The helicon mechanism of relaxation of a relativistic electron beam in a plasma

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The problem of the interaction of a relativistic electron beam with a plasma located in a weak ($H^2 \lesssim n_0 T$) magnetic field is considered. It is found that the main cause of the collective slowing down of the beam may be its instability against helicon excitation. It is shown that the interaction of the beam with the helicons is described with sufficient accuracy by a quasilinear theory (the nonlinear processes involving helicons are unimportant in the problem under consideration). The relaxation length for a beam steadily injected into the plasma, as well as the spectrum of the helicons excitable by the beam, is found in the framework of the quasilinear approximation.

1. INTRODUCTION

In the present paper we investigate the question how the presence of a magnetic field in a plasma influences the nature of the relaxation of a relativistic electron beam. The formulation of the problem is connected with the problem of the beam heating of a dense plasma confined in the transverse direction by walls[1)]. The magnetic field in this case only helps to decrease the thermal conductivity, and is relatively weak, being such that

$$\beta = 8\pi n_0 T/H^2 \gtrsim 1. \tag{1}$$

We shall henceforth restrict ourselves to such fields.

Let us first recall the results pertaining to the relaxation of a beam in a plasma without a magnetic field[2]. The relaxation mechanism is connected here with the excitation of Langmuir oscillations because of the beam instability[2)]. The generation of these oscillations during the steady injection of the beam into the plasma is balanced by their transfer to the long-wave part of the spectrum, owing to induced scattering by the plasma ions. The estimate for the relaxation length of the beam has the form

$$l \sim \frac{c}{\omega_p}\frac{n_0}{n_b}\frac{m}{M}\left(\frac{mc^2}{T}\right)^2\left(\frac{E_b}{mc^2}\right)^3. \tag{2}$$

Here n_0 and n_b are the plasma and beam densities respectively, $\omega_p \equiv (4\pi n_0 e^2/m)^{1/2}$ is the electron plasma frequency, T is the plasma temperature, and E_b is the energy of the beam electrons.

Let us now turn to the case when the external magnetic field is different from zero. It is not difficult to verify that for $\beta > 1$ the magnetic field has a slight effect on both the dispersion law for the Langmuir oscillations interacting with the beam and on the probability of scattering of these oscillations by the ions. Therefore, in a weak magnetic field ($\beta > 1$) the relaxation pattern can change only because of the excitation of these oscillations, which are absent when H = 0. From among these oscillations, we investigate helicons in detail in the present paper. As will be shown, their role in the relaxation process can turn out to be decisive.

It follows from the linear theory that helicons are excited by the beam less intensely than Langmuir oscillations. Nevertheless, under conditions when the energy density of the Langmuir oscillations is limited by the nonlinear processes, the helicons, and not the Langmuir waves, may be responsible for the relaxation of the beam (this circumstance was pointed out in[3]. In conformity with the foregoing, we shall solve the relaxation problem in the following fashion: we shall first consider the interaction of the beam with the helicons, completely neglecting the influence of all the other (including the Langmuir) oscillations, and then determine more accurately the conditions of applicability of this approach.

In the second section of the paper we investigate qualitatively the formulated problem. An estimate is obtained there for the relaxation length of the beam in the plasma due to helicon excitation, and it is shown that the relaxation amounts mostly to the scattering of the beam electrons, the energy losses being relatively small. Furthermore, it is established that the beam-helicon interaction can be described with sufficient accuracy in the framework of a quasi-linear approximation. In the third section we obtain an analytic solution to the quasi-linear problem of the steady injection of a beam into a plasma. In the final (fourth) section we formulate the conditions of applicability of the results of the paper.

2. QUALITATIVE INVESTIGATION

In a magnetic field H_z, the condition for a beam electron to interact with a wave is of the form

$$\omega_k - k_z v_z - n\Omega = 0, \tag{3}$$

where $\Omega \equiv \omega_H mc^2/E_b$ is the cyclotron frequency for relativistic electrons. This relation follows from the laws of conservation of energy and momentum in an elementary event of emission (or absorption) of a wave by a particle. The quantity $\hbar\omega_k$ is then the energy of the emitted wave, while $\hbar n\Omega$ is the change in the "transverse" energy of the particle upon emitting the wave. As is evident from the dispersion relation for helicons.

$$\omega_k = \omega_H k|k_z|c^2/\omega_p^2, \tag{4}$$

the phase velocity of these waves in a weak magnetic field ($\omega_H \ll \omega_p$) is low compared to the velocity of light. Therefore, for helicons, a Cerenkov (n = 0) resonance with the beam electrons is impossible. If, however, $n \neq 0$, then under the resonance condition (3) we can neglect the quantity ω_k. In other words, the change in the "transverse" energy of the particle is large compared to the energy loss due to the emission of the wave. This means that the action of the helicons on the beam leads to near-elastic scattering of the particles.

