Surface-charge oscillations in superconductors

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(Submitted June 9, 1973)
Zh. Eksp. Teor. Fiz. **65**, 2016–2022 (November 1973)

It is predicted that low-frequency "plasma" oscillations analogous to Josephson plasma oscillations in tunnel junctions exist in superconducting films and thin filaments. An attempt is made to explain with their aid the recently observed generation of a high-frequency field in superconducting films in the resistive state.

Superconducting thin films and filaments have unique electromagnetic properties connected with the inertia of the superconducting carriers (kinetic inductance). Allowance for the electric fields produced by the uncompensated charges and emerging to the surrounding space leads to the conclusion that natural field oscillations exist in such systems. The frequency of these oscillations, while containing alongside with other parameters also the characteristic "plasma" factor $(4\pi N_S e^2/m)^{1/2}$ (N_S is a concentration of the superconducting electrons), is nonetheless quite small, and tends in particular to zero as $k \to 0$ (k is the wave vector). The corresponding waves considered in the present papers will be called plasma oscillations ("miniplasmons"), in analogy with the Josephson plasma oscillations in tunnel junctions of superconductors.[1] Formally, they are similar to surface plasmons in thin films of normal metals[2,3], but unlike the latter they have a lower frequency and much weaker damping.

The concept of kinetic inductance[4] 1) follows from the definition of the definition of the superconducting current $j = N_S e v_S$, where v_S is the superfluid velocity, and from the relation

$$\partial v_s/\partial t = eE/m. \tag{1}$$

From this we find that the electric field E is proportional to the time derivative of the current, $E = \mathscr{L} \times (dI/dt)$, and the quantity $\mathscr{L}$ ("superconducting" or kinetic inductance) is inversely proportional to N_S and to the conductor cross section area S. At sufficiently small N_S and S, the effects of kinetic inductance predominate over effects of the ordinary (geometrical) inductance and the normal losses.2)

1. We consider a superconducting filament with transverse dimension d that is small in comparison with the correlation length ξ of superconductivity theory and with the penetration depth δ. In this case, a state of one-dimensional superfluid motion is realized, in which all the quantities—current, superluid velocity, etc.—depend only on one spatial coordinate x and generally speaking on the time t. We assume the time variation to be sufficiently small, and in particular, we assume that the characteristic field-variation frequency is small in comparsion with the gap $\Delta \sim [T_c(T_c - T)]^{1/2}$.

The expression for the current in the superconductor, taking into account the nonlinear effects and the normal losses, is

$$I = [N_s e v_s (1 - v_s^2/v_c^2) + \sigma_n E] S, \tag{2}$$

where S is the cross section area of the filament, v_c is the critical velocity of the condensate corresponding to the vanishing of the Euler parameter (the critical velocity determined from the maximal value of the superconducting current is equal to $v_m = v_c/\sqrt{3}$).

If I depends on x then, in accordance with the continuity equation

$$\partial I/\partial x + \partial Q/\partial t = 0 \tag{3}$$

charge will accumulate in definite sections of the filament. Here $Q = Q(x, t)$ is the linear density of the uncompensated charge at the point x and at the instant t. At frequencies that are small in comparison with the plasma frequency ω_0 of the metal, this charge will emerge to a surface (in a layer on the order of the Debye length) and produce in the surrounding space a field **E**. The value of the field can be obtained in the quasistatic approximation from the condition

$$\operatorname{div} \mathbf{E} = 4\pi Q(x, t)\,\delta(\rho), \tag{4}$$

where $\delta(\rho)$ is a two-dimensional δ function in the (y, z) plane. Solving Eq. (4) and determining the field $E = E_x(x, t)$ on the surface of the conductor, we easily obtain

$$E = -\frac{\partial \varphi}{\partial x}, \quad \varphi(x, t) = \int_{-\infty}^{\infty} \alpha(x - x') Q(x', t)\, dx', \tag{5}$$

where $\alpha(x)$ is given by

$$\alpha(x) = (1 - e^{-|x|/d})/|x|, \tag{6}$$

or in the k representation

$$\alpha(k) = \frac{1}{\pi} \int \frac{d^2\varkappa}{k^2 + \varkappa^2} = \ln \frac{k^2 + \varkappa_m^2}{k^2}, \quad \varkappa_m = d^{-1}. \tag{7}$$

