Excitation of a nonlinear regular plasma wave by a charged particle beam

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A nonlinear theory is developed for the excitation of a one-dimensional plasma wave with a fixed phase by a highly relativistic high-density beam under the conditions of beam-plasma instability. It is shown that, as a result of the relativistic increase in the mass of the electrons, the phase bunching effects are substantially reduced and do not lead to nonlinear stabilization of the instability for low field amplitudes. The Cerenkov resonance between the beam and the field, which is a wave with variable phase velocity, is therefore maintained throughout the instability process so that the beam loses a significant fraction of the energy associated with directed motion through the excitation of plasma oscillations. The electron-ion instability (excitation of low-frequency oscillations by relativistic electron beams) and the instability of an oscillator beam under the conditions of the anomalous Doppler effect (generation of an electromagnetic wave by a relativistic electron beam moving parallel to an external magnetic field) are investigated in the nonlinear approximation. The maximum field energy is found for both cases.

1. INTRODUCTION

One of the basic problems in the practical utilization of beam instabilities is to find methods of increasing the efficiency of interaction between a charged-particle beam and plasma.[1-3] This is so because even in the case of a small thermal spread ($\text{Im}\,\omega \gg kv_T$, k is the wave vector, v_T is the thermal velocity) the increase in the amplitude of the field excited by the beam is stabilized by effects associated with the capture of the beam particles by the wave.[1)] The result of this is that the fraction of energy lost by the low-density beam ($\gamma_0\nu^{1/3} \ll 1$, $\nu = n_b/n_p$; n_b and n_p are the densities of the beam and plasma, respectively, and γ_0 is the beam energy in units of mc^2) through the excitation of oscillations turns out to be relatively small[6]

$$E^2/8\pi = n_b m v_0^2 \gamma_0^2 (\nu/2)^{1/3}. \tag{1}$$

The energy density of the field excited by the beam in the plasma increases with increasing beam energy and when $\gamma_0\nu^{1/3} \sim 1$ it turns out to be comparable with the initial energy density of the beam.[12,13] At the same time, as a result of the relativistic increase in mass, the phase bunching effect in the particle beam which was investigated in[7-9] by numerical integration of the equations of motion is substantially reduced and, as shown below, when the condition

$$\gamma_0\nu^{1/3} \gg 1 \tag{2}$$

is satisfied, it does not lead to the nonlinear stabilization of the instability at low field amplitude. The beam particles are in phase with the wave for a sufficiently long interval of time $t \sim \gamma_0^{3/2}/\omega_b$ (ω_b is the Langmuir frequency of the beam) and do not succeed in transferring to the field an energy comparable with the initial beam energy. The directed slowing down of the beam is accompanied by a change in the phase velocity of the wave, which is determined by the beam velocity v(t):

$$E(t,z) = \sum_k E_k(t) \exp\left[ik\int_0^t v(\tau)\,d\tau - ikz\right].$$

The relative effectiveness of nonlinear phase bunching and directed slowing down of electrons as functions of the beam and plasma parameters is used in Sec. 2 to estimate the instability of a relativistic beam in the form of a sequence of charged bunches. The results are generalized to the case of a continuous beam in Sec. 3.

The same process of directed slowing down of the beam governs the electron energy loss due to instability, which appears during the passage of the relativistic beam through the ion core (Sec. 4). The energy density of the electromagnetic wave excited by a beam of oscillators moving at a velocity in excess of the velocity of light (under conditions of the anomalous Doppler effect) is determined in Sec. 5. The results established in Sec. 5 generalize the nonrelativistic formulas reported previously in[4].

2. EXCITATION OF A PLASMA WAVE BY A SEQUENCE OF RELATIVISTIC BUNCHES

Consider the excitation of a one-dimensional plasma wave by an electron beam in the form of a sequence of charged planes (surface charge density σe) moving through plasma with velocity v_0 and separated by a distance l. The set of equations describing the motion of the beam particles in the self-consistent field E(t, z) is[6]

$$\frac{\partial}{\partial z}\left(\frac{\partial^2 E}{\partial t^2} + \omega_p^2 E\right) = -4\pi\sigma e \frac{\partial^2}{\partial t^2} \sum_{s=-\infty}^{\infty} \delta(z - sl - z_s(t));$$
$$\frac{d}{dt}[\dot z_s(t)\gamma_s(t)] = -\frac{e}{m}\text{Re}\,E[t, z_s(t)], \tag{3}$$

where z_s and $\gamma_s = (1 - \dot z_s/c^2)^{-1/2}$ are the coordinate and energy (in units of mc^2) of the s-th bunch, and ω_p is the plasma frequency.