Let us derive the law of variation of the angular spread $\Delta\theta$ of the particles for the case of steady injection of the beam into the plasma. For this purpose, let us estimate the distance from the plasma boundary over which the energy density of the oscillations excited by a beam with the angular spread $\Delta\theta$ attains a level substantially exceeding the thermal energy density:

$$z \sim \Lambda v_{gr}/\gamma_b.$$

Here Λ is the Coulomb logarithm, v_{gr} is the component of the group velocity of the wave along the z axis, and γ_b is the instability increment. For γ_b, in turn, we have the estimate (see Sec. 3):

$$\gamma_b \sim \omega_H \frac{n_b}{n_0}\frac{mc^2}{E_b}\frac{kc}{\Omega}\frac{1}{\Delta\theta^2}. \tag{5}$$

Hence

$$z \sim \Lambda \frac{c}{\omega_p}\frac{n_0}{n_b}\frac{\omega_H}{\omega_p}\Delta\theta^2.$$

If the nonlinear wave-wave interaction is negligibly weak, then this relation can be simultaneously regarded as the dependence of the angular spread of the beam on the z coordinate. Thus, in the quasi-linear approximation[3)]

$$\Delta\theta(z) \sim (z/l_h)^{1/2}, \tag{6}$$

where the quantity

$$l_h \equiv \Lambda \frac{c}{\omega_p}\frac{\omega_H}{\omega_p}\frac{n_0}{n_b} \tag{7}$$

is the relaxation length of the beam.

Let us find the energy density U_h of the oscillations at a distance z from the plasma boundary from the law of conservation of momentum flux. At the entrance to the plasma, there are no oscillations, and the momentum flux of the beam electrons is equal to $n_b v_b p_b$, where v_b and p_b are the electron velocity and momentum respectively. Taking into account the fact that the relaxation leads mostly to an increase in the angular spread of the beam, we obtain

$$n_b v_b p_b = n_b v_b p_b [1-\Delta\theta^2(z)] + v_{gr} k_z \frac{U_h}{\omega_{\mathbf{k}}}$$

($\omega_{\mathbf{k}}$ and k_z are respectively characteristic values of the frequency and the longitudinal component of the wave vector of the oscillations). It follows from this that

$$U_h(z) \sim n_b v_b p_b \Delta\theta^2(z). \tag{8}$$

At a distance l_h from the plasma boundary the energy density of the oscillations is comparable to the energy density of the beam, and the energy flux of the oscillations, which is equal to $v_{gr}U_h$, remains substantially less than the energy flux of the beam. i.e., the relative electron-energy loss $\Delta E/(E_b - mc^2)$ turns out to be small:

$$\frac{\Delta E}{E_b - mc^2} \sim \left(\frac{\omega_H}{\omega_p}\right)^2 \frac{mc^2}{E_b} \ll 1 \tag{9}$$

(we have allowed for the fact that when $\Delta\theta \approx 1$ the beam excites oscillations with wave vector $k \approx \Omega/v_b \sim \Omega/c$).

Let us now estimate the effect of the nonlinear processes on the course of the relaxation. The computation of the probabilities of these processes shows that the dominant process is the decay of the helicons:

$$\omega_{\mathbf{k}} = \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2}, \quad \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2,$$

where $\omega_{\mathbf{k}}$ is given by the formula (4). The increment $\gamma_{h \to h+h}$ of the decay instability is equal in order of magnitude to $\omega_{\mathbf{k}} U_h/H^2$ (see, for example,[4]). Using the expression (8), we can easily verify that $\gamma_{h \to h+h}$ is comparable to γ_b only at the final stage of the relaxation (when $\Delta\theta \sim 1$). Thus, we have shown that the important part of the relaxation process can indeed be described by the equations of the quasi-linear theory. The solution of these equations is obtained in Sec. 3.

3. SOLUTION OF THE QUASI-LINEAR PROBLEM

We shall consider the problem of the steady injection of a beam into a plasma in the following formulation. We shall assume that the plasma, in which there is a uniform magnetic field H_z, occupies the half-space $z > 0$. Into the plasma along the z axis is injected a monoenergetic relativistic ($E_b - mc^2 \gtrsim mc^2$) electron beam of infinite extent in the transverse direction. The beam-electron distribution function $F(p, \theta, z)$ at the entrance to the plasma is specified in the form

$$F(p,\theta,z)|_{z=0} = \frac{1}{2\pi}\frac{n_b}{p_b^2}\delta(p-p_b)\Phi_0(\theta) \tag{10}$$

(we shall assume, for simplicity, that the function $\Phi_0(\theta)$ is a monotonically decreasing function — see the figure). Here and below we use a spherical system of coordinates (p, θ, φ) in momentum space, the z axis being the polar axis of this system.

