Couette flow of a nematic liquid

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The stability of Couette flow of a nematic liquid with respect to infinitesimal perturbations is investigated as a function of the viscosity coefficient α_3 (the Leslie coefficient) and the Reynolds number R. It is shown that at $\alpha_3 < 0$ the Couette flow of the mesophase is stable at all R. When $\alpha_3 > 0$ there exists a critical number $R_c(\alpha_3)$ above which the stationary motion becomes unstable. The dependence of the number R_c on the magnetic field that stabilizes the stationary flow of the liquid crystal is determined.

1. As is well known, the flow of an isotropic liquid between two moving parallel planes remains stable with respect to infinitesimally small perturbations at arbitrarily large Reynolds numbers. Stable undamped nonstationary motion was produced experimentally in this case starting with Reynolds numbers of approximately 1500 [1].

In an anisotropic (nematic) liquid, Couette flow has certain singularities connected with the fact that the mesophase is characterized by several viscosities α_j and elastic moduli K_{ii}. The distribution of the "director" and of the flow velocities in the mesophase are interrelated and are determined by the values of α_j and K_{ii} [2–4]. Analysis shows that an important role is played here by the viscosities α_3 and $\beta_2 = (\alpha_3 + \alpha_4 + \alpha_6)/2$, or by the ratios $\epsilon = \alpha_3/\alpha_2$ and $\delta = -\beta_2/\alpha_2$. The shear viscosity α_3 enters in the relation between the moments of the forces acting on the director and the rates of deformation in the layer of the nematic liquid. The coefficient β_2 determines the "viscous" stresses in the Navier-Stokes equation. Owing to the elasticity of the liquid crystal, an inhomogeneous distribution of the director $\mathbf{n}$ gives rise to a molecular field acting on the orientation of the molecules. At a corresponding balance of the moments of the forces due to the flow and to the inhomogeneous distribution of $\mathbf{n}$, an equilibrium orientation of $\mathbf{n}$ is established in the mesophase flow. This balance can be upset by small perturbations in a definite range of the material parameters ϵ and $\lambda_{ii} = \rho K_{ii}\alpha_2^{-2}$ and of the number $R = \rho u l |\alpha_2|^{-1}$, which plays here the role of the Reynolds number (ρ is the density, l is the thickness of the layer, and u is the relative velocity of the planes). Increasing deviations of $\mathbf{n}$ from the equilibrium position lead to an interruption of the stationary flow and to the onset of turbulence in the nematic liquid at finite and relatively small R.

A recent communication [5] reported synthesis of new liquid crystals, the viscosity anisotropy of which seems to vary noticeably with temperature. Thus, it becomes possible to investigate different regimes of motion in corresponding intervals of the values of ϵ and R. Gähwiller [5] observed at a certain temperature T_0 turbulization of the flow of the mesophase between parallel plates, one of which was stationary and the other moved along the x axis with constant velocity u. In such a motion with a constant velocity gradient $dv_0/dz = u/l$ (the plates had coordinates $z = 0$ and $z = l$), the director is oriented at an angle θ_0 to the x axis in the xz plane [2], with

$$\mathrm{tg}^2(2\theta_0) = 4\varepsilon/(1-\varepsilon)^2.$$

Usually $\alpha_3 < 0$, $\alpha_2 < 0$, $\epsilon \ll 1$, and $\theta_0^2 \approx \epsilon \ll 1$. The observed effect is attributed in [5] to the vanishing of the coefficient α_3 at the temperature T_0.

It is shown in the present paper that at $\epsilon < 0$ there exists a critical number $R_c(\epsilon)$ characterizing the loss of stability. At the point $\epsilon = 0$, the function $R_c(\epsilon)$ becomes infinite. At positive ϵ, the Couette flow of an anisotropic liquid is stable at all values of R.

2. We consider the nematic-liquid layer whose geometry was described above. We assume that $\alpha_1 = 0$ and $|\epsilon| \ll 1$. Under these conditions, the unperturbed stationary flow is described, in accordance with [3,4], by the equations

$$\frac{d}{dz}\left[(\alpha_4+\alpha_6-2\alpha_2\theta_0^2)\frac{dv_0}{dz}\right] = 0,\qquad K_{11}\frac{d^2\theta_0}{dz^2} = (\alpha_3-\alpha_2\theta_0^2)\frac{dv_0}{dz} \tag{1}$$

with boundary conditions $\theta_0(0) = \theta_0(l) = 0$, $v_0(0) = 0$, $v_0(l) = u$. The first equation of (1) is the Navier-Stokes equation, and the second is the equilibrium equation for the director. It is assumed in (1) that $\theta_0^2 \ll 1$.