We shall consider slowly varying amplitude and phase by substituting[2)]

$$E(t, z) = \varepsilon(t) \exp[i\omega_p t - ikz + i\vartheta(t)] \tag{4}$$

and take an average of the equation for the field over the space period, so that we obtain

$$\frac{d}{dt}(\dot z\gamma) = -\frac{e}{m}\varepsilon\cos\Phi,$$
$$\dot\varepsilon = 2\pi e n_b v\cos\Phi, \qquad \varepsilon\dot\vartheta = 2\pi e n_b v \sin\Phi, \tag{5}$$

where $n_b = \sigma/l$ is the effective beam density, $\Phi(t) = kz(t) - \omega_p t - \vartheta(t)$, $k = 2\pi/l = \omega_p/v_0$, $v = \dot z$.

Integrating Eq. (5) subject to the initial conditions $\epsilon(0) = \Phi(0) = 0$, we obtain the following integrals of motion[6]

$$n_b mc^2\gamma+\frac{\varepsilon^2}{4\pi}=n_b mc^2\gamma_0; \tag{6}$$

$$\nu^{1/3}w\sin\Phi=w^2+\frac{c^2}{v_0^2}\left\{\left[1+\frac{v_0^2}{c^2}(\gamma_0-w^2)^2\right]^{1/2}-\gamma_0\right\}, \tag{7}$$

where $w^2 = \epsilon^2/4\pi n_b mc^2$.

After some algebraic rearrangement, Eq. (7) may be written in the more convenient form

$$w^3+2\frac{v_0^2}{c^2}\gamma_0^2\alpha w^2-\frac{v_0^2}{c^2}\gamma_0^2\alpha^2 w-2\gamma_0^3\alpha=0, \tag{8}$$

where $\alpha = \nu^{1/2}\sin\Phi$.

Different states of nonlinear stabilization of instability [right-hand side of Eq. (5), which gives the field amplitude, equal to zero] are defined by the ratio between the first and fourth terms in Eq. (8). When the beam is "weakly" relativistic ($\gamma_0\nu^{1/3} \ll 1$) the effect associated with the phase motion of the bunch relative to the field is the dominant one. Substituting $\sin\Phi = 1$ in this case, we have[6]

$$w_{max}\approx\gamma_0(4\nu)^{1/6}. \tag{9}$$

Accordingly, the field energy density is given by Eq. (1).

In the opposite limiting case, when Eq. (2) is satisfied, it follows from Eq. (8) that

$$\sin\Phi=w^3/2\nu^{1/2}\gamma_0^3. \tag{10}$$

Substituting $w_{max} \lesssim \gamma_0^{1/2}$ into Eq. (10) [which, according to Eq. (6), corresponds to almost total loss of energy by the beam], we find that[3)]

$$\sin\Phi_{max}\approx(4\gamma_0^3\nu)^{-1/2}\ll 1. \tag{11}$$

The instability of a high-density relativistic beam is not, therefore, accompanied by an appreciable phase shift of the bunches relative to the wave, and the kinetic energy of the electrons is completely transformed into the plasma wave field energy.

3. INSTABILITY OF A HIGH-CURRENT RELATIVISTIC BEAN IN PLASMA

We shall use the following set of equations to describe the interaction of a continuous relativistic beam with plasma:

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial z}+eE\frac{\partial f}{\partial p}=0,$$
$$\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right)E=-4\pi e\frac{\partial}{\partial t}\int_{-\infty}^{\infty}(f-f_0)v\,dp, \tag{12}$$

where $f(t, z, p)$ is the function describing the electron momentum distribution in the beam.

The solution of Eq. (12) will be sought in the form

$$f(t,z,p)=f_0(t,p)+\frac{1}{2}\sum_k[f_k(t,p)e^{-ikz}+f_k^*(t,p)e^{ikz}], \tag{13}$$
$$E(t,z)=\frac{1}{2}\sum_k[E_k(t)e^{-ikz}+E_k^*(t)e^{ikz}],$$

where f_0 is the phonon distribution function and f_k are the amplitudes of small oscillating additional terms.[4)]

Substituting Eq. (13) in Eq. (12) and averaging with respect to z, we obtain

$$\frac{\partial f_k}{\partial t}-ikf_k=-eE_k\frac{\partial f_0}{\partial p}, \tag{14}$$

$$\frac{\partial f_0}{\partial t}+\frac{e}{4}\sum_k\left(E_k^*\frac{\partial f_k}{\partial p}+E_k\frac{\partial f_k^*}{\partial p}\right)=0, \tag{15}$$

$$\left(\frac{d^2}{dt^2}+\omega_p^2\right)E_k=-4\pi e\frac{d}{dt}\int_{-\infty}^{\infty}vf_k\,dp. \tag{16}$$