We shall restrict ourselves to the investigation of beams with not too small angular spreads $\Delta\theta$, so that the excited instability can be regarded as a kinetic instability. As applied to the problem under consideration, this restriction can be written as follows:

$$\Delta\theta \gg \left(\frac{n_b}{n_0}\frac{k_\perp c}{\Omega}\right)^{1/4}. \tag{11}$$

Here $k_\perp$ is a characteristic value of the transverse (with respect to the magnetic field) component of the wave vector. As will be shown below (see formula (28)), $k_\perp \sim \Omega c^{-1}\Delta\theta^{-2/3}$.

For the description of the relaxation process let us use the system of quasi-linear equations:

$$v\cos\theta\frac{\partial F}{\partial z} = \frac{1}{p^2}\frac{\partial}{\partial p}p^2\left(D_{pp}\frac{\partial F}{\partial p} + \frac{1}{p}D_{p\theta}\frac{\partial F}{\partial\theta}\right) + \frac{1}{p\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\left(D_{\theta p}\frac{\partial F}{\partial p} + \frac{1}{p}D_{\theta\theta}\frac{\partial F}{\partial\theta}\right), \tag{12}$$

$$\frac{\partial\omega_{\mathbf{k}}}{\partial k_z}\frac{\partial W_{\mathbf{k}}}{\partial z} = 2\gamma_b W_{\mathbf{k}}. \tag{13}$$

We denote by $W_{\mathbf{k}} \equiv W_{\mathbf{k}}(z)$ the spectral energy density of the oscillations, and by $v \equiv v(p)$ the electron velocity. The components of the diffusion tensor $D_{\alpha\beta}$ and the increment γ_b of the instability are given by the following formulas:

$$D_{\alpha\beta} = 8\pi^2 e^2 \sum_{n=-\infty}^{\infty}\int W_{\mathbf{k}}\left(\frac{1}{\omega}\frac{\partial}{\partial\omega}\omega^2\varepsilon_{\mu\nu}a_\mu^* a_\nu|_{\omega=\omega_{\mathbf{k}}}\right)^{-1} \times \mathrm{Re}\, A_\alpha^{(n)*}A_\beta^{(n)}\delta(\omega_{\mathbf{k}} - k_z v\cos\theta - n\Omega)\,d\mathbf{k}, \tag{14}$$

$$\gamma_b = 4\pi^2 e^2\left(\frac{1}{\omega}\frac{\partial}{\partial\omega}\omega^2\varepsilon_{\mu\nu}a_\mu^* a_\nu|_{\omega=\omega_{\mathbf{k}}}\right)^{-1} \times \mathrm{Re}\sum_{n=-\infty}^{\infty}\int\left(|A_p^{(n)}|^2 v\frac{\partial F}{\partial p} + A_p^{(n)*}A_\theta^{(n)}\frac{v}{p}\frac{\partial F}{\partial\theta}\right) \cdot\delta(\omega_{\mathbf{k}} - k_z v\cos\theta - n\Omega)\,d\mathbf{p}. \tag{15}$$

Here $\mathbf{a}$ is the polarization vector of the wave and $\epsilon_{\mu\nu}$ is the permittivity tensor of the plasma. Furthermore, we introduce the notation

$$\mathbf{A}^{(n)} = \frac{1}{2\pi\omega_{\mathbf{k}}}\int_0^{2\pi}\{\mathbf{a}\omega_{\mathbf{k}} + [\mathbf{v}[\mathbf{ka}]]\}\exp\left(-in\varphi + i\frac{k_\perp v\sin\theta}{\Omega}\sin\varphi\right)d\varphi,$$

where the angle φ in momentum space is measured from the direction of the vector $\mathbf{k}_\perp$. Using the explicit expression for the polarization vector of the helicon, we obtain

$$A_p^{(n)} = \left(1+\frac{k^2}{k_z^2}\right)^{-1/2}\left(J_n \frac{n\Omega}{k_\perp v}\left|\frac{k}{k_z}\right| + \sin\theta J_n'\right), \tag{16}$$

$$A_\theta^{(n)} = -A_p^{(n)} \frac{1}{\sin\theta}\frac{k_z v}{\omega_k}, \tag{17}$$

where $J_n \equiv J_n(k_\perp v\Omega^{-1}\sin\theta)$ is a Bessel function.