It follows from (1) that dv_0/dz and θ_0 are, generally speaking, functions of z. However, in the region of values of ϵ, λ_{11}, and R corresponding to small θ_0^2, the velocity gradient dv_0/dz can be regarded as approximately constant, and $\theta_0(z)$ can be obtained from the equilibrium equation for the director, in which $dv_0/dz \approx u/l$. The last equation then takes the form

$$\frac{\lambda_{11}}{R}\frac{d^2\theta_0}{dz^2} = -\varepsilon + \theta_0^2. \tag{2}$$

If $1 \gg \epsilon \gg (\lambda_{11}/R)^2$, then it follows from (2) that $\theta_0 \approx \epsilon^{1/2}$ (the Leslie solution) everywhere with the exception of a narrow layer next to the wall, with thickness on the order of

$$z_0 = l(2\lambda_{11}^2/R^2|\varepsilon|)^{1/4},$$

in which the orientation of the director changes exponentially like $\exp(-z/z_0)$ from the value $\theta_0 \approx \epsilon^{1/2}$ in the main layer to $\theta_0 = 0$ on the solid surface.

If $|\epsilon| \ll (\lambda_{11}/R)^2 \ll 1$, then according to (2) the function $\theta_0(z)$ takes the approximate form

$$\theta_0(z) \approx -\frac{\varepsilon R}{2\lambda_{11}}\left[\left(\frac{z}{l}\right)^2 - \frac{z}{l}\right].$$

At $\epsilon < 0$ and $(\lambda_{11}/R)^2 \ll |\epsilon| \ll 1$ the distribution of $\theta_0(z)$ is described by the Weierstrass function

$$3(2|\varepsilon|)^{1/2}\wp\left(\frac{z+C}{z_0};\ -\frac{2}{3},\ g_3\right),$$

where C and g_3 are determined from the boundary conditions. In this case the $\theta_0(z)$ dependence is periodic with a period on the order of z_0.

Superimposed on the investigated stationary solution obtained above are nonstationary small perturbations

$$v_{1x}(x,z,t),\ v_{1z}(x,z,t),\ \theta_1(x,z,t).$$

The onset of the instability is characterized by the appearance of solutions

$$\Psi(x,z,t)\sim\psi(z)\exp[i(kx-\omega t)]$$

with complex frequency $\omega=\omega'+i\omega''$, $\omega''>0$, for the system of equations describing the mesophase.

We note that the problem of the onset of instability is two-dimensional. The reason is that, in the considered first-order approximation, the deviations of the director and of the flow velocity from the plane xz are proportional to the wave number p corresponding to the relation $(v_{1x}, v_{1z}, \theta_1)\sim\exp(ipy)$. Yet finite values of p, as can be readily seen, lead to an additional damping of the perturbations of v_{1x}, v_{1z}, and θ_1. Thus, within a short time interval after the termination of the stationary regime (while v_{1x}, v_{1z}, and θ_1 are small), the perturbations are characterized by the numbers $p=0$, and the deviations from the plane xz can be neglected.

Linearizing the exact equations [3,4] with respect to v_{1x}, v_{1z}, and θ_1, and neglecting the quantity θ_0 in the Navier-Stokes equations (just as in the determination of the stationary solution), we obtain for v_{1x}, v_{1z}, and θ_1 the following system of equations:

$$\begin{gathered}\frac{\partial v_{1x}}{\partial x}+\frac{\partial v_{1z}}{\partial z}=0,\\
\rho\left(\frac{\partial v_{1x}}{\partial t}+v_0\frac{\partial v_{1x}}{\partial z}+\frac{u}{l}v_{1z}\right)=-\frac{\partial P}{\partial x}+\beta_2\frac{\partial^2 v_{1x}}{\partial z^2}\\
+\left(\frac{1}{2}\alpha_4+\frac{3}{2}\alpha_6-\alpha_2\right)\frac{\partial^2 v_{1x}}{\partial x^2}+(\alpha_6-\alpha_2)\frac{u}{l}\frac{\partial\theta_1}{\partial x},\\
\rho\left(\frac{\partial v_{1z}}{\partial t}+v_0\frac{\partial v_{1z}}{\partial x}\right)=-\frac{\partial P}{\partial z}+\frac{1}{2}(\alpha_4-\alpha_6)\frac{\partial^2 v_{1z}}{\partial z^2}\\
+\left(\frac{1}{2}\alpha_4+\frac{1}{2}\alpha_6-\alpha_2\right)\frac{\partial^2 v_{1z}}{\partial x^2}+\alpha_6\frac{u}{l}\frac{\partial\theta_1}{\partial z}+\alpha_2\left(\frac{\partial^2\theta_1}{\partial t\,\partial x}+v_0\frac{\partial^2\theta_1}{\partial x^2}\right),\\
K_{11}\frac{\partial^2\theta_1}{\partial z^2}+K_{33}\frac{\partial^2\theta_1}{\partial x^2}+\alpha_2\left(\frac{\partial\theta_1}{\partial t}+v_0\frac{\partial\theta_1}{\partial x}+2\theta_0\frac{u}{l}\theta_1\right)=\alpha_2\frac{\partial v_{1z}}{\partial x}+\alpha_3\frac{\partial v_{1x}}{\partial z}.\end{gathered}\tag{3}$$

