in light units, equal to the distance r to the observation point. After a time $\tau \sim r/c$, the residual field in this idealized case relaxes according to the law t^{-2l-2}. The same law determines the relaxation of the fields at distances $r \sim r_g$. The appearance at steep wave-pocket fronts of tails with a power damping law is a general feature of the passage of waves through a space-curvature barrier created by a gravitating body.

1. INTRODUCTION

The rapid progress made in astrophysics has generated an increasing interest in "black holes"—bodies undergoing the phenomenon of gravitational collapse. The exact solution to the problem of the gravitational collapse of a spherically symmetric mass distribution is well known [1]. In the space surrounding such an object, the gravitational and electric fields do not vary. Real astronomical objects are always nearly spherically symmetric formations. In connection with the search for celestial bodies undergoing gravitational collapse, the problem of the fields outside such an object arouses great interest. Interest in the problem surged particularly when Regge and Wheeler [2] discovered that the static solutions for the multipole components of the gravitational field diverge as $r \to r_g$, where r_g is the gravitational radius of the body (see also [3,4]).

The problem of the fields of a collapsing object has been considered by Ginzburg and Ozernoĭ for the electromagnetic field [5] and by Doroshkevich, Zel'dovich, and Novikov for the gravitational field [4,6]. The static solutions for the spherically nonsymmetric part of the field as we approach the gravitational radius r_g vary according to the law (see [4]):

$$\psi \sim \ln(1 - r_g/r). \qquad (1.1)$$

The requirement that the fields have no singularities in the dynamical process as $r \to r_g$ leads, as has been shown in [5,6], to the important conclusion that the nonspherical part of the fields outside matter vanishes asymptotically. Estimates made in [5,6] show that this decrease occurs according to, or faster than, the law $\sim 1/t$.

A more detailed mathematical analysis carried out in [7] shows that the fields near r_g are by no means quasi-static in nature. The requirement that the field on the stellar surface be analytic in the proper time s of the surface leads to a boundary condition for the wave generated by the surface—the amplitude of this wave varies according to an exponential law. In this case, however, the effect of anomalous tunneling through a space-curvature barrier—an effect which is similar to the anomalous-scattering effect in quantum mechanics[8]—is not manifested. The anomalous-tunneling effect leads, as will be shown below, to the appearance of a power "tail" after the passage of the exponential front. The period during which this power law remains valid is equal to $\tau \sim r$, where r is the distance from the star to the observer.

After a time $\tau \sim r$, the amplitude $\psi(r, t)$ of the field becomes a quantity of the order of a static multipole field: $\psi_{st} \sim \mu/r^{l+1}$. This part of the field relaxes to zero according to a different law. In a real situation, this relaxation law is determined by the physical conditions of the observation. The law according to which this residual relaxation occurs in the case when the collapsing star is significantly nearer to the observation point than any other celestial body has been estimated for $t \gg r$ by Price [9].

The purpose of the present paper is to investigate the behavior of the fields of a slightly nonspherical object during its collapse. Notice that the slightness of the deviations from spherical symmetry is a characteristic feature of sufficiently dense astrophysical objects of mass $M \gtrsim M_\odot$, so that the investigation of the slightly nonspherical case is physically justified.

2. FORMULATION OF THE PROBLEM AND THE BOUNDARY CONDITIONS

The fields in the space surrounding the star are assumed to be sufficiently weak, so that we can neglect their influence on the basic Schwarzschild metric and consider the linear—in the field amplitude—problem of the propagation of gravitational, electromagnetic, scalar, etc., perturbations in the space curved by the gravitational field of the stellar mass. After simple transformations (see Appendix I), it is possible to express the fields in terms of the field function $\psi(r, t)$. The equation for the partial harmonic $\psi_l(r, t)$ of the field function has the form

$$\partial^2\psi/\partial t^2 = \partial^2\psi/\partial x^2 - U_l(x)\psi, \qquad (2.1)$$

where we have introduced the "characteristic" variable

$$x = r + r_g \ln\frac{r - r_g}{r_g},$$

and chosen the system of units in which the velocity of light $c = 1$.