Notice that the formulas (16) and (17) are valid only for $n \neq 0$, since in deriving them we neglected the quantity ω_k in comparison with $n\Omega$. As to the terms with $n = 0$, they do not, as has already been noted in the preceding section, make any contribution to the quasi-linear equations.

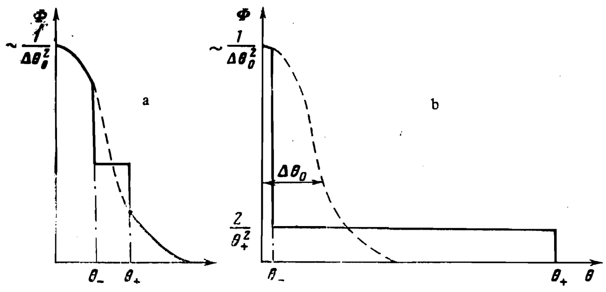

FIG. 1. The dynamics of beam relaxation (the dashed curves represent the initial distribution function): a) the initial phase of the relaxation; b) the asymptotic phase. $\Delta\theta_0$ is a characteristic value of the initial angular spread of the beam.

With the aid of the formula (17) it is not difficult to obtain the following relations between the components of the diffusion tensor:

$$\frac{D_{pp}}{D_{p\theta}} \lesssim \frac{\omega_k}{k_z v} \ll 1, \quad \frac{D_{p\theta}}{D_{\theta\theta}} \lesssim \frac{\omega_k}{k_z v} \ll 1.$$

It is evident from these relations that the angular spread of the beam increases much more rapidly in the course of the relaxation than the momentum spread. Therefore, the problem can be reduced to that of finding the angular distribution $\Phi(\theta, z)$ of the beam electrons:

$$\Phi = \frac{2\pi}{n_b}\int_0^\infty F p^2\, dp. \tag{18}$$

The equation for Φ is obtained by integrating (12) over p with allowance for the condition $\Delta p/p_b \ll \Delta\theta$:

$$v_b\cos\theta \frac{\partial\Phi}{\partial z} = \frac{1}{p^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta D_{\theta\theta}\Big|_{p=p_b}\frac{\partial\Phi}{\partial\theta}. \tag{19}$$

The boundary condition at z = 0 has the form

$$\Phi(\theta, z)|_{z=0} = \Phi_0(\theta). \tag{20}$$

The increment γ_b of the instability is expressible up to terms of order $\omega_k/k_z v$ in terms of the function Φ as follows[4)]:

$$\gamma_b = -\Omega_b\frac{\pi}{2}\frac{n_b}{n_0}\left(1+\frac{k_z^2}{k^2}\right)\frac{k_z}{|k_z|}\frac{kv_b}{\Omega_b} \times\sum_{n=-\infty}^{\infty}\int_0^\pi |A_p^{(n)}|^2\frac{\partial\Phi}{\partial\theta}\delta\left(\frac{k_z v_b}{\Omega_b}\cos\theta+n\right)d\theta, \tag{21}$$

where $\Omega_b \equiv \Omega(p_b)$.

To simplify the notation, it is convenient to go over to the dimensionless variables

$$z = \frac{1}{\pi}\frac{n_0}{n_b}\frac{c\omega_H}{\omega_p^2}\frac{c}{v_b}x,$$

$$k_z = \frac{\Omega_b}{v_b}\xi, \quad k_\perp = \frac{\Omega_b}{v_b}\eta,$$

$$W_k = \frac{1}{2\pi}n_b p_b v_b\left(\frac{v_b}{\Omega_b}\right)^3 I.$$

In the new variables the system of quasi-linear equations assumes the form

$$\cos\theta\frac{\partial\Phi}{\partial x} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta} \times\sum_{n=-\infty}^{\infty}\int\frac{n^2Q_n^2}{n^2+\eta^2\cos^2\theta}I\delta(\xi\cos\theta+n)\,d\xi\,\eta\,d\eta\frac{\partial\Phi}{\partial\theta}, \tag{22}$$

$$\frac{\partial I}{\partial x} = -I\sum_{n=-\infty}^{\infty}\int_0^\pi\frac{n^2Q_n^2}{2n^2+\eta^2\cos^2\theta}\delta(\xi\cos\theta+n)\frac{\partial\Phi}{\partial\theta}d\theta, \tag{23}$$

where

$$Q_n = \left(1+\frac{\eta^2\cos^2\theta}{n^2}\right)^{1/2}\frac{n}{\eta}J_n(\eta\sin\theta)+\sin\theta J_n'(\eta\sin\theta).$$

We shall obtain the solution to the system (22) and (23) at that stage of the relaxation where the angular spread of the beam is still small ($\Delta\theta \ll 1$). In that case, however, $\Delta\theta$ may substantially exceed the initial angular spread $\Delta\theta_0$ of the beam.