Here P is the pressure, $v_0=uz/l$, and θ_0 is determined by (2). The boundary conditions at $z=0$ and $z=l$ assume the usual form

$$v_{1x}=v_{1z}=0,\ \partial v_{1z}/\partial z=0,\ \theta_1=0.\tag{4}$$

We change over from a system of equations for v_{1x}, v_{1z}, and θ_1 to an equation for $\Psi\equiv\theta_1$. Introducing the operator

$$\hat{L}=\lambda_{11}\frac{d^2}{dz^2}-\lambda_{33}k^2-i\lambda_{11}\Lambda l^{-2}Z-2\theta_0 l^{-2}R,$$

$$Z=z/l+\Omega,$$

where $\Omega=-\omega/uk$ and $\Lambda=Rkl/\lambda_{11}$, we find from (3) that the function $\psi(z)$ should satisfy the relation

$$\begin{gathered}\left\{(k^2-\varepsilon f(z))\left[\frac{d^4}{dz^4}-f(z)\frac{d^2}{dz^2}+k^2g(z)\right]\hat{L}-2i\varepsilon Ml^{-3}\left(\frac{d^2}{dz^2}-k^2\right)\frac{d}{dz}\hat{L}\right.\\
\left.-iMk^2l^{-3}\left[k^2+\varepsilon\left(\frac{d^2}{dz^2}-f(z)\right)\right]\left(\frac{d}{dz}-k^2lZ\right)\right\}\psi(z)=0;\end{gathered}\tag{5}$$

$$M=Rkl/\delta,\quad f(z)=(2+1/\delta)k^2+iMl^{-2}Z,\quad g(z)=f(z)-k^2.$$

Equation (5) is accurate to quantities proportional to ϵ^2. The small "parameter" is in this case the quantity

$$\left|\varepsilon\frac{d^2}{dz^2}\Big/k^2\right|.$$

This ratio, as will be shown below, is small at the point where the stability is lost, provided the modulus of ϵ is small enough. In the calculation of the critical values R_c and k_c, we can therefore neglect the corresponding corrections in (5).

The boundary conditions that must be satisfied by the solution of (5) are obtained from (3) and (4) and take the form

$$\begin{gathered}\psi=0,\quad\left[k^2-\varepsilon\left(\frac{d^2}{dz^2}-f(z)\right)\right]\hat{L}\psi=0,\\
\left\{\left[k^2-\varepsilon\left(\frac{d^2}{dz^2}-f(z)\right)\right]\frac{d}{dz}+i\varepsilon Ml^{-3}\right\}\hat{L}\psi=0.\end{gathered}\tag{6}$$

The dispersion equation that determines the function $\omega(k)$ is obtained by substituting in (6) the function $\psi(z)$, which is a combination of linearly independent solutions $\psi_j(z)$ of Eq. (5).

3. The functions $\psi_j(z)$ depend on the relations between the dimensionless parameters λ_{ii}, δ, ϵ, kl, Ω, R. In real substances we have $\lambda_{ii}<1$, $|\epsilon|\ll\delta<1$. At fixed λ_{ii}, δ, and ϵ it is of interest to consider the region of values of kl and R in which the imaginary part Ω'' of the reduced frequency $\Omega=\Omega'+i\Omega''$ is small. An analysis of (5) and (6) shows that in this region of values of kl and R the most important solutions $\psi_j(z)$ are those that depend exponentially on the quantities with large absolute values

$$(iM)^{1/2}Z^{3/2},\quad(i\Lambda)^{1/2}Z^{3/2}$$

and can be expressed in the form

$$\psi_j(z)\sim\exp(S_j(z)).\tag{7}$$

We consider in this section the region $\epsilon>0$, $\epsilon\gg(\lambda_{11}/R)^2$. The results of Sec. 2 show that in this case the elasticity of the liquid crystal, and accordingly the terms of λ_{ii} in Eq. (5), can be neglected everywhere in the layer with the exception of a small region of thickness z_0 at the wall. In the main layer we have $\theta_0\approx\epsilon^{1/2}$, and θ_1 does not vanish on the boundary of the main layer. It is therefore necessary to exclude from the boundary conditions (6) the condition $\psi=0$, and the remaining two should be satisfied by virtue of the equalities $v_{1z}=0$ and $\partial v_{1z}/\partial z=0$ at $z=0$ and $z=l$.