We shall suppose that at the initial time the thermal spread in the beam is sufficiently small and the distribution function is close to a delta function:

$$f_0(0,p)=n_b\delta(p-p_0). \tag{17}$$

The development of instability is accompanied by a reduction in the directed momentum and an increase in the electron temperature in the beam so that the distribution function is given by

$$f_0(t,p)=\frac{n_b(t)}{\sqrt{\pi}p_T(t)}\exp\left\{-\frac{1}{p_T^2(t)}[p-p(t)]^2\right\}. \tag{18}$$

Since according to Eq. (18) the state of the beam is uniquely determined by the moments of the distribution function, we can substantially simplify the problem by considering the equations for $n_b(t)$, $p_T(t)$, $p(t)$ instead of the complicated set of equations given by Eqs. (14) and (15).

Let us transform Eq. (16) by eliminating the time derivative on the right-hand side with the aid of Eq. (14)

$$\left(\frac{d^2}{dt^2}+\omega_p^2\right)E_k=-4\pi e\int_{-\infty}^{\infty}\left(ikv^2f_k+eE_k\frac{dv}{dp}f_0\right)dp. \tag{19}$$

The second term under the integral sign in Eq. (19) is a small addition to the plasma frequency and can be omitted without loss of generality. Next, we compare the right-hand sides of Eqs. (15) and (19) and eliminate the function f_k. The result is

$$\frac{d}{dt}\int_{-\infty}^{\infty}mc^3\left(\frac{p}{mc}-\operatorname{arctg}\frac{p}{mc}\right)f_0\,dp=\frac{i}{16\pi}\sum_k\frac{1}{k}\left(E_k^*\frac{d^2E_k}{dt^2}-E_k\frac{d^2E_k^*}{dt^2}\right). \tag{20}$$

The equation which determines the thermal spread in the beam will be taken to be the relation which follows from Eq. (15), namely,

$$\frac{d}{dt}\int_{-\infty}^{\infty}[p-p(t)]^2f_0\,dp=\frac{e}{2}\int_{-\infty}^{\infty}[p-p(t)]\sum_k(f_kE_k^*+f_k^*E_k)\,dp. \tag{21}$$

The function f_k on the right-hand sides of Eqs. (16) and (21) is given by Eq. (14):

$$f_k(t,p)=-e\int_0^t E_k(\tau)\frac{\partial f_0}{\partial p}\exp\left(ik\int_\tau^t v\,dt_1\right)d\tau. \tag{22}$$

Substituting for $f_0(t, p)$ from Eq. (18) into Eqs. (16), (20), and (21), using Eq. (22), and evaluating the integrals with respect to the momenta on the assumption that $p(t) \gg p_T(t)$, we obtain the following closed set of equations:

$$n_b\frac{d}{dt}\left[mc^3\left(\frac{p}{mc}-\operatorname{arctg}\frac{p}{mc}\right)+v\frac{n_bp_T^2}{2m\gamma^3}\right]=\frac{i}{16\pi}\sum_k\frac{1}{k}\left(E_k^*\frac{d^2E_k}{dt^2}-E_k\frac{d^2E_k^*}{dt^2}\right), \tag{23}$$

$$\left(\frac{d^2}{dt^2}+\omega_p^2\right)E_k=-4\pi e^2ikn_b\int_0^t E_k(\tau)\exp[i\Phi_k(t)-i\Phi_k(\tau)]\times[ikv^2(t-\tau)+2v]\frac{dv}{dp}d\tau,$$

$$\frac{dp_T^2}{dt}=2e^2\operatorname{Re}\int_0^t E_k^*(t)E_k(\tau)\exp[i\Phi_k(t)-i\Phi_k(\tau)]d\tau, \tag{24}$$

$$\Phi_k(t)=k\int_0^t v(t_1)\,dt_1,$$

where v(t) is the directed beam velocity, $dp/dv = m\gamma^3$, and the density n_b which by Eq. (15) is a constant of motion has been taken out from under the time integral.