Let us make the assumption, which will be confirmed by the result, that when $\Delta\theta \ll 1$ the dominant contribution to the right-hand sides of Eqs. (22) and (23) are made by the terms with $n = -1$. This enables us to substantially simplify the basic system of equations:

$$\frac{\partial\Phi}{\partial x} = \frac{1}{\theta}\frac{\partial}{\partial\theta}\frac{1}{\theta}\int_0^\infty\frac{Q_{-1}^2}{1+\eta^2}I\eta\,d\eta\frac{\partial\Phi}{\partial\theta}, \tag{24}$$

$$\frac{\partial I}{\partial x} = -I\frac{Q_{-1}^2}{2+\eta^2}\frac{1}{\theta}\frac{\partial\Phi}{\partial\theta}. \tag{25}$$

Here I is a fraction of the variables θ, η, and x, and, moreover,

$$I(\theta, \eta, x) \equiv I(\xi, \eta, x)|_{\xi=(\cos\theta)^{-1}}.$$

The idea of the solution of the system (24) and (25) consists in the use of the large parameter Λ, which is equal to the logarithm of the ratio of the energy density of the oscillations excited by the beam to the thermal-noise energy density (see[6, 7]). As can be seen from Eq. (25),

$$I = I_T\exp\left(-\frac{Q_{-1}^2}{2+\eta^2}\int_0\frac{1}{\theta}\frac{\partial\Phi}{\partial\theta}dx\right), \tag{26}$$

where I_T is the spectral energy density of the thermal noise. Since the integral occurring in the exponent is very large ($\propto\Lambda$), the quantity I as a function of η has a sharp peak at the point $\eta = \eta_m$ corresponding to the maximum of the function $Q_{-1}^2/(2+\eta^2)$. Thus, we can set with sufficient accuracy

$$I(\theta, \eta, x) = \psi(\theta, x)\delta(\eta - \eta_m(\theta)). \tag{27}$$

For $\theta \ll 1$ the quantity η_m is easily found analytically:

$$\eta_m = 4^{1/3}\theta^{-2/3}. \tag{28}$$

Substituting the formulas (27) and (28) into Eqs. (24) and (25), we obtain

$$\frac{\partial\Phi}{\partial x} = 4^{-2/3}\frac{1}{\theta}\frac{\partial}{\partial\theta}\theta^{1/3}\psi\frac{\partial\Phi}{\partial\theta}, \tag{29}$$

$$\frac{\partial\psi}{\partial x} = -\frac{1}{4}\theta\frac{\partial\Phi}{\partial\theta}\psi. \tag{30}$$

The system (29) and (30) possesses an integral that allows us to find the spectrum of the oscillations from the known distribution function $\Phi(\theta, x)$:

$$\Phi - \Phi_0 + 4^{1/3}\frac{1}{\theta}\frac{\partial}{\partial\theta}\theta^{-2/3}\psi = 0. \tag{31}$$

Hence

$$\psi = 4^{-1/3}\theta^{2/3}\int_0^\theta(\Phi_0 - \Phi)\theta\,d\theta. \tag{32}$$

In order to find the function $\Phi(\theta, x)$, we note that the Eqs. (29) and (30) are similar in structure to the system of equations considered by Ivanov and Rudakov[7]. Using this similarity, we can assert that for those values of θ at which the quantity Ψ substantially exceeds the thermal level, the derivative $\partial\Phi/\partial\theta$ will inevitably be small ($\partial\Phi/\partial\theta \propto \Lambda^{-1}$). Let us denote the limits of that region where the noise level is high by θ_- and θ_+. Then for $\theta_-(x) < \theta < \theta_+(x)$ the function Φ has the form of a plateau. For all other values of θ, it is, as is evident from the formula (31), equal to Φ_0. Denoting the height of the plateau by P(x), we obtain

$$\Phi(\theta, x) = \begin{cases} \Phi_0(\theta), & \theta < \theta_-(x) \\ P(x), & \theta_-(x) < \theta < \theta_+(x). \\ \Phi_0(\theta), & \theta > \theta_+(x) \end{cases} \tag{33}$$

The height of the plateau is determined from the condition of particle-number conservation:

$$P(x) = \frac{2}{\theta_+^2 - \theta_-^2} \int_{\theta_-}^{\theta_+} \Phi_0 \theta \, d\theta. \tag{34}$$