The function $\alpha(x_i - x_j)$ is actually the coefficient of the reciprocal capacitance between the points x_i and x_j. Since the integral (7) diverges as $\kappa \to \infty$, we have introduced a cutoff radius (κ_m) equal in order of magnitude to the reciprocal filament diameter d. Inverting the relation (5), we obtain

$$Q(x, t) = \int_{-\infty}^{\infty} \lambda(x - x') E(x', t)\, dx', \tag{8}$$

where $\lambda(k) = -(ik\alpha(k))^{-1}$ in the Fourier representation.

Substituting (8) in (3) and taking (2) into account, we obtain a closed relation for the quantity $v_S(x, t)$:

$$\frac{\partial}{\partial x}\left[\frac{N_s e^2 S}{m} v_s \left(1 - \frac{v_s^2}{v_c^2}\right) + \sigma_n S \frac{\partial v_s}{\partial t}\right] + \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} \lambda(x - x') v_s(x', t)\, dx' = 0. \tag{9}$$

The obtained equation determines the possible states of the superfluid motion in the one-dimensional case. In particular, the homogeneous state

$$v_s = \text{const} = v_0, \quad I = I_0 = N_s e v_0 (1 - v_0^2/v_c^2) S \tag{10}$$

is one of its particular solutions. Assuming then

$$v_s = v_0 + w e^{ikx - i\omega t}, \tag{11}$$

we obtain the spectrum of the small oscillations (w) superimposed on the homogeneous state. Substitution of (11) in (9) leads to a dispersion equation for the small oscillations

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$$\omega^2+k^2\alpha(k)\,(i\omega\sigma_n-N_se^2\eta(I_0)/m)S=0 \qquad (12)$$

We have introduced here the function

$$\eta(I_0)=1-3v_0^2/v_c^2=1-v_0^2/v_m^2, \qquad (13)$$

which vanishes at the critical current ($I_0 = I_c$). We see from (12) that at $v_0 > v_m$ the frequency is pure imaginary ($\mathrm{Im}\,\omega > 0$). This means instability of the homogeneous state on the decreasing branch of the pair-braking curve, $dI_0/dv_0 < 0$.[6] At $v_0 > v_m$, the solution has a vibrational character with a small imaginary part ($\mathrm{Im}\,\omega < 0$). Putting $\omega = \bar\omega - i\gamma$, we obtain at small k:

$$\bar\omega=\left[\frac{N_se^2S}{m}\eta(I_0)\right]^{1/2}\alpha^{1/2}(k)\,k=v(k)\,k, \qquad (14)$$

$$\gamma={}^1/_2k^2\alpha(k)\,\sigma_nS. \qquad (15)$$

According to (6), at small k the quantity $\alpha(k)$ takes the form $\alpha \approx \ln[(kd)^{-2}] \sim 2\ln(L/d)$, where L is the total length of the filament. Since the logarithm is a slow function, we can approximately assume α to be constant. In this approximation, formula (14) describes waves with a linear spectrum, the propagation rate of which along the filament is small in comparison with the velocity c of the electromagnetic waves in vacuum: $v/c \approx S^{1/2}/\delta \ll 1$, where $\delta = (mc^2/4\pi N_Se^2)^{1/2}$. The damping is proportional to the square of the wave vector and is small in comparison with the frequency as $k \to 0$. In "dirty" systems ($\sigma_n \to 0$), and also at small S, the damping effect becomes relatively negligible, since γ decreases more rapidly than $\bar\omega$ when these parameters decrease.3)

2. The foregoing analysis can be automatically extended to include the case of thin films, but it becomes necessary then to take into account effects connected with electrodynamic delay.