We now introduce the slowly varying field amplitude $\epsilon_k(t)$ by substituting

$$\varepsilon_k(t)=E_k(t)\exp[-i\Phi_k(t)],\quad \dot{\varepsilon}_k\ll\dot{\Phi}\varepsilon_k$$

and if we neglect the second derivatives of the amplitude we can rewrite Eq. (23) in the form

$$n_b mc^3\left(\frac{p}{mc}-\operatorname{arctg}\frac{p}{mc}\right)+v\left(\frac{n_b p_T^2}{2m\gamma^3}+\sum_k\frac{|\varepsilon_k|^2}{8\pi}\right)=S_0, \tag{25}$$

$$ik\left(2v\frac{d\varepsilon_k}{dt}+\varepsilon_k\frac{dv}{dt}\right)+(\omega_p^2-k^2v^2)\varepsilon_k=\int_0 \omega_b^2k^2v^2(t-\tau)\varepsilon_k(\tau)\frac{d\tau}{\gamma^3}, \tag{26}$$

$$\frac{dp_T^2}{dt}=2e^2\operatorname{Re}\sum_k\int_0^t\varepsilon_k^*(t)\varepsilon_k(\tau)d\tau, \tag{27}$$

where ω_b is the Langmuir frequency of the beam.

The integro-differential equation given by Eq. (26) may be reduced to a differential equation by differentiating with respect to time t:

$$ik\frac{d^2}{dt^2}\left(2v\frac{d\varepsilon_k}{dt}+\varepsilon_k\frac{dv}{dt}\right)+\frac{d^2}{dt^2}[(\omega_p^2-k^2v^2)\varepsilon_k]=\omega_b^2k^2v^2\frac{1}{\gamma^3}\varepsilon_k. \tag{28}$$

We shall assume that the contribution of the thermal energy flux to the integral of motion is sufficiently small and will neglect the term proportional to p_T^2 in Eq. (25) (the validity of this approximation will be estimated below). This ensures that the closed set of equations reduces to a set of equations for the amplitudes ϵ_k given by Eq. (28), in which the velocity v(t) is related to the field amplitude by Eq. (25).

During the linear stage of the instability[1-3] [$v(t) = v_0$ and $\gamma(t) = \gamma_0$] the solution of Eq. (28) takes the form $\epsilon_k(t) = \epsilon_{0k}\exp(\lambda_k t)$, where λ_k satisfies the equation

$$2ikv_0\lambda_k^3+(\omega_p^2-k^2v_0^2)\lambda_k^2-\omega_b^2k^2v_0^2/\gamma_0^3=0. \tag{29}$$

The maximum growth rate ($\operatorname{Re}\lambda_0 > 0$) occurs for the harmonic $k_0 = \omega_p/v_0$ (the width of the wave packet is $\Delta k \sim |\lambda_0|/v_0$)

$$\lambda_0=\frac{\sqrt{3}-i}{2^{4/3}}\left(\frac{\nu}{\gamma_0^3}\right)^{1/3}\omega_p. \tag{30}$$

Moreover, the harmonics for which $kv_0 < \omega_p$ turn out to be unstable:

$$\lambda_k\approx\left(\frac{k^2v_0^2}{\omega_p^2-k^2v_0^2}\frac{\omega_b^2}{\gamma_0^3}\right)^{1/2}, \tag{31}$$

and oscillations with $kv_0-\omega_p \gtrsim |\lambda_0|$ are not excited during the linear stage.

According to Eq. (25), the increase in the field amplitude is accompanied by a reduction in the beam velocity. Accordingly, there is a change in the energy distribution over the wave-number spectrum. The increase in the amplitude of unstable "fast" harmonics ($\omega_p/k > v_0$) is slowed down and the "slow" harmonics ($\omega_p/k < v_0$) enter resonance with the beam during the nonlinear stage [$v(t) < v_0$] and are amplified with maximum growth rate. The final result is that the spectrum of oscillations excited during the nonlinear stage is shifted into the slow-wave region, which is qualitatively confirmed by numerical calculations.[8]

The analytic nonlinear solutions of Eq. (28) cannot be found for an arbitrary number of harmonics. We shall therefore consider the case where the beam excites in the plasma a wave with a fixed value of the wave vector k_0 which, according to[8], occurs in a sufficiently thinned-out spectrum [$|kv_0-\omega_p| > (\nu/\gamma_0^3)^{1/3}\omega_p$] or in the presence of external modulation[5)].