Let us represent the solution of Eq. (2.1) in the form of a Fourier integral:

$$\psi(t,x)=\int_{-\infty}^{\infty}dkf_k\psi_k(x)e^{-ikt}. \qquad (2.2)$$

Then we obtain for ψ_k a Schrödinger-type equation:

$$\psi_k''+[k^2-U_l(x)]\psi_k=0. \qquad (2.3)$$

The potential U_l for the various fields is given in the Appendix I. For example, for the case of the electromagnetic field

$$U_l(x)=\frac{l(l+1)}{r^2(x)}-\frac{l(l+1)r_g}{r^3(x)}, \quad l=1,2,\ldots$$

For all the fields the potentials $U_l(x)$ have the following universal asymptotic form:

$$U_l(x\to-\infty)\approx\frac{l(l+1)}{r_g^2}\exp\left(\frac{x}{r_g}\right), \qquad (2.4a)$$

$$U_l(x\to+\infty)\approx\frac{l(l+1)}{x^2}+\frac{2l(l+1)r_g}{x^3}\ln\frac{x}{r_g}. \qquad (2.4b)$$

Notice that as x varies from $-\infty$ to $+\infty$, U_l increases monotonically, reaches a maximum at some $x=x_0$, and then decreases monotonically to zero, remaining always positive in the entire interval of variation of x.

The complete solution of the problem presupposes the simultaneous solution of the problem of the stellar motion, as well as of the problem of the fields inside the star. The fields on the stellar suface goes over continuously into the fields outside the star, which is how the solutions are matched. We shall assume that on the stellar surface the field and the field function are smooth functions of the proper time s of a particle moving together with the surface. In particular, there is nothing to distinguish for the particles the moment $s=0$ when the line $r=r_g$ intersects the surface; therefore, for $s\ll 1$, we can write for the field function $\psi_{sur}(s)$ on the surface:

$$\psi_{sur}(s)=a+bs+cs^2+\ldots \qquad (2.5)$$

In the relativistic phase of the collapse, for $r\to r_g$ and $x\to-\infty$, the equation of motion of the surface in the Lemaitre coordinates R, τ (see, for example, [1]) is, up to small terms, the equation of a straight line:

$$R=\alpha\tau. \qquad (2.6)$$

The value of the constant α depends on the specific conditions, and is limited only by the condition $|\alpha|<1$. The transition to the Schwarzschild time t and to the coordinate x is made with the aid of the equation of the trajectory of the surface:

$$t+x=\frac{s}{\sqrt{\alpha^2+1}}=\beta s, \quad t\to+\infty, \quad x\to-\infty \qquad (2.7)$$

and

$$r-r_g=r_ge^{x/r_g}=r_ge^{-t/r_g}=\frac{\alpha-1}{\sqrt{\alpha^2+1}}s. \qquad (2.8)$$

Let us describe the initial and boundary conditions which must be satisfied by the solution to Eq. (2.1) in the case under investigation.

1. The initial data are prescribed for $t=t_0$ in the region $x\geq x_0$. Here x_0 is the point at which the surface is located at the moment $t=t_0$.

2. Only an outgoing wave exists at $x\to+\infty$ and $t>t_0$. We write the general solution to Eq. (2.1) that satisfies this requirement in the form (see (2.2)):

$$\psi_k(x)=A_k(x)[e^{ikx}+B_k(x)e^{-ikx}]. \qquad (2.9)$$

The quantities $A_k(x)$ and $B_k(x)$ satisfy the following boundary conditions:

$$A_k(x\to-\infty)=1,\ B_k(x\to+\infty)=0. \qquad (2.10)$$

Notice that $A_k(+\infty)$ and $B_k(-\infty)$ have respectively the meaning of transmission and reflection coefficients.

3. On the trajectory $x(t)$ of the surface, the function $\psi(t, x(t))$ is a regular function of the proper time s of a particle on the surface. If we choose the moment t_0 such that the proper time $s(t_0)$, reckoned from the moment the surface of the body passes the point $r=r_g$, is small, then the $\psi(s)$ dependence has the form (2.5), while the trajectory of the surface is described by the formulas (2.6)–(2.8). In this case the point x_0 turns out to be located in the region where the potential $U_l(x)$ is exponentially small. The conditions 1, 2, and 3 allow us to uniquely determine the solution. The construction of the functions $A_k(x)$ and $B_k(x)$ is presented in the Appendix II, while the quantities f_k are computed from the initial and boundary data in Appendix III.

3. RESULTS AND DISCUSSION

We represented the solution of our problem in the form (2.2) and (2.9):

$$\psi(t,x)=\int_{-\infty}^{\infty}f_k\psi_k(x)e^{-ikt}dk,$$
$$\psi_k(x)=A_k(x)[e^{ikx}+B_k(x)e^{-ikx}]. \qquad (3.1)$$

The standard solution $\psi_k(x)$ in the region $|x|\gg r_g$ is computed in the Appendix II. The results for large values of x for the regions $|kx|\gg 1$ and $|kx|\ll 1$ (see (II.21)–(II.23)) are significantly different.