With allowance for the formulas (33) and (34), the expression (32) for the spectral function Ψ assumes the form

$$\psi = 4^{-1/3}\theta^{2/3} \left(\int_{\theta_-}^{\theta} \Phi_0 \theta \, d\theta - \frac{\theta^2 - \theta_-^2}{\theta_+^2 - \theta_-^2} \int_{\theta_-}^{\theta_+} \Phi_0 \theta \, d\theta \right). \tag{35}$$

The equations for θ_- and θ_+ are obtained by integrating the relation (30) over the intervals $\theta_- - 0 < \theta < \theta_- + 0$ and $\theta_+ - 0 < \theta < \theta_+ + 0$. Taking into account the fact that the logarithm of the ratio of the energy density of the oscillations excited by the beam to the thermal-noise energy density is, to a good degree of accuracy, equal to Λ, we obtain

$$\frac{d\theta_-}{dx} = \frac{1}{4\Lambda} \theta_- \left[\frac{2}{\theta_+^2 - \theta_-^2} \int_{\theta_-}^{\theta_+} \Phi_0 \theta \, d\theta - \Phi_0(\theta_-) \right], \tag{36}$$

$$\frac{d\theta_+}{dx} = \frac{1}{4\Lambda} \theta_+ \left[\frac{2}{\theta_+^2 - \theta_-^2} \int_{\theta_-}^{\theta_+} \Phi_0 \theta \, d\theta - \Phi_0(\theta_+) \right]. \tag{37}$$

The boundary conditions for θ_- and θ_+ have the form

$$\theta_-(0) = \theta_+(0) = \theta_0,$$

where θ_0 is the point at which the function $\theta \partial\Phi_0/\partial\theta$ has its minimum.

Analysis of Eqs. (36) and (37) shows that in the course of the relaxation the function $\theta_-(x)$ decreases, while $\theta(x)$ increases. A qualitative picture of the relaxation is shown in the figure. At the initial stage, the behavior of the quantites θ_- and θ_+ strongly depends on the details of the function $\Phi_0(\theta)$, and cannot be found analytically. However, the asymptotic behavior of the relaxation turns out to be quite simple:

$$\theta_- = \theta^* \exp\left(-\frac{\Phi_0(0)}{4\Lambda} x \right), \quad \theta_+ = \left(\frac{x}{4\Lambda} \right)^{1/2}. \tag{38}$$

Here θ^* is a constant depending on the specific form of $\Phi_0(\theta)$. The solution (38) is applicable when $x \gg \Lambda\Delta\theta_0^2$, i.e., when $\theta_- \ll \Delta\theta_0$ and $\theta_+ \gg \Delta\theta_0$. In the asymptotic regime, instead of the formula (35), we can use a simpler expression for the spectral function

$$\psi(\theta, x) = 4^{-1/3}\theta^{2/3} \begin{cases} 1 - \theta^2/\theta_+^2, & 0 < \theta < \theta_+ \\ 0, & \theta > \theta_+ \end{cases}. \tag{39}$$

Hence we obtain the following formula for the oscillation-energy flux S in dimensional variables:

$$S = \frac{9 \cdot 4^{1/3}}{20} \left(\frac{\omega_H}{\omega_p} \right)^2 n_b m c^2 v_b \theta_+^{4/3}. \tag{40}$$

In conclusion, let us note that the obtained solution to the quasi-linear problem completely agrees with the qualitative estimates made in Sec. 2.

4. DISCUSSION OF THE RESULTS

Let us enumerate the conditions which should be fulfilled before the beam-relaxation mechanism considered in Sec. 2 and 3 could be realized.

We solved the problem, neglecting helicon attenuation by the plasma electrons and ions. The attenuation by the ions is exponentially weak for waves whose phase velocity is higher than the thermal velocity of the ions:

$$\frac{\omega_k}{k} = c \frac{\omega_H}{\omega_p} \frac{k_z c}{\omega_p} > \left(\frac{T}{M} \right)^{1/2}.$$

Substituting into this expression the value $k_z \sim \Omega_b/c$, we obtain the following limitation on the magnitude of the magnetic field:

$$\omega_H > \omega_p \left(\frac{E_b}{mc^2} \right)^{1/2} \left(\frac{T}{Mc^2} \right)^{1/4}. \tag{41}$$

Notice that helicons can be excited by the beam even in the case when the inverse inequality obtains. In this case, though, there should be a buildup of oscillations with large k_z, which are responsible for the high-order resonances ($k_z c = n\Omega$, $n > 1$). The problem of beam relaxation in such a situation is solvable in exactly the same fashion as was done in Sec. 2 and 3.