We shall suppose that the harmonic $k_0 = \omega_p/v_0$ is excited, so that the growth rate is determined by Eq. (30). As the beam is slowed down [$v(t) < v_0$] the resonance condition is violated and the second term in Eq. (28) becomes comparable with the first when the beam velocity falls to the value

$$\omega_p^2-k_0^2v^2\sim(\nu/\gamma_0^3)^{1/3}\omega_p^2,\quad \gamma\sim(\nu/\gamma_0^3)^{-1/6}. \tag{32}$$

Since the change in the velocity at this time is still small, and the instability continues to develop at a lower growth rate, in the analysis of the nonlinear stage of the beam-plasma interaction (during which most of the beam energy losses occur) we can neglect in Eq. (28) the terms containing the third derivative and consider the equations

$$\frac{d^2}{dt^2}[(v_0^2-v^2)\varepsilon]=\omega_b^2\frac{v^2}{\gamma^3}\varepsilon, \tag{33}$$

$$n_bmc^3\left(\frac{p}{mc}-\operatorname{arctg}\frac{p}{mc}\right)+\frac{v\varepsilon^2}{8\pi}=S_0. \tag{34}$$

Expressing the field amplitude $\epsilon(t)$ in terms of the velocity v(t) obtained from Eq. (34), and substituting the result in Eq. (33), we obtain the nonlinear equation for the function v(t), the first integral of which is

$$\left(\frac{dv}{dt}\right)^2=4\omega_b^2v^2\frac{\varepsilon^2}{8\pi}\frac{{}^1/_6mn_b(v_0^2-v^2)^2(v^2+v_0^2/2)+I(v)}{[(3v^2+v_0^2)\varepsilon^2/8\pi+mn_b\gamma^3v^2(v_0^2-v^2)]^2}$$
$$I(v)=\int_v^{v_0}(3v^2+v_0^2)\frac{v\varepsilon^2}{8\pi}\frac{dv}{\gamma^3}. \tag{35}$$

In deriving Eq. (35), we used the relation

$$\frac{d\varepsilon^2}{dt}=-\frac{\varepsilon^2}{v}\frac{dv}{dt}-\frac{d}{dt}(8\pi n_bmc^2\gamma), \tag{36}$$

which follows from Eq. (34).

According to Eq. (35), the derivative of the beam velocity vanishes at points $v_1 = v_0$ and $v_2 = 0$ which determine the maximum and minimum values of the beam velocity. The beam therefore excites in the plasma a wave with variable phase velocity $v_{ph} = v(t)$ which is synchronous with the beam at each instant of time and completely loses its energy of directed motion.

It is clear that this effect can be interpreted as the successive excitation by the beam of a spectrum of waves with frequencies $\omega(t) = k_0v(t)$ in the interval $\omega_p \geq \omega(t) \geq 0$. Equation (34) then gives the energy distribution over the oscillation spectrum and is the differential characteristic of the spectrum. To estimate the effectiveness of the beam-plasma interaction as a function of the beam energy, it is useful to consider the spectral characteristics defined as follows:

$$W_\varepsilon=\int_0^{p_0}v\frac{|\varepsilon|^2}{8\pi}dp=\frac{1}{4}n_bm^2c^4(\gamma_0^2-1-\ln\gamma_0^2). \tag{37}$$

Comparing this with the analogous quantity for the beam

$$W_b=n_bmc^2\int_0(\gamma-1)vdp={}^1/_2n_bm^2c^4(\gamma_0-1)^2, \tag{38}$$

we see that

$$1\geqslant W_\varepsilon/W_b\geqslant{}^1/_2, \tag{39}$$

where the left-hand inequality is satisfied in the nonrelativistic case and the right-hand inequality occurs for $\gamma_0 \gg 1$.

Equation (35) has been obtained on the assumption that the thermal spread in the beam is sufficiently small

$$4\pi n_b p_T^2 \ll m\gamma^3 |\varepsilon|^2. \quad (40)$$

Substituting $p_T \sim e\varepsilon t$ in Eq. (40) [using Eq. (27)], we obtain the following restriction on the time interval t during which the approximation given by Eq. (35) is valid:

$$t \ll \gamma_0^{3/2}/\omega_b. \quad (41)$$

when $t \sim \gamma_0^{3/2}/\omega_b$ the contribution of the thermal flux to the energy integral (25) is comparable with the field energy density.