Let us separate out the dominant terms having singularities in the region of small values of k:

$$\psi_k(x)=\psi_1(x)+\text{const}\,e^{ikx}(kr_g)^{l+2}\ln(kr_g),\ |kx|\gg 1, \qquad (3.2)$$

$$\psi_k(x)=\psi_2(x)+\text{const}\,(kx)^{l+1}(kr_g)^{l+2}\ln(kr_g),\ |kx|\ll 1. \qquad (3.3)$$

Here ψ_1 and ψ_2 are terms that are regular in the variable k. Notice that (3.2) corresponds to a transmitted wave, while (3.3) corresponds to a reflected wave.

The spectral function f_k is found in the Appendix III, and has the form (see (III.8))

$$f_k=\text{const}\frac{1-e^{ikt_1}}{(kr_g)^2}. \qquad (3.4)$$

The constant in (3.4) is proportional to the magnitude of the multipole moment produced by the static field at a distance $r=R$ from the collapsing mass to the nearest stars that, at this distance, curve space more strongly than the collapsing star.

Let us now compute the integral (3.1). Notice first of all that there are two characteristic times in the problem: the time $\tau_0=r_g$ and the time x (let us recall that the velocity of light $c=1$). It is natural, in computing (3.1), to shift the contour of the integration over k into the complex k plane. For positive $\tau=t-x$, the exponentials allow us to shift the contour into the lower half-plane. The contribution of the regular parts of $\psi_k(x)$ then yields, on account of the residues at the poles of f_k at $k=-in/2r_g$, $n=0, 1, \ldots$, the expression

$$\psi_1(t,x)\sim\exp(-\tau n/2r_g). \qquad (3.5)$$

The formula (3.5) is not quantitatively applicable when $\tau \leq \tau_0$. This formula only indicates that the quantity $\psi(t, x)$ changes rapidly by some constant value during the period τ_0.

The contribution of the terms having branch cuts and described by the formulas (3.2) and (3.3) depends on the relation between the quantities τ and x. If $\tau_0 \ll \tau \ll x$, then the dominant role in (3.1) is played by the term

$$\int_{-\infty}^{+\infty} e^{-ik(t-x)} A_k(x)\, dk$$

and the asymptotic form of ψ_k for $k(t-x) \sim 1$, i.e., for $|kx| \gg 1$, is important. The major role is now played by the integral over the branch cut arising in the lower half-plane because of the logarithmic factor in (3.2):

$$\psi \approx \text{const} \int_{-\infty}^{+\infty} dk e^{-ik(t-x)} (kr_g)^l (1-e^{ikt_1}) \ln(kr_g).$$

We obtain for different relations between t_1 and $\tau = t - x$:

$$\psi \approx \frac{\text{const}}{\tau^{l+1}}, \quad r_g \ll \tau \ll t_1 \ll x, \tag{3.6}$$

$$\psi \approx \frac{\text{const} \cdot t_1}{\tau^{l+2}}, \quad r_g \ll t_1 \ll \tau \ll x. \tag{3.7}$$

It is significant that for both $\tau \lesssim r_g$ and $r_g < \tau \ll x$ the variation laws (3.5)–(3.7) do not, in the region of their validity, include any dependence on the distance to the star.

The physical wave fields g(r, t) are constructed from the field function $\psi(x, t)$ with the aid of formulas given in the Appendix I. These fields decay according to the law

$$g(x, t) \approx \frac{\text{const}\,(\tau_0/\tau)^{l+2}}{r}, \quad \tau_0 \ll \tau \ll x.$$

The stellar fields decrease in a time $t \sim x$ to values of the order of the static-multipole field:

$$\psi_{st} \sim 1/x^{l+1}. \tag{3.8}$$

The relaxation of this part of the field depends on the physical conditions. For the idealized case when there are no other bodies, the law of variation of the field for these asymptotic times is

$$\psi \sim \text{const} \cdot x^{l+1}/t^{2l+2}. \tag{3.9}$$

Owing to the fact that the maximum value of this law is the quantity (3.8), this part of the relaxation of the field does not carry away energy.

The law (3.9) changes if at distances $R_1 \sim t$, the spatial curvature is due to extraneous masses. According to the physical meaning, the signal responsible for (3.9) is the result of the reflection of the initial field pulse from the spatial curvature in the region of distances greater than the distance x from the star to the observer. In the real astronomical situation, we can expect to find other celestial bodies between the collapsar and the observation point. Let the nearest sphere on which such bodies curve the space more strongly than the mass of the collapsar be located at $x = R_1$. Then the fields at the observation point will vary at first according to the law (3.5), and then, when $\tau < R_1$, according to the law (3.6). After this, when $\tau \gtrsim R_1$, there will appear signals (echo) reflected by the regions of space curved by the masses surrounding the collapsar. The variation law (3.5) was discovered in [7]. The law (3.9) according to which the residual relaxation occurs in the idealized case was obtained by Price [9].