When finding solutions of the form (7) we can simplify (5) significantly, inasmuch as the following estimates hold in the interval of the values of δ, ϵ, kl, Ω, and R of interest to us:

$$|\Omega|\sim1,\ \varepsilon\ll(kl)^2/M\ll(kl)^2\ll1,$$

$$|f|\sim|g|\sim|\hat{L}|\sim|d^2/dz^2|\sim l^{-2}M.$$

As a result, (5) takes the form

$$\begin{gathered}\left\{lZ\frac{d^4}{dz^4}+4\frac{d^3}{dz^3}-lZ[f(z)-12\varepsilon\theta_0M(kl)^{-3}(lZ)^{-2}]\frac{d^2}{dz^2}\right.\\
\left.-2iMl^{-2}Z\frac{d}{dz}+iMk^2l^{-1}Z^2\right\}\psi(z)=0.\end{gathered}\tag{8}$$

With the aid of (8) we obtain the solutions $\psi_{1,2}(z)$:

$$\psi_{1,2}(z)\approx\left[Z^{-9/4}\pm\frac{16}{9}(iM)^{-1/2}Z^{-15/4}\right]\exp(\pm S(z)),$$

$$S(z)\approx S_0(z)+(iM)^{1/2}\left[\Omega_1Z^{1/2}-i\left(1+\frac{1}{\delta}\right)M^{-1}(kl)^2Z^{1/2}-4i\varepsilon\theta_0(kl)^{-3}Z^{-3/2}\right],\tag{9}$$

$$S_0(z)={}^2/_3(iM)^{1/2}Z^{3/2},$$

where Ω_0 is the solution of the dispersion equation neglecting the small corrections to S(z), which are proportional to $(kl)^2/M$ and $\epsilon\theta_0(kl)^{-3}$, while $\Omega=\Omega_0+\Omega_1$ is the solution with these corrections taken into account. The remaining solutions of (8), accurate to small quantities of order $(kl)^2/M$ and $\epsilon\theta_0(kl)^{-3}$, are equal to

$$\psi_3(z)\approx 1,\quad \psi_4(z)\approx Z^{-1}. \tag{10}$$

Expressions (9) and (10) for the functions $\psi_j(z)$ are valid at values of z such that $|Z|\sim 1$ (near the solid walls), and become meaningless at $|Z|\ll 1$. We note also that in the interval of values of z of interest to us (near the walls) the single-valued function S(z) is defined for complex values of Ω satisfying the condition

$$\Omega''-\left(1+\frac{1}{\delta}\right)M^{-1}(kl)^2+12\varepsilon\theta_0(kl)^{-3}\,\mathrm{Im}\,Z^{-2}>0. \tag{11}$$

The dispersion equation for Ω_0 can be easily obtained with the aid of the boundary conditions (6), which actually are of the following form in the approximation under consideration:

$$\psi=0,\ d\psi/dz=0,\ \text{at}\ \ z=0,\ z=l. \tag{12}$$

Substituting in (12) a linear combination of the functions (9) and (10), we obtain, accurate to quantities of order M^{-1}, an equation for Ω_0:

$$\begin{aligned}\mathrm{tg}[S_0(l)-S_0(0)]&=\mathrm{tg}\left\{\frac{2}{3}\left(\frac{M}{i}\right)^{1/2}[(1+\Omega_0)^{3/2}-\Omega_0^{3/2}]\right\}\\ &=\frac{16}{9}\left(\frac{i}{M}\right)^{1/2}[(1+\Omega_0)^{-3/2}-\Omega_0^{-3/2}].\end{aligned} \tag{13}$$

Equation (13) can be approximately written also in the form

$$(1+\Omega_0)^{3/2}-\Omega_0^{3/2}\approx\frac{\sqrt{i}}{2}\left\{3n\pi M^{-1/2}+\sqrt{i}\frac{16}{3}M^{-1}[(1+\Omega_0)^{-3/2}-\Omega_0^{-3/2}]\right\}, \tag{13a}$$

where n = 1, 2, ... Eqs. (13) or (13a) can be solved with respect to Ω_0 not for all values of the parameter M. For any branch n there exists a critical value M_n^* above which the solution Ω_{0n} does not exist, and below which the solution is

$$\Omega_{0n}=-\frac{1}{2}+i\Omega_{0n}'', \tag{14}$$

with $\Omega_{0n}''\geq 0$. The quantity Ω_{0n}'' vanishes at the critical value of the parameter $M_n=M_n^*$. From (13) and (13a) we obtain approximately

$$M_n^*\approx(3n\pi+64/9n\pi)^2. \tag{15}$$

The condition $\Omega''>0$ leads to damping of the perturbations. Therefore the value of $k_n^*(R)$, which is determined at fixed R from relation (15), is of greatest interest from the point of view of the possible instability. It is necessary, however, to take into account the corrections to S(z) from (9). Substituting S(z) from (9) in place of $S_0(z)$ in (13), we obtain values of $\Omega_n''(k)$ and $\Omega_{1n}''(k)$ that differ from zero at $k=k_n^*(R)$

$$\Omega_n''(k_n^*)=\Omega_{1n}''(k_n^*)\approx\left(1+\frac{1}{\delta}\right)\frac{(k_n^*l)^2}{M_n^*}+16\frac{\varepsilon\theta_0}{(k_n^*l)^3}. \tag{16}$$