The equation analogous to (9) takes in the two-dimensional case the form

$$\mathrm{div}\left[\frac{N_se^2d}{m}\mathbf{v}_s\left(1-\frac{v_s^2}{v_c^2}\right)+\sigma_nd\frac{\partial\mathbf{v}_s}{\partial t}\right]+\frac{\partial^2}{\partial t^2}\int d^2\rho'\lambda(\rho-\rho')\,\mathbf{v}_s(\rho',t)=0, \qquad (16)$$

where d is the film thickness and $\lambda(k) = -ik/2\pi k$. The frequency and damping of the waves are given by

$$\bar\omega=\left(\frac{2\pi N_se^2d}{m}\right)^{1/2}\eta^{1/2}(I_0)\sqrt{k}, \qquad (17)$$

$$\gamma=\pi\sigma_ndk. \qquad (18)$$

At small k, the phase velocity $v(k) = \bar\omega/k$ increases like $1/\sqrt{k}$ and at sufficiently small k it exceeds the speed of light c. It is clear that in the latter case the foregoing analysis is insufficient.4)

For a more correct description, we write down the equations for the potentials in the Lorentz gauge ($\mathrm{div}\,\mathbf{A} + c^{-1}\,\partial\varphi/dt = 0$) in the form

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi=-4\pi q(\rho,t)\,\delta(z),$$
$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}=-\frac{4\pi}{c}\mathbf{j}(\rho,t)\,\delta(z), \qquad (19)$$

where q and j are the charge density and current density integrated over the thickness of the film. The quantities q and j satisfy the continuity equation

$$\partial q/\partial t+\mathrm{div}\,\mathbf{j}=0$$

and

$$\mathbf{j}=N_se\mathbf{v}_s\left(1-\frac{v_s^2}{v_c^2}\right)d+\sigma_nd\mathbf{E}. \qquad (20)$$

Solving (19) in the momentum representation and representing $\mathbf{v}_S$ in the form $\mathbf{v}_0 + \mathbf{w}\exp[i\mathbf{k}\cdot\boldsymbol{\rho} - i\omega t]$, we obtain

$$(c^2k^2-\omega^2)^{1/2}\mathbf{w}=\frac{2\pi N_se^2d}{mc}\left(\frac{c^2}{\omega^2}\mathbf{k}(\mathbf{k}\mathbf{u})-\mathbf{u}\right),$$
$$\mathbf{u}=\mathbf{w}(1-v_0^2/v_c^2)-2\mathbf{v}_0(\mathbf{v}_0\mathbf{w})/v_c^2-im\omega\sigma_n\mathbf{w}/N_se^2. \qquad (21)$$

For longitudinal oscillations (the vectors k, w, and u parallel to one another) we obtain

$$\bar\omega=[2\Omega(\sqrt{c^2k^2+\Omega^2}-\Omega)]^{1/2}, \qquad (22)$$

$$\gamma=\bar\omega^4\tau/(\bar\omega^2+2\Omega^2). \qquad (23)$$

We have introduced here the notation (see (13) for the definition of η)

$$\Omega=\frac{\pi N_se^2d}{mc}\eta(I_0),\qquad \tau=\frac{m\sigma_n}{2N_se^2}\eta^{-1}(I_0). \qquad (24)$$

At $k \gg \Omega/c$, formulas (22) and (23) coincide with (17) and (18). In the limit of small k ($k \ll \Omega/c$), to the contrary, we have

$$\bar\omega\approx ck,\quad \gamma\approx\tau(ck)^4/2\Omega^2. \qquad (25)$$

We see therefore that the propagation velocity of long waves actually coincides with the speed of light c.

If we have a superconducting strip of finite width (d_1), then the transition from the behavior typical of a film to the behavior corresponding to the case of a "filament" is realized in the case of values $d_1 \lesssim \delta^2/d$, inasmuch as the quantity $\Lambda = \delta^2/d$ has the meaning of the skin depth of penetration for superconducting films.[7-9] At small d, even relatively broad strips ($d_1 \gg \delta$) may behave not like "films" but like "filaments."

3. Let us demonstrate the analogy between the considered phenomena and "plasma" waves in Josephson tunnel junctions. The role of v_S in the latter case is played by the quantity φ, which is the coherent phase difference between two superconductors separated by a thin insulating layer. We have the equations (see[10])

$$I=I_c\sin\varphi+\frac{\hbar}{2eR}\dot\varphi,\quad \dot\varphi=\frac{2ed}{\hbar}E, \qquad (26)$$

$$Q=\varepsilon SE/4\pi,\quad I+\dot Q=0. \qquad (27)$$

In this case S is the area of the junction, d is the thickness of the insulating gap, ϵ is its dielectric constant, I_c is the critical Josephson current, and R is the resistance of the junction to single-particle current.