Near the collapsing star, at distances $r \sim r_g$, the region of validity of the formulas (3.6) and (3.7) narrows down. At $r \sim r_g$, the rapid – with a characteristic time $\tau_0 \approx r_g$ – variation of the field is followed by relaxation according to the law (3.9).

APPENDIX I

There arises in the derivation of Eq. (2.3) the problem of the separation of the angular variables in the wave equations for the scalar, vector, and tensor fields. This problem is most easily solved by the Lifshitz method [10], modified in such a way as to be applicable to the group O(3) [11]. We shall give the final formulas relating the components of the physical fields with the corresponding field function $\psi_l(r, t)$.

A. The Electromagnetic Field

Let us construct from the components F_{ik} of the electromagnetic field-strength tensor the following combinations:

$$iF_{02} \mp F_{03}/\sin\theta = 2E_\pm, \quad F_{01} = E,$$
$$iF_{12} \mp F_{13}/\sin\theta = 2H_\pm, \quad F_{23}/\sin\theta = iH.$$

The partial harmonics $(E_\pm)_l$ and $(H_\pm)_l$ are easily expressible in terms of E_l and H_l, namely,

$$\alpha(H_+ - H_-)_l = -\frac{\partial H_l}{\partial r}, \quad \alpha(E_+ - E_-)_l = -\frac{\partial H_l}{\partial t};$$

$$\frac{\alpha}{r(r-r_g)}(E_+ + E_-)_l = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_l), \quad \alpha\frac{r-r_g}{r^3}(H_+ + H_-)_l = -\frac{\partial E_l}{\partial t}$$

($\alpha \equiv \sqrt{l(l+1)}$). The functions E_l and H_l satisfy one and the same equation:

$$\frac{d^2\psi}{dx^2} + [k^2 - U_l(x)]\psi = 0, \quad U_l(x) = \frac{r-r_g}{r^3} l(l+1),$$

where ψ coincides either with E or with H, depending on whether the field source is an electric or a magnetic multipole.

B. Gravitational Perturbations

Let us represent the perturbed metric in the form $g_{ik} = g_{ik}^{(0)} + h_{ik}$, where the h_{ik} are small perturbations of the Schwarzschild metric $g_{ik}^{(0)}$. The equations for the h_{ik} break up into two independent groups (see [2,11]). To the first group pertain the metric perturbations connected with the matter-density perturbation. The second group of equations describes the metric perturbations which do not give rise to perturbations in the matter density. For the second type of perturbations in free space, we have:

$$h_b{}^a = \Phi_l(r, t) S_b{}^a(\theta, \varphi), \quad h_b{}^1 = G_l(r, t) S_b(\theta, \varphi),$$
$$h_1{}^1 = 0, \quad h_a{}^a = h_2{}^2 + h_3{}^3 = 0, \quad a, \ b = 2, 3,$$
$$G_l = \frac{r-r_g}{r^2}\psi_l, \quad l(l+1)\Phi_l = \frac{r-r_g}{r^2}\left(\frac{d\psi_l}{dr} + \frac{1}{r}\psi_l\right).$$

For the expressions for the spherical functions $S_b^a(\theta, \varphi)$ and $S_a(\theta, \varphi)$, see [11].

The field function $\psi_l(k, x)$ satisfies the following equation:

$$\frac{d^2\psi}{dx^2} + [k^2 - U_l(x)]\psi = 0,$$
$$U_l(x) = \frac{l(l+1)}{r^2} - \frac{[l(l+1)+3]r_g}{r^3} + \frac{3r_g^2}{r^4}.$$

It can be shown that an arbitrary static solution to the equations for the second type of gravitational perturbations for $l = 2, 3, \ldots$ corresponds to coordinate perturbations of the metric and has the form

$$G_l=R_l'(r)r^2(1-r_g/r),\quad \Phi_l=R_l(r), \tag{I.1}$$

where $R_l(r)$ is an arbitrary function of r. The corrections to the metric $g_{ik}^{(0)}$ that correspond to the static solution (I.1) can always be made to vanish by performing a suitable coordinate transformation.