It is seen from (16) that the condition (11) is satisfied. According to (16), $\Omega_1''(k_1^*)>0$ for arbitrary R, although it does have a minimum of the order of $\epsilon^{3/5}$ at $R\sim 10\epsilon^{-3/10}$ (here $\theta=\epsilon^{1/2}$ and $\delta\sim 1/3$). Thus, in the considered region of values $\epsilon\gg(\lambda_{11}/R)^2$ the stationary flow of a nematic liquid is stable.

4. We now consider the region $|\epsilon|\ll(\lambda_{11}/R)^2$. In this case the elasticity of the liquid crystal is quite appreciable in the entire layer, and θ_0 is a smooth function of z (see Sec. 2). The regions of the values of kl and R in which the imaginary part Ω'' is small correspond here to a large value of the parameter Λ. When finding the solutions of (5) in the form (7), we use the following estimates:

$$|\Omega|\sim 1,\ |\varepsilon|\ll(kl)^2\Lambda^{-1}\ll(kl)^2\ll 1,\ |\theta_0|\ll kl,$$
$$|f|\sim|g|\sim\lambda_{11}\Lambda l^{-2}\ll l^{-2},\quad\left|\frac{d^2}{dz^2}\right|\sim\Lambda l^{-2}.$$

Actually, the parameter ϵ can now be neglected everywhere in (5) except in the operator $\hat{L}$, where $|\theta_0|\sim\epsilon R/\lambda_{11}$. In this approximation, Eq. (5) takes the form

$$\left\{\frac{d^6}{dz^6}-[i\Lambda l^{-2}Z+(\lambda_{33}/\lambda_{11})k^2-\varepsilon(R/\lambda_{11})^2l^{-4}(z^2-lz)]\frac{d^4}{dz^4}-4i\Lambda l^{-3}\frac{d^3}{dz^3}\right\}\psi(z)=0. \tag{17}$$

The solutions of (17) at values of z not too close to $l/2$ are given by the following expressions:

$$\begin{aligned}&\psi_{1,2}(z)\approx\left[Z^{-1/4}\pm\frac{5}{48}(i\Lambda)^{-1/2}Z^{-7/4}\right]\exp(\pm S(z)),\\ &S(z)\approx S_0(z)+(i\Lambda)^{1/2}\{\Omega_1Z^{1/2}-i(\lambda_{33}/\lambda_{11})\Lambda^{-1}(kl)^2Z^{1/2}\\ &\qquad+i\varepsilon\Lambda(kl)^{-2}[\Omega_0(1+\Omega_0)Z^{1/2}+{}^1/_5Z^{5/2}]\},\\ &S_0(z)=(i\Lambda)^{1/2}Z^{3/2}.\end{aligned} \tag{18}$$

Here Ω_0 is the solution of the dispersion equation neglecting the small corrections to S(z), proportional to $(kl)^2\Lambda^{-1}$ and $\epsilon\Lambda(kl)^{-2}$; $\Omega=\Omega_0+\Omega_1$ is the solution with allowance for these corrections; $\psi_6(z)$ and the pre-exponential factors in $\psi_{1,2}(z)$ are written down accurate to $(kl)^2\Lambda^{-1}$, $\epsilon\Lambda(kl)^{-2}$; Λ^{-1}; $\psi_3(z)$, $\psi_4(z)$, $\psi_5(z)$ are accurate to $(kl)^2\Lambda^{-1}$ and $\epsilon\Lambda(kl)^{-2}$. The single-valued function S(z) is defined for complex Ω satisfying the conditions

$$\begin{aligned}&\Omega''-\frac{\lambda_{33}}{\lambda_{11}}\frac{(kl)^2}{\Lambda}+\frac{\varepsilon\Lambda}{(kl)^2}\Omega_0'>0\quad\text{if}\ \varepsilon<0,\\ &\Omega''-\frac{\lambda_{33}}{\lambda_{11}}\frac{(kl)^2}{\Lambda}+\frac{\varepsilon\Lambda}{(kl)^2}(1+\Omega_0')>0\quad\text{if}\ \varepsilon>0.\end{aligned} \tag{19}$$

The dispersion equation for Ω_0 is obtained with the aid of conditions (6), which in this approximation take the form

$$\psi=0,\quad\frac{d^2\psi}{dz^2}=0,\quad\left[\frac{d^3}{dz^3}-i\Lambda Zl^{-2}\frac{d}{dz}\right]\psi=0 \tag{20}$$

at z = 0 and $z=l$.