The logarithmic decrement of the attenuation by the electrons for helicons interacting with the beam is given by the following formula:

$$\gamma_e = -\left(\frac{\pi}{2} \right)^{1/2} \omega_k \frac{k_\perp}{k} \frac{k_\perp}{\omega_H} \left(\frac{T}{m} \right)^{1/2}.$$

Comparing γ_e with the increment (5) of the beam instability, we obtain the condition allowing us to neglect the damping by the electrons during the entire relaxation process:

$$\frac{n_b}{n_0} > \left(\frac{\omega_H}{\omega_p} \right)^2 \left(\frac{mc^2}{E_b} \right)^2 \left(\frac{T}{mc^2} \right)^{1/2} \tag{42}$$

(we have used the explicit expression for the wave vector of the most unstable oscillations).

Under the conditions being considered the beam, together with the helicons, of course excites in the plasma Langmuir oscillations as well. Let us discuss the question as to when we can neglect the influence of the Langmuir waves on the variation of the angular spread of the beam and on the spectrum of the helicons excitable by the beam. For this influence to be negligible, it is naturally necessary that the relaxation length l_h (see (7)) be less than the quantity l (see (2)):

$$\frac{\omega_H}{\omega_p} < \frac{1}{\Lambda} \frac{m}{M} \left(\frac{mc^2}{T} \right)^2 \left(\frac{E_b}{mc^2} \right)^3. \tag{43}$$

Besides the condition (43), there is still a limitation on the initial angular spread of the beam:

$$\Delta\theta_0 > (l_h/l)^{1/5}.$$

Let us explain the meaning of the last limitation. In the "Langmuir" mechanism of relaxation the angular spread of the beam varies according to the following law[5)]:

$$\Delta\theta \sim (z/l)^{1/5}. \tag{44}$$

On the other hand, in relaxation on the helicons we have $\Delta\theta \sim (z/l_h)^{1/2}$ (see (6)). It can be seen from the relations (6) and (44) that the relaxation of a beam with a small angular spread ($\Delta\theta_0 < (l_h/l)^{1/3}$) takes place in two stages. In the first phase (right up to $\Delta\theta \sim (l_h/l)^{1/3}$) the angular spread increases because of the interaction of the electrons with the Langmuir oscillations; in the second phase because of the interaction with the helicons. If, on the other hand, $\Delta\theta_0 > (l_h/l)^{1/3}$, then the first phase of the relaxation is unimportant.

It should be noted that even in the case when the angular spread of the beam is due to the excitation of helicons the relaxation with respect to energy can take place owing to the Langmuir oscillations. The corresponding estimate for the energy losses of the beam is of the form

$$\frac{\Delta E}{E_b} \sim \begin{cases} (l_h/l)^{1/3}, & \Delta\theta_0 < (l_h/l)^{1/3} \\ (l_h/l)/\Delta\theta_0^2, & \Delta\theta_0 > (l_h/l)^{1/3} \end{cases}. \qquad (45)$$

This result is easily obtainable from the relation connecting the energy loss due to the excitation of Langmuir oscillations with the magnitude of the angular spread of the beam[2]:

$$\frac{d}{dz}\frac{\Delta E}{E_b} = \frac{1}{l}\frac{1}{\Delta\theta^4}. \qquad (46)$$

(The cause of the difference between this formula and the result obtained in[2] is indicated in the footnote to the formula (44).)

Let us now show that under the conditions of interest to us we can neglect the nonlinear interaction between the helicons and the Langmuir oscillations. We have in mind the decay process $l \to l + h$, where the symbols l and h stand for Langmuir waves and helicons, respectively. The calculation of the probability of this process yields the following estimate for the increment of the decay instability:

$$\gamma_{l\to l+h} \sim \omega_p \frac{U_l}{n_0 mc^2}\left(\frac{k_h c}{\omega_p}\right)^3.$$

Using this estimate and the expression for the energy density of the Langmuir oscillations excitable by the beam (see[2]),

$$U_l \sim n_b mc^2 \left(\frac{T}{mc^2}\right)^2 \frac{M}{m}\frac{mc^2}{E_b}\frac{1}{\Delta\theta^2}$$

we can easily verify that the relation $\gamma_{l\to l+h} < \gamma_b$ is automatically fulfilled.

Let us make another remark about the interaction of the beam with the low-frequency oscillations in which the motion of the ions is important. As can be seen from the condition (3) for resonance, the z component of the wave vector of the wave interacting with the beam is bounded from below by the quantity Ω_b/c. Therefore, when

$$\frac{T}{mc^2} \gg \frac{m}{M}\left(\frac{E_b}{mc^2}\right)^2$$

the inequality

$$k_z (T/M)^{1/2} \gg \omega_H m/M, \qquad (47)$$

which allows us to regard the motion of the ions in the oscillations as unmagnetized, is fulfilled. Furthermore, it is not difficult to show that for $\beta > 1$ the inequality (47) allows us to generally neglect the contribution of the ions to the dispersion relation[6]. This means that under the indicated conditions, the low-frequency (ion) oscillations do not influence the relaxation of the beam.