The formulas giving the connection between the function $\psi_l(k, x)$ and the components h_{ik} for the gravitational perturbations of the first kind are considerably more unwieldy. The explicit expressions are given in, for example, [12,13]. The potential $U_l(x)$ in this case is

$$U_l(x)=\frac{r-r_g}{r}\frac{\nu^2(\nu+2)r^3+3\nu^2r_gr^2+9\nu r_g^2r+9r_g^3}{r^3(\nu r+3r_g)^2},$$
$$\nu=(l-1)(l+2).$$

APPENDIX II

Construction of the Solution

To solve Eq. (2.3), it is convenient for us to use the method of phase functions [14]. Let us represent the potential $U_l(x)$ in the form

$$U_l(x)=V_0(x)+V(x), \tag{II.1}$$

$$V_0=\begin{cases}0, & x<a\\ l(l+1)/x^2, & x>a\end{cases}. \tag{II.2}$$

We choose the value $x=a$ from the condition that the asymptotic expression (2.5b) should be valid for $U_l(x>a)$. Consequently,

$$V(x>a)=\frac{2l(l+1)r_g}{x^3}\ln\left(\frac{x}{r_g}\right)+O\left(\frac{1}{x^3}\right).$$

Let us introduce the functions $u_1(x)$ and $u_2(x)$, which are solutions to Eq. (2.3) with the potential (II.2):

$$u''+[k^2-V_0(x)]u=0. \tag{II.3}$$

These functions satisfy the same boundary conditions satisfied by $\psi_k(x)$ (see (2.9) and (2.10)), and have the form

$$u_1(x)=\begin{cases}e^{ikx}+b_1e^{-ikx}, & x<a\\ a_1h_l^{(1)}(kx), & x>a\end{cases}, \tag{II.4}$$

$$u_2(x)=\begin{cases}a_2e^{-ikx}, & x<a\\ h_l^{(2)}(kx)+b_2h_l^{(1)}(kx), & x>a\end{cases} \tag{II.5}$$

Here $h_l^{(1)}(z)$ and $h_l^{(2)}(z)$ are the Riccati-Hankel functions. The coefficients a_1, a_2, b_1, and b_2 are determined from the joining conditions for the solutions $u_{1,2}$ at $x=a$:

$$b_1=\frac{p_2^*(ka)}{p_1(ka)}e^{2ika},\quad b_2=-\frac{p_2(ka)}{p_1(ka)},\quad a_1=a_2=\frac{2e^{ika}}{p_1(ka)},$$

$$p_1(z)=h_l^{(1)}(z)-ih_l^{(1)\prime}(z),\quad p_2(z)=h_l^{(2)}(z)-ih_l^{(2)\prime}(z)$$

(see, for example, [14]). Notice that as $ka\to 0$

$$b_1=-1+i\gamma_1(ka)+i\gamma_2(ka)^2+\ldots,\quad b_2=1+i\varepsilon_1(ka)+\varepsilon_2(ka)^2+\ldots,$$

$$a_1=a_2=\delta_1(ka)^{l+1}+i\delta_2(ka)^{l+2}+\ldots;\quad \gamma_1=-\frac{2(l+1)}{l},\quad \gamma_2=\frac{l+1}{l}\gamma_1,$$

$$\varepsilon_1=\frac{2(l+1)}{l(2l-1)!!(2l+1)!!},\quad \varepsilon_2=-\frac{2l+1}{l(l+1)}\varepsilon_1,\quad \delta_1=\frac{2}{l(2l-1)!!}$$

$$\delta_2=\frac{l+1}{l}\delta_1.$$

Further, we seek the solution to Eq. (2.3) in the form

$$\psi_k(x)=\tilde{A}(x)[u_1+\tilde{B}(x)u_2] \tag{II.6a}$$

with the condition supplementing the definition of $\tilde{A}(x)$ and $\tilde{B}(x)$:

$$\psi_k'(x)=\tilde{A}(x)[u_1'+\tilde{B}(x)u_2']. \tag{II.6b}$$

Substituting (II.6a) and (II.6b) into (2.3), and taking into account the fact that u_1 and u_2 satisfy Eq. (II.3), we find

$$\tilde{B}'(x)=-\frac{V(x)}{2ika_1}[u_1+\tilde{B}(x)u_2]^2, \tag{II.7}$$

$$\tilde{B}(+\infty)=0, \tag{II.8}$$

and also

$$\tilde{A}'(x)=\tilde{A}(x)\frac{V(x)}{2ika_1}u_2[u_1+\tilde{B}(x)u_2], \tag{II.9}$$

$$\tilde{A}(-\infty)=1. \tag{II.10}$$

The boundary conditions (II.8) and (II.10) follow directly from (2.8), (2.9), (II.4), and (II.5).