Substituting in (2) a linear combination of the functions (18), we obtain, accurate to Λ^{-1}, an equation for Ω_0:

$$\begin{aligned}\mathrm{tg}[S_0(l)-S_0(0)]&=\mathrm{tg}\left\{\frac{2}{3}\left(\frac{\Lambda}{i}\right)^{1/2}[(1+\Omega_0)^{3/2}-\Omega_0^{3/2}]\right\}\\ &={}^5/_{48}(i/\Lambda)^{1/2}[(1+\Omega_0)^{-3/2}-\Omega_0^{-3/2}].\end{aligned} \tag{21}$$

Equation (21) has solutions of the type (14) when the parameter Λ is less than or equal to a certain critical value, even approximately for the branch n by

$$\Lambda_n^*\approx(3n\pi+5/12n\pi)^2. \tag{22}$$

At $k=k_n^*(R)$ as determined from (22) we have $\Omega_{0n}'=-1/2$ and $\Omega_{0n}''=0$. Including quantities of order $(kl)^2\Lambda^{-1}$ and $\epsilon\Lambda(kl)^{-2}$ in S(z), we find with the aid of (21), in which we put S(z) from (18) in place of $S_0(z)$, that

$$\Omega_n''(k_n^*)=\Omega_{1n}''(k_n^*)\approx\frac{\lambda_{33}}{\lambda_{11}}\frac{(k_n^*l)^2}{\Lambda_n^*}+\frac{\varepsilon\Lambda_n^*}{5(k_n^*l)^2}. \tag{23}$$

It is seen from (23) that the conditions (19) are satisfied.

Let us analyze expression (23). At $\epsilon>0$ the imaginary part $\Omega_n''(k_n^*)$ is positive, and consequently in this case ($\epsilon\ll\lambda_{11}^2/R^2$) the stationary flow is likewise stable against infinitesimally small perturbations. At $\epsilon<0$, however, the quantity $\Omega_n''(k_n^*)$ can vanish. The corresponding value of R_n is

$$R_{nc}\approx 3n\pi\lambda_{11}|5\lambda_{33}/\varepsilon\lambda_{11}|^{1/4}, \tag{24}$$

with the critical values $R=R_c$ and $k^{-1}=k_c^{-1}$, above which the interruption of the stationary regime and the growth of the perturbations become possible, are equal to

$$R_c = R_{1c} \approx 3\pi\lambda_{11}|5\lambda_{33}/\lambda_{11}\varepsilon|^{1/4},$$
$$k_c = k_1^*(R_{1c}) \approx 3\pi|\lambda_{11}\varepsilon/5\lambda_{33}|^{1/4}. \tag{25}$$

The applicability of formulas (25) is limited by the relation $|\epsilon| < (\lambda_{11}/R_c)^2$ or $|\epsilon| < 10^{-4}$. Nonetheless, even in this narrow region of values of ϵ the critical numbers R_c can be smaller than unity: $R_c \sim 10^{-2}$ at $|\epsilon| \sim 10^{-4}$ ($\lambda_{ii} \sim 10^{-4}$ in real substances).

If $R \gtrsim R_c$ but $kR \lesssim k_c R_c \approx (3\pi)^2 \lambda_{11}$, the growing nonstationary perturbations are characterized by the frequency

$$\omega(k) = -uk\Omega(k) \approx \frac{uk}{2} + iuk_0\left[\frac{\Delta k}{6k_0} + 4\left|\frac{\lambda_{33}\varepsilon}{5\lambda_{11}}\right|^{1/2}\frac{\Delta R}{R_c}\right], \tag{26}$$

where

$$\Delta R = R - R_c, \quad \Delta k = k - k_0, \quad k_0 = \frac{k_c R_c}{R} \approx k_c\left(1 - \frac{\Delta R}{R_c}\right),$$
$$-24|\lambda_{33}\varepsilon/5\lambda_{11}|^{1/2}\Delta R/R_c \leqslant \Delta k/k_0 \leqslant 0.$$

As seen from (26), the interval of wave numbers k in which $\omega''(k) > 0$ broadens with increasing ΔR. We emphasize that we are dealing here with the branch with $n = 1$. Branches with $n > 1$ at the same value of R attenuate at all values of k. We note also that allowance for the relation $(v_{1x}, v_{1z}, \theta_1) \sim \exp(ipy)$ leads, as noted above, to an additional damping of the small perturbations, namely

$$\omega(k, p) \approx \omega(k) - i\lambda_{22}ulR_c^{-1}p^2,$$

where $\omega(k)$ is given by (26). Thus, growing (but small) perturbations near the instability threshold are characterized by the number $p = 0$.