4. ELECTRON-ION INSTABILITY

Consider the low-frequency electron-ion instability in the nonlinear approximation,[14] using the method developed in Sec. 3. The nonlinear analysis of this effect given in[15] for various models of electron and ion beams has shown that the most important effect is the slowing down of the electron beam. We shall therefore restrict our attention to the nonlinearity of the electron motion, and will treat the ions in the linear approximation (the estimate is given below). The equations describing the motion of the electrons will be taken in the form given by Eqs. (14)—(16), replacing the plasma frequency ω_p^2 in the field equation (16) by the natural frequency of the ions[6] $\omega_i^2 = 4\pi e^2 n_i/M$. Since the time necessary for the development of the instability is large in comparison with the Langmuir frequency of the electrons

$$\partial f_1/\partial t \ll k v_0 f_1, \quad k v_0 \sim \omega_b/\gamma_0^{3/2}, \quad (42)$$

we write the solution of Eq. (14) in the form of an expansion into a series:

$$f_1 = -e\left[\frac{i}{kv} + \frac{1}{(kv)^2}\frac{\partial}{\partial t} - \frac{i}{(kv)^3}\frac{\partial^2}{\partial t^2}\right]\left(E\frac{\partial f_0}{\partial p}\right). \quad (43)$$

Substituting Eq. (43) in Eqs. (15) and (16), and assuming that $f_0 = n_b\delta[p - p(t)]$, we obtain a closed set of equations for the beam field and momentum:

$$2i\frac{d^3}{dt^3}\left(\frac{\omega_b^2 E}{k^3 v^3 \gamma^3}\right) + \frac{d^2}{dt^2}\left[\left(1 - \frac{\omega_b^2}{k^2 v^2 \gamma^3}\right)E\right] + \omega_i^2 E = 0, \quad (44)$$

$$\frac{dp}{dt} = -\frac{e^2}{2k^2 m}\left[E^*\frac{d}{dt}\left(\frac{E}{v^3\gamma^3}\right) + E\frac{d}{dt}\left(\frac{E^*}{v^3\gamma^3}\right)\right]. \quad (45)$$

Integrating Eq. (45), we obtain an equation for the constant of the motion

$$|E|^2 = \frac{k^2 m^2}{e^2} v^6 \gamma^6 \left(\frac{1}{v^2} - \frac{1}{v_0^2}\right). \quad (46)$$

In the linear approximation, assuming that $v = v_0$, $k = \omega_b/v_0\gamma_0^{3/2}$, and $E = E_0 e^{\lambda t}$, we obtain

$$\lambda = \frac{\sqrt{3}+i}{2^{4/3}}\frac{\omega_i^{2/3}\omega_b^{1/3}}{\gamma_0^{1/2}} \quad (47)$$

As the field amplitude increases, the beam energy decreases and the second term in Eq. (44) becomes comparable with the first for a relatively small change in the beam velocity:

$$\Delta v \sim \gamma_0 (\omega_i/\omega_b)^{2/3} v_0. \quad (48)$$

Since the instability is not stabilized for small nonlinearity, we shall, as in Sec. 3, neglect the term proportional to the third derivative. Moreover, if we now express the field amplitude in terms of the velocity given by Eq. (46), we obtain the following nonlinear equation:

$$\frac{d^2}{dt^2}[(v_0^2 - v^2)^{1/2}(v_0^2\gamma_0^3 - v^2\gamma^3)] = \omega_i^2 v^2 \gamma^3 (v_0^2 - v^2)^{1/2}. \quad (49)$$

Using the first integral of Eq. (49)

$$\left(v\frac{dv}{dt}\right)^2 = \omega_i^2 c^4 \frac{\gamma_0^2 - \gamma^2}{\gamma^3}\frac{\gamma(\gamma_0+\gamma)(\gamma_0^2+\gamma^2-2) + (\gamma_0^2-1)[(\gamma_0^2-1)\gamma - 2\gamma_0]}{[3\gamma^3(\gamma_0+\gamma) + (\gamma_0^2-1)(\gamma_0^2+\gamma^2+\gamma\gamma_0)]^2}, \quad (50)$$

we can determine the extremal values of the beam velocity by setting $v(v_{ext}) = 0$. Since the right-hand side of Eq. (50) vanishes only for $v_{ext} = \pm v_0$, the development of the instability is accompanied by an initial reduction of the beam velocity to zero. The velocity then changes sign and increases in absolute magnitude back to the initial value.[15] By analyzing Eq. (46), we can investigate the dependence of the field energy density on the beam velocity (energy). With decreasing velocity the field amplitude increases, reaching the maximum value

$$|E|^2/16\pi = n_b m v_0^2 \gamma_0/27 \quad (51)$$

when

$$v_m^2 = \frac{2v_0^2}{3-\beta_0^2}, \quad \gamma_m = \gamma_0\left(1 - \frac{\beta_0^2}{3}\right), \quad \beta_0 = \frac{v_0}{c}. \quad (52)$$

Further reduction in the beam velocity is accompanied by a reduction in the field energy density, so that $E_{min} = 0$ for $v = 0$ and the beam energy is completely transformed into the energy of the ion oscillations.