Let us investigate the behavior of the quantities $\tilde{A}(x)$ and $\tilde{B}(x)$ when $|kr_g|\ll 1$. In the region $x>a$, substituting the specific forms of u_1 and u_2 from (II.4) and (II.5) into (II.7), we obtain

$$\tilde{B}'(x>a)=-\frac{V}{2ika_1}[a_1h_l^{(1)}(kx)+\tilde{B}(h_l^{(2)}(kx)+b_2h_l^{(1)}(kx))]^2.$$

Representing $\tilde{B}$ in the form

$$\tilde{B}(x>a)=a_1\beta(x), \tag{II.11}$$

we find that

$$\beta'(x)=-\frac{V}{2ik}[h_l^{(1)}(kx)+\beta(h_l^{(2)}(kx)+b_2h_l^{(1)}(kx))]^2,\quad \beta(+\infty)=0. \tag{II.12}$$

This system can easily be iterated. The first iteration is

$$\beta(x)=\int_x^{+\infty}\frac{V(y)}{2ik}[h_l^{(1)}(ky)]^2\,dy. \tag{II.13}$$

In the region $x<a$, we obtain upon substituting the specific forms of u_1 and u_2 from (II.4) and (II.5) into (II.7)

$$\tilde{B}'(x<a)=-\frac{V}{2ika_1}[e^{ikx}+b_1e^{-ikx}+\tilde{B}a_1e^{-ikx}]^2,$$
$$B(a)=a_1\beta(a).$$

Let us represent $\tilde{B}(x<a)$ in the form

$$\tilde{B}(x<a)=s(x)/a_1, \tag{II.14}$$

Then we obtain for the quantity $s(x)$ an equation with the corresponding boundary condition:

$$s'(x)=-\frac{V}{2ik}[e^{ikx}+b_1e^{-ikx}+s(x)e^{-ikx}]^2,\quad s(a)=a_1^2\beta(a). \tag{II.15}$$

We seek the solution to this system in the form of a series in k and k ln k:

$$s(x)=\sum_{n=1}^{2l+2}(2ik)^n s_{n,0}(x)+\sum_{\substack{n=2l+3\\ m=0}}^{\infty}(2ik)^n\ln^m k\, s_{n,m}(x) \tag{II.16}$$

The function $s_{1,0}$ satisfies the Riccati equation:

$$s_{1,0}'(x)=-V\left(x-\frac{l+1}{l}a+s_{1,0}\right)^2,$$

while the $s_{n\geq 2,m}$ satisfy linear equations of the first order, e.g.,

$$s_{2l+3,1}'(x)=-2V\left[x-\frac{l+1}{l}a+s_{1,0}\right]s_{2l+3,1}.$$

The expressions (II.11), (II.13), and (II.14)–(II.16) solve the problem of the construction of the solution. We find it convenient for the purpose of discussing the results to represent the solution $\psi_k(x)$ in the form

$$\psi_k(x)=A_k(x)[e^{ikx}+B_k(x)e^{-ikx}]. \tag{II.17}$$

Let us give the expressions of $A_k(x)$ and $B_k(x)$ in terms of the functions $\beta(x)$ and $s(x)$ for $x>a$:

$$A_k=N_s(a)L_\beta(x)a_1R^{(1)}[1+b_2\beta(x)],$$
$$A_kB_k=N_s(a)L_\beta(x)a_1R^{(2)}\beta(x). \tag{II.18}$$

Correspondingly, in the region $x<a$

$$A_k(x<a)=N_s(x),$$
$$B_k(x<a)=b_1+s(x). \tag{II.19}$$

Here

$$N_s(x)=\exp\left\{\int_{-\infty}^{x}\frac{V}{2ik}e^{-ikx}[e^{ikx}+b_1e^{ikx}+s(x)e^{-ikx}]dx\right\},$$
$$L_\beta(x)=\exp\left\{\int_{a}^{x}\frac{V}{2ik}(h_l^{(2)}+b_2h_l^{(1)})[h_l^{(1)}+\beta(x)(h_l^{(2)}+b_2h_l^{(1)})]dx\right\} \tag{II.20}$$

Moreover, we have used $h_l^{(1)}$ and $h_l^{(2)}$ represented in the form

$$h_l^{(1)}(kx)=e^{ikx}R^{(1)}(kx),\quad h_l^{(2)}(kx)=e^{-ikx}R^{(2)}(kx),$$

where

$$R^{(1)}=i^{-l-1}\sum_{n=0}^{l}\frac{(-1)^n(l+n)!}{n!(l-n)!}\frac{1}{(2ikx)^n},$$
$$R^{(2)}=i^{l+1}\sum_{n=0}^{l}\frac{(l+n)!}{n!(l-n)!}\frac{1}{(2ikx)^n}.$$

The quantities a_1, b_1, and b_2 are defined by the expressions (3.7).