The instability that develops at $R > R_c$ can be suppressed by a magnetic field H applied along the x axis, for in this case the threshold value of $R_c(H)$ increases. Indeed, in the presence of a field we have

$$\hat{L}(H) = \hat{L} - \left(\frac{\rho\chi_a}{\alpha_2^2}\right)H^2,$$
$$S_H(z) = S(z) - i\left(\frac{i}{\Lambda}\right)^{1/2}\frac{\chi_a(lH)^2}{K_{11}}Z^{1/2},$$

where χ_a is the anisotropy of the magnetic susceptibility. Accordingly, we obtain in place of (23)

$$\Omega_{1H}''(k_1^*) = \Omega_1''(k_1^*) + \chi_a(lH)^2/3\pi^2K_{11}, \tag{27}$$

and the critical number $R_c(H)$ is given by

$$R_c(H) \approx R_c\left\{\frac{1}{18\pi^2}\left|\frac{5\lambda_{11}}{\lambda_{33}\varepsilon}\right|^{1/2}\frac{\chi_a(lH)^2}{K_{11}} + \left[1 + \left(\frac{1}{18\pi^2}\left|\frac{5\lambda_{11}}{\lambda_{33}\varepsilon}\right|^{1/2}\frac{\chi_a(lH)^2}{K_{11}}\right)^2\right]^{1/2}\right\}^{1/2}. \tag{28}$$

It is seen from (28) that at $\lambda_{11} \sim 10^{-4}$, $|\epsilon| \sim 10^{-4}$, ($R_c \sim 10^{-2}$), $l \sim 10^{-2}$ cm, $K_{11} \sim K_{33} \sim 10^{-7}$ dyne and $\chi_a \sim 10^{-7}$ erg-cm^{-3} G^{-2} a magnetic field $H \sim 10^4$ G raises the instability threshold appreciably.

5. We now discuss the region $\epsilon < 0$, $1 \gg |\epsilon| \gg (\lambda_{11}/R)^2$. In this region, unlike the cases considered above, at R larger than a certain number $\tilde{R}_c$ there are no solutions $\theta_0(z)$ such that $|\theta_0|_{max} \lesssim |\epsilon|^{1/2}$. Indeed, the condition for Eq. (2) to have a solution with one maximum ($\theta_0^2 \ll 1$) is $\theta_0(l/2) = \theta_{min}$ or

$$\frac{l}{2} = -l\left(\frac{\lambda_{11}}{2R}\right)^{1/2}\int_0^{\theta_{min}}\left[|\varepsilon|(\theta - \theta_{min}) + \frac{1}{3}(\theta^3 - \theta_{min}^3)\right]^{-1/2}d\theta$$
$$= \left(\frac{3}{8}\right)^{1/4}z_0\left(\frac{1}{1+\xi}\right)^{1/4}F\left\{2\,\mathrm{arctg}\left(\frac{\xi}{3(1+\xi)}\right)^{1/2}, \frac{1}{2} + \frac{1}{4}\left(\frac{3\xi}{1+\xi}\right)^{1/2}\right\}, \tag{29}$$

where $F\{\varphi, r\}$ is an incomplete elliptic integral of the first kind and $\xi = \theta_{min}^2|\epsilon|^{-1}$. The right-hand side of (29) has a maximum at $\xi \approx 1.5$. Accordingly, Eq. (29) has no solutions at $R > \tilde{R}_c \approx 4.8\lambda_{11}|\epsilon|^{-1/2}$. At $R < \tilde{R}_c$ Eq. (29) has, generally speaking, two solutions. The stability of one of these solutions, which is the smallest in amplitude ($\sim R\epsilon/\lambda_{11}$), was investigated in Sec. 4. The second solution has a larger amplitude on the order of $-\lambda_{11}/R$ and does not vanish at $\epsilon = 0$. This solution is apparently not realized in experiment, since it corresponds to a larger energy dissipation.

We have considered so far stationary distributions of the director with $\theta_0^2 \ll 1$. But the solution of Eq. (2), according to Sec. 2, is generally speaking a Weierstrass function which, as is well known, has a pole and two zeroes in the parallelegram of the periods. Actually, in the solutions considered so far (at $R < \tilde{R}_c$) the constant C and the invariant g_3 were chosen such that the zeroes of the Weierstrass function coincided with the points $z = 0$ and $z = l$, and the pole did not lie on the segment $(0, l)$. Actually, however, the function $\theta_0(z)$ has no poles, inasmuch as at $\theta_0^2 > \delta$ the derivative dv_0/dz in (1) can no longer be regarded as constant, and at $\theta_0^2 \sim 1$ the equations themselves no longer hold. The distributions of $\theta_0(z)$ at $R \gg \tilde{R}_c$ have the following qualitative character: $\theta_0(z)$, which is a periodic function with period on the order of z_0, assumes negative values on the order of $-|\epsilon|^{1/2}$ on the segments $\Delta z \sim z_0$ and positive values on the order of unity on the segments $\Delta z \sim |\epsilon|^{1/4}z_0 \ll z_0$. As a result the difference $S(l) - S(0)$ at $kl \ll 1$ is, according to (18), of the order of