Let us now estimate the validity of the linear approximation as applied to the ions. Since the condition for the linearization of the equations of motion for the ions is $\partial v_i/\partial t \gg \partial v_i^2/\partial z$ where

$$v_i \sim eE/M\lambda, \quad \partial v_i/\partial t \sim \lambda v_i, \quad \partial v_i^2/\partial z \sim k v_i^2, \quad (53)$$

and substituting for λ and E from Eqs. (47) and (51), we find that

$$(n_i/n_b)^{2/3} \gg (m/M)^{1/3}. \quad (54)$$

5. EXCITATION OF A REGULAR ELECTROMAGNETIC WAVE BY A BEAM OF RELATIVISTIC OSCILLATORS

In conclusion, we consider the excitation of an electromagnetic wave by a relativistic electron beam moving down a constant external magnetic field $\mathbf{H}_0$ with velocity v_0 which is greater than the velocity of light in the slowing down system with an effective refractive index n, where $v_0 > c/n$ (anomalous Doppler effect).[16]

The solution of the self-consistent set of equations

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{v}\nabla)\mathbf{p} = e\mathbf{E} + \frac{e}{c}[\mathbf{v}\times\mathbf{H}+\mathbf{H}_0], \quad \frac{\partial n_b}{\partial t} + \operatorname{div}\mathbf{j} = 0,$$
$$\operatorname{rot}\mathbf{H} = \frac{n^2}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}; \quad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \quad (55)$$

will be sought in the form of circularly polarized waves propagating along the magnetic field:

$$H_y - iH_x = n(E_x + iE_y) = nE(t)\exp[i\Phi + i\varphi(t)],$$
$$v_x + iv_y = v_\perp(t)\exp[i\Phi + i\vartheta(t)], \quad (56)$$

where **p** and **v** are, respectively, the momentum and velocity of the beam, **E** and **H** are the electric and magnetic components of the self-consistent field $\mathbf{j} = en_b\mathbf{v}$, $\Phi = \omega(t - nz/c)$, and ω is the wave frequency. Assuming, moreover, that $v_z(t, z) = v_\parallel(t)$ and $n_b(t)z) = n_0$ (the continuity equation is then automatically satisfied), we obtain the following set of equations in total derivatives with respect to time and for the slowly-varying amplitudes and phases:

$$\frac{d}{dt}\gamma v_\perp = \frac{e}{m}(1 - \beta_\parallel n)E\cos\eta, \quad (57)$$

$$\frac{d}{dt}\gamma v_{\parallel}=\frac{e}{mc}nv_{\perp}E\cos\eta, \tag{58}$$

$$\frac{dE}{dt}=-2\pi en_0\frac{v_{\perp}}{n^2}\cos\eta, \tag{59}$$

$$\frac{d\eta}{dt}=\frac{\omega_H}{\gamma}+\omega(1-\beta_{\parallel}n)-\mathrm{tg}\,\eta\frac{d}{dt}\ln(\gamma v_{\perp}E), \tag{60}$$

where

$$\omega_H=eH_0/mc,\quad \eta=\vartheta-\varphi,\quad \gamma=(1-\beta_{\parallel}^2-\beta_{\perp}^2)^{-1/2},\quad \beta^2=|\mathbf{v}|^2/c^2.$$

In terms of the same variables, the change in the beam energy with time is described by the equation

$$\frac{d\gamma}{dt}=\frac{e}{mc^2}v_{\perp}E\cos\eta. \tag{61}$$

Subtracting Eq. (58) from Eq. (61) term by term, we obtain[7)]

$$\frac{d}{dt}(\beta_{\parallel}n-1)\gamma=\frac{e}{mc^2}(n^2-1)v_{\perp}E\cos\eta. \tag{62}$$

The set of equations given by Eqs. (57), (59), (60), and (62) may be written in a more compact form by substituting

$$a=\beta_{\perp}\gamma,\quad b=(\beta_{\parallel}n-1)\gamma,\quad \Omega=\omega/\omega_H,\quad ds=\omega_H dt/\gamma.$$

Accordingly, we have

$$\begin{gathered}\dot a=-b\varepsilon\cos\eta,\quad \dot\varepsilon=-q^2a\cos\eta,\quad \dot b=(n^2-1)a\varepsilon\cos\eta,\\ \dot\eta=1-\Omega b-\mathrm{tg}\,\eta\frac{d}{ds}\ln(\varepsilon a).\end{gathered} \tag{63}$$