It follows from the formulas (II.17)–(II.20) that $\psi_k(x)$ as a function of k has a logarithmic branch point at $k=0$. The magnitude of the discontinuity of the function $\psi_k(x)$ in crossing the branch cut depends on k and x. Let us give the explicit forms of the logarithmic terms contained in $\psi_k(x>a)$ in two asymptotic regions.

1. $|kx|\gg 1$, $|kr_g|\ll 1$:

$$\psi_k(x)=\frac{2i^{-l-1}c(x)}{l(2l-1)!!}(ka)^{l+1}(1+\tilde{p}_1kr_g\ln kr_g)e^{ikx}+\ldots \tag{II.21}$$

Here we have omitted the terms that are small in the parameters $1/kx$ and kr_g.

2. $|kx|\ll 1$:

$$\psi_k(x)=-\frac{2i}{l}\left(\frac{a}{r_g}\right)^{l+1}\left(\frac{r_g}{x}\right)^{l}(kr_g)+\ldots$$
$$+\frac{2ic(x)}{l(2l-1)!!(2l+1)!!}\left(\frac{ax}{r_g^2}\right)^{l+1}\tilde{p}_2(kr_g)^{2l+3}\ln(kr_g)+\ldots,$$
$$c(x)=\exp\left\{\int_{-\infty}^{a}V\left[x-\frac{l+1}{l}a+s_{1,0}\right]dx-\right.$$
$$\left.-\frac{1}{2l+1}\int_a xV\,dx-\frac{l+1}{l(2l+1)}\int_a\left(\frac{a}{x}\right)^{2l}aV\,dx\right\}, \tag{II.22}$$

where $\tilde{p}_1$ and $\tilde{p}_2$ are constants depending on l. In (II.22), we have omitted the terms that are small in the parameter $|kx|$. Notice that $\psi_k(x)$ in (II.22) can be smoothly continued into the region of values of $x<a$. The dependence of $\psi_k(x<a)$ on k is the same as in the expression (II.22). For $x\gg a\gg r_g$, the asymptotic form of $c(x)$ is

$$c(x)\approx c_0a^{-l-1}, \tag{II.23}$$

and therefore the asymptotic forms of the solutions (II.21) and (II.22) do not depend on the quantity a.

APPENDIX III

Computation of the Spectral Function

The spectral function f_k is, in the representation of the solution

$$\psi(t,x)=\int_{-\infty}^{\infty}f_k\psi_k(x)e^{-ikt}dk \tag{III.1}$$

determined from the joining condition at the boundary of matter and from the initial data described in Sec. 2. Let the coordinate x_s of the stellar surface at the moment t_0 correspond to the late phase of the collapse: $-x_s(t_0)\gg 1$. For $t>t_0$, $x_s(t)$ will always be in the region where the formulas (2.6)–(2.8) can be used. The joining to matter requires that for $t>t_0$

$$\psi(t,x_s(t))=\int_{-\infty}f_k\psi_k(x_s(t))e^{-ikt}dk=a+b\tau+c\tau^2+\ldots \tag{III.2}$$

The solution $\psi_k(x)$ for large negative values of the coordinate $x=x_s$ is (see Appendix II)

$$\psi_k\approx A_k(e^{ikx}+B_ke^{-ikx}), \tag{III.3}$$

and for $x\to-\infty$, $A_k=1$ and $B_k=-1+O(k)$ are constants not depending on the coordinates. Substituting the dependence (2.8), $x_s(t)$, into (III.2), we obtain

$$\int_{-\infty}^{\infty}f_k(e^{-2ikt}+B_ke^{-ik\beta_s})dk=a+be^{-t/r_g}+\ldots \tag{III.4}$$

The term containing B_k turns out to be a regular function of s, and therefore the main condition that arises in the joining of it to the field at the stellar boundary is the requirement that the function

$$\int_{-\infty}^{\infty}f_ke^{-2ikt}dk=\tilde{a}+\tilde{b}e^{-t/r_g}+\tilde{c}e^{-2t/r_g}+\ldots \tag{III.5}$$

be a regular function in the variable $\tau=e^{-t}$.