$$S(l) - S(0) \sim \frac{1}{z_0}\left[\left(\frac{R}{\lambda_{11}}\right)^{1/2}|\varepsilon|^{1/4}z_0 + \left(-i\frac{Rl\omega}{\lambda_{11}u}\right)^{1/2}z_0\right] \sim in\pi. \tag{30}$$

From (30) we obtain an estimate for ω'':

$$\omega_n'' \sim i\frac{u}{l}\left(|\varepsilon|^{1/2} - (n\pi)^2\frac{\lambda_{11}}{R}\right). \tag{31}$$

It is seen from (31) that at $R \gg \tilde{R}_c$ the stationary motions considered here are unstable with respect to perturbations with $n \ll (R/\tilde{R}_c)^{1/2}$.

6. The results obtained above show that in the region $\epsilon > 0$ the Couette flow of a nematic liquid is stable against infinitesimally small perturbations. The instability of the stationary motion relative to small perturbations of finite amplitude should appear in the mesophase, apparently, the same as in an isotropic liquid.

At $\epsilon < 0$ the stationary motion characterized by the distribution $\theta_0(z)$ with one minimum is stable only at $R < R_c$. The critical number R_c increases without limit as $|\epsilon| \to 0$ like $R_c \sim 10\lambda_{11}|\epsilon|^{-1/4}$. In order of magnitude, this expression for $R_c(\epsilon)$ is valid in the interval $0 < |\epsilon| < 10^{-4}$, which is practically independent of the values of the other parameters (λ_{11}, δ) that enter in the problem. In spite of the narrowness of this interval, the quantity R_c varies here in fact from infinity to values much less than unity. It appears that this is the cause of the experimental fact [5] that a change of the value of R in the vicinity of $\epsilon = 0$ by a factor of almost 100 did not lead to vanishing of the turbulence.

Suppression of turbulence by a magnetic field was observed experimentally [5]. We note that the critical value of the magnetic field H_c at which the instability should vanish in the case when $R > R_c(H = 0) = R_c$, is given in accordance with (28) by the expressions

$$H_c \approx 6\pi\left|\frac{\lambda_{33}\varepsilon}{5\lambda_{11}}\right|^{1/4}\left(\frac{K_{11}}{\chi_a l^2}\right)^{1/2}\left(\frac{R - R_c}{R_c}\right)^{1/2}, \quad \text{if} \quad \frac{R - R_c}{R_c} \ll 1,$$
$$H_c \approx 3\pi\left|\frac{\lambda_{33}\varepsilon}{5\lambda_{11}}\right|^{1/4}\left(\frac{K_{11}}{\chi_a l^2}\right)^{1/2}\left(\frac{R}{R_c}\right)^{1/2}, \quad \text{if} \quad \frac{R}{R_c} \gg 1.$$

The peculiarity of the onset of instability in a nematic liquid lies also in the fact that at $R > R_c$, as shown by the dispersion relations (26), the imaginary part $\omega''(k)$ depends linearly on the proximity of the wave numbers k, which characterize the perturbations to the critical value k_c. The good velocity of the packet of perturbations $\partial\omega/\partial k$ is therefore meaningless in this case ($\partial\omega/\partial k$ is a complex quantity here), and the perturbations cannot be carried along the flow of the liquid crystal.

In the region $\epsilon < -10^{-4}$ and $R \ll \lambda_{11}|\epsilon|^{-1/2} < 10\lambda_{11}|\epsilon|^{-1/4}$ the stationary flow is stable, and $\theta_0(z)$ has one minimum. If $R \gg \lambda_{11}|\epsilon|^{-1/2}$ at the same values of ϵ, then this motion should become turbulent, for it is impossible to realize in this case stationary distributions of $v_0(z)$ and $\theta_0(z)$ with several minima (maxima) of $\theta_0(z)$. The value $R = \tilde{R}_c \approx 5\lambda_{11}|\epsilon|^{-1/2}$ at which the stationary solutions $\theta_0(z)$ with one minimum vanish should be regarded in the region $|\epsilon| \gg 10^{-4}$ as a threshold number corresponding to the instability of the nematic-liquid Couette flow.

It can be assumed that other types of stationary motion of an anisotropic liquid, in a definite range of the material parameters, also have a specifically low turbulence threshold.

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