In the present paper we have restricted ourselves to the investigation of the relaxation of a beam in the case when the magnetic field in the system is relatively weak ($\beta < 1$). It should, however, be noted that we used this limitation only where we discussed the possibility of neglecting the interaction of the beam with all the oscillations except the helicons. By itself, however, the helicon mechanism of beam relaxation may also prove to be important when $\beta < 1$, since all the limitations formulated by us are of the nature of sufficient conditions.

In conclusion, let us recall that all the results were obtained for the case of a beam that is unbounded in the transverse direction, i.e., it is implied that the inequality $R \gg l_h$, where R is the beam radius, is fulfilled. In the opposite case ($R < l_h$), the waves propagating in the radial direction leave the region of interaction with the beam before they have time to intensify. However, the limitation $R \gg l_h$ can turn out to be unimportant if, because of the radial inhomogeneity in the plasma concentration and in the magnetic field, the plasma column is a wave guide. For helicons, such a situation is easily realizable, the qualitative picture of the relaxation remaining the same as in the case of the unbounded beam.

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[1)]The problem of dense-plasma confinement by walls has been considered by Chebotaev et al. [1]

[2)]To avoid any misunderstanding, let us note that we are dealing here with an isothermal plasma, in which the buildup of ion sound is impossible

[3)]Notice that the formula (6) makes sense only when $\Delta\theta \gg \Delta\theta_0$, where $\Delta\theta_0$ is the angular spread of the beam at the entrance to the plasma.

[4)]From the formula (21) and the expression (16) for $A_p^{(n)}$ it follows, in particular, that an electron beam with a monotonically decreasing distribution function Φ does not excite oscillations that propagate strictly along the magnetic field. The situation changes if, instead of an electron beam, we take an ion beam. The quasi-linear relaxation of an ion beam has been investigated by Rowlands et al.[5] under the assumption that the oscillation spectrum is one-dimensional ($k_\perp = 0$).

[5)]The difference between this formula and the result of[2] ($\Delta\theta \sim (z/l)^{1/4}$) is due to the fact that in[2] the spectrum of the Langmuir oscillations in the region $k \gtrsim \omega_p/c$ is practically assumed to be isotropic. A more accurate investigation performed in[8] has shown that in fact the oscillations are concentrated in a relatively narrow region of k space. The estimate presented here has been written with allowance for this fact.

[6)]Let us recall that we are dealing with an isothermal plasma, in which ion-sound vibrations are impossible.

[1] P. Z. Chebotaev, D. D. Ryutov, M. D. Spektor, and G. E. Vekshteĭn, Doklad na VI Evropeĭskoĭ konferentsii po fizike plazmy i upravlyaemomu termoyadernomu sintezu (Paper presented at the 6-th Eurpean Conference on Plasma Physics and Controlled Thermonuclear Fusion), Moscow, 1973.

[2] B. N. Breĭzman, D. D. Ryutov, and P. Z. Chebotaev, Zh. Eksp. Teor. Fiz. 62, 1409 (1972)[Sov. Phys. -JETP 35, 741 (1972)].

[3] B. N. Breĭman, Doklad na VI Evropeĭskoĭ konferentsii po fizike plazmy i upravlyaemomu termoyadernomu sintezu (Paper presented at the 6-th European Conference on Plasma Physics and Controlled Thermonuclear Fusion), Moscow, 1973.

[4]M. A. Lifshits and V. N. Tsytovich, Zh. Eksp. Teor. Fiz. 62, 606 (1972)[Sov. Phys. -JETP 35, 321 (1972)].
[5]D. Rowlands, V. D. Shapiro, and V. I. Shevchenko, in: Vzaimodeĭstvie zaryazhennykh chastits s plazmoĭ (The Interaction of Charged Particles with a Plasma), Naukova dumka, 1967, p. 53.
[6]W. E. Drummond, Phys. Fluids, 5, 1133 (1962).
[7]A. A. Ivanov and L. I. Rudakov, Zh. Eksp. Teor. Fiz. 51, 1922 (1966)[Sov. Phys. -JETP 24, 1027 (1967)].
[8]B. N. Breĭzman, V. E. Zakharov, and S. L. Musher, Zh. Eksp. Teor. Fiz. 64, 1297 (1973)[Sov. Phys. -JETP 37, (1973)].

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