Integrating these equations subject to the initial conditions $\eta(0) = a(0) = 0$, $\epsilon(0) = \epsilon_0$, $b(0) = \Omega^{-1}$ (condition for the anomalous Doppler effect), we obtain the following integrals:

$$(n^2-1)a^2+b^2=\Omega^{-2}, \tag{64}$$

$$\varepsilon^2-\varepsilon_0^2=\frac{2q^2}{n^2-1}\left(\frac{1}{\Omega}-b\right), \tag{65}$$

$$\sin\eta=(1-\Omega b)^2/2a\varepsilon\Omega(n^2-1). \tag{66}$$

These relationships enable us to reduce the set of equations given by Eq. (63) to a first-order equation. In view of Eq. (64), we substitute

$$(n^2-1)^{1/2}\Omega a=\sin\psi,\quad \Omega b=\cos\psi.$$

The result of this is the following equation for the function $\psi(s)$:

$$\dot\psi=-(n^2-1)^{1/2}\varepsilon\cos\eta. \tag{67}$$

Substituting Eqs. (65) and (67), and expressing η in terms of ψ from Eq. (66), we have

$$\dot\psi^2=\varepsilon_0^2(n^2-1)+\frac{2q^2}{\Omega}(1-\cos\psi)-\frac{(1-\cos\psi)^4}{4\sin^2\psi}. \tag{68}$$

The maximum value of the function $\psi(s)$ is obtained from the condition $\psi(\psi_m) = 0$. Since $|\psi_m| \ll 1$ (when $q^2 \ll 1$ and $\epsilon_0 \ll 1$), we have

$$\psi_m=\begin{cases}2\varepsilon_0^{1/3}(n^2-1)^{1/6}, & q/\Omega\ll\varepsilon_0^{2/3}(n^2-1)^{1/6}\\ 2(4q^2/\Omega)^{1/4}, & q/\Omega\gg\varepsilon_0^{2/3}(n^2-1)^{1/6}\end{cases} \tag{69}$$

where the first case corresponds to the instability of the charged particle in the field of an external transverse wave.

Substituting for ψ_m in Eq. (65), we obtain the energy density of the electromagnetic field excited by the beam in the absence of the external wave $\epsilon_0 = 0$:

$$\varepsilon_{max}^2=\frac{1}{n^2-1}\left(\frac{4q^2}{\Omega}\right)^{1/2}. \tag{70}$$

Returning to the dimensional variables, we can rewrite this formula in the form

$$\frac{n^2E_{max}^2}{8\pi}=\frac{n_0mc^2(\beta_{0\parallel}n-1)\gamma_0}{n^2-1}\left[(\beta_{0\parallel}n-1)\gamma_0\frac{8\pi n_0mc^2}{n^2H_0^2}\right]^{1/2}. \tag{71}$$

The nonlinear stabilization of the instability arises as a result of the detuning of the phase resonance between the beam and the wave due to the longitudinal slowing down of the beam. In contrast to the above Cerenkov instability, in the present case the phase velocity of the wave is unaltered ($v_{ph} = c/n$) and the instability is stabilized even at small field amplitude.

1)The stabilization of the increase in amplitude of the regular field as a result of capture of a beam of oscillators by the wave was first discussed in [4]. Similar effects were investigated for Cerenkov instability in an unbounded plasma in [5-9] and in the case of the injection of a beam into unbounded plasma in [10,11].

2)This approximation is valid when $\nu^{1/3}/\gamma_0 \ll 1$.

3)It is important to note that when the equation $\omega^2_{max}=\gamma_0-1$ is exactly satisfied, the second and fourth terms in Eq. (8) compensate one another and comparison of the first and third terms gives a condition which is stronger than that given by Eq. (11), namely, $(\gamma_0\nu)^{1/2} \gg 1$. This is also the case for a continuous beam (Sec. 3) when there is a substantial reduction in the growth rate during the nonlinear stage of instability.

4)The condition that the self-modulation depth of the beam is small, $f_0 \gg |f_k|$, for which one can neglect effects associated with the phase shift of the particles, is equivalent to the requirement that $(\gamma_0^3\nu)^{1/6} \gg 1$.

5)We note that, for a relativistic beam, the condition that the spectrum is "thinned-out" is less stringent than in the nonrelativistic case [8] because of the reduction in the width of the wave packet by the factor $\gamma_0 \gg 1$.

6)The condition that the modulation depth is "small," $f_0 \gg |f_1|$, now takes the form $\gamma_0 \gg 1$.

7)We note that, when n = 1, the function $(1-\beta_{\parallel})\gamma$ is a constant of motion and there is a nonlinear resonance between the beam and the field. [17,18]

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