The exponent in (III.5) allows us to deform the contour into the lower half of the complex k plane. From (III.5), we find that the function f_k in the lower half-plane can have only a simple pole at the point $k=0$, and that the nearest singularity is located on the imaginary k axis at the point $k=-in/2r_g$, where n is an integer. The condition that the field function should be a regular function does not impose any limitations on $\tilde{a}$, $\tilde{b}$, etc.

To obtain more detailed information, it is necessary to use the initial data at the moment $t=t_0$. Let us first of all consider the problem of the radiation of a star whose surface is at rest at the point $x=x_0$. Let the field $\varphi(x_0,t)$ on the stellar surface be a given function of the time t. Then, representing the field in the form (III.1), we obtain

$$\varphi(x_0,t)=\int_{-\infty}^{\infty}f_k\psi_k(x_0)e^{-ikt}dk, \tag{III.6}$$

$$f_k=\frac{\varphi_k(x_0)}{\psi_k(x_0)},\quad \varphi_k(x_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(x_0,t)e^{ikt}dt. \tag{III.7}$$

In particular, let us choose the function $\varphi(x_0,t)$ such that at $t=t_0$ the solution (III.1) and (III.7) to the problem coincides with the initial data. On account of causality, this condition only determines $\varphi(x_0,t)$ for $t<t_0$. For $t>t_0$, the choice of $\varphi(x_0,t)$ is dictated by the requirement that the set f_k in the formula (III.7) give the solution to the problem of the field of the collapsing star, i.e., that the solution (III.1) satisfy (III.4). For such a choice the quantity f_k in (III.7) evidently does not depend on the value of x_0. Let us now investigate the behavior of f_k as $k \to 0$. To the left of the barrier, where the formula (III.3) is valid, we obtain

$$f_k \approx \varphi_k(x_0) / 2ikx_0.$$

If at $t \to -\infty$ the field in the space surrounding the star was a static field, then for such a field to be producible, it is necessary that at $t \to -\infty$

$$\varphi(x_0, t) \approx q_l x_0.$$

This is a well-known formula for the field of a static multipole near the gravitational radius [2,4]. For $\varphi_k(x_0)$ and f_k, we find

$$\varphi_k(x_0) = \frac{q_l x_0}{k - i0}\left(\frac{e^{ikt_0}}{2\pi i}\right) + \ldots,$$

$$f_k = \frac{-q_l}{k - i0}\left(\frac{e^{ikt_0}}{4\pi(k - i0)}\right) + \ldots.$$

If by chance the static field does not exist at $t_1 \to -\infty$, but exists from the moment $t = t_1$, then we obtain

$$\varphi_k(x_0) = \frac{q_l x_0}{k - i0}\frac{1}{2\pi i}(e^{ikt_0} - e^{ikt_1}). \qquad \text{(III.8)}$$

and, correspondingly, a simple pole for f_k in the upper half-plane. Evidently, an exotic choice of the initial data could lead to the appearance of a zero of integral order in f_k for $k \to 0$. Thus, in the upper half-plane, f_k has for $k \to 0$ a pole of order two if there existed at first a static metric, and a pole of order one if the static field vanishes at large distances $R > |t_1|$. Furthermore, f_k for $k \to +i0$ may contain, for example, a branching of the form $k^\sigma \ln k$ if the variation law $\varphi(x_0, t)$ has for $t \to -\infty$ the form

$$\varphi(x_0, t \to -\infty) \to \text{const} / t^{\sigma+1}. \qquad \text{(III.9)}$$

We do not, however, see any physical basis for such a law. Notice that $\psi_k(x_0)$ has zeros at the points k_n:

$$\psi_{k_n}(x_0) = 0, \qquad k_n \approx \pm\pi n / x_0, \qquad n = 1, 2, \ldots \qquad \text{(III.10)}$$

These zeros correspond to the natural frequencies of a "resonator" consisting of a wall at $x = x_0$ and a potential barrier to the right of x_0. For the problem, solved by the formulas (III.6) and (III.7), of the radiation of a stationary star, this leads to poles in f_k and to the appearance of lines in the radiation spectrum. The transparency of the barrier leads to the broadening of the lines, and implies that all these poles are located in the lower half-plane. The poles (III.10) describe standing waves in the "resonator."

For the problem of the field of a collapsing star, the analyticity condition for the field on the surface of matter requires the absence of poles in the lower half-plane. This implies that $\varphi_{k_n}(x_0) = 0$, so that the residues at the poles in this case are equal to zero.

For a plausible physical situation, we can expect that the field has a static character at a distance of the order of the distance R from the collapsing star to the nearest stars. The function f_k for small k then has the form (III.8), where $|t_1| \sim R$.

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