Confining ourselves to the case of a one-dimensional system (Sec. 1), we see that the first pair of equations in (26) is analogous to formulas (1) and (2), whereas relations (27) correspond to the formulas (8) and (3) given above. Combining (26) and (27), we obtain in the usual manner the equation for the phase shift in the junction[10]

$$I_c\sin\varphi+\frac{\hbar}{2eR}\dot\varphi+\frac{\hbar C}{2e}\ddot\varphi=0, \qquad (28)$$

where $C = \epsilon S/4\pi d$ is the capacitance of the junction. If the ohmic losses are small ($\omega RC \gg 1$), then Eq. (28) describes "Josephson plasma resonance," which are current oscillations with frequency $\omega_J = (2eI_c/\hbar C)^{1/2}$. Equation (28) is analogous to relation (9) given above for the superfluid velocity.

Unlike the case considered in this paper, the oscillation frequency for the tunnel junction is finite as $k \to 0$.[1] Nonetheless, this is valid only for the "Meissner" state of the junction.[10] When vortices enter into the junction, thresholdless modes of the small-oscillation spectrum also appear.[11,12]

4. We present in conclusion estimates of the frequency of the considered "plasma" oscillations and the feasi-

bility of observing them in experiment. The most interesting in this respect is the case of a "filament" (n = 1), since the predicted waves have in this case a practically linear dispersion and a small propagation velocity (v ≪ c). Considering a filament of length L, we obtain on the basis of (14) the following estimate of the frequency

$$\omega\sim\omega_0\left(\frac{N_s}{N}\right)^{1/2}\frac{S^{1/2}}{L}\ln\frac{L}{\sqrt{S}}\eta^{1/2}(I_0), \tag{29}$$

where $\omega_0 = (4\pi Ne^2/m)^{1/2}$ is the plasma frequency for the metal and $\omega_0 \sim 10^{16}$ sec^{-1}. Since the density N_s of the superconducting electrons decreases with decreasing mean free path (l) (see[13]), ω is small for "alloys" with sufficiently small l. Simultaneously, as already noted, the wave damping also decreases in such systems, since γ is proportional in accordance with (15) to the normal conductivity σ_n. The case of extremely "dirty" sytems ($l \ll \xi_0$) is therefore of greatest interest. Thus, at parameter values

$$\frac{l}{\xi_0}\sim10^{-3}\quad \frac{T_c-T}{T_c}\sim10^{-2},\quad \eta\sim10^{-1},\quad L=10\text{ cm},\quad S=10^{-10}\text{ cm}^2$$

we obtain the estimate $\omega \sim 10^8$ sec^{-1}.

Churilov, Dmitriev, and Beskorsyĭ[5] have reported observation of radiation in superconducting films in resistor state, i.e., when current flows through them. Although no explanation was offered of this phenomenon, it was attributed in natural fashion to the decreasing current-voltage characteristic of thin films, observed in,[14,15] and to the relaxation oscillations of the current in the superconducting channel, which are connected with the pair-breaking mechanism.[10,15,16] The soft mode of the "plasma" oscillations considered in the present paper suggests that plasma-wave excitation can serve as one of the possible explanations of the observed phenomenon. Let us estimate the degree to which such an assumption can correspond to reality.

The films (strips) investigated in[5] satisfy the criterion $d\cdot d_1 < \delta^2$ (see Sec. 2) and should be regarded in this sense as thin filaments (one-dimensional samples). Substituting the parameters of[5] in (29), we obtain

$$\omega\sim10^{11}\,\text{sec}^{-1}\left(\frac{T_c-T}{T_c}\right)^{1/2}\eta^{1/2}(I_0). \tag{30}$$

The experimentally obtained frequencies were 10^8–10^9 sec^{-1}.[5] Although the additional factors in (30) are small in comparison with unity (η tends to zero as $I \to I_c$), it is nevertheless difficult to imagine them to be able to cause a frequency decrease by two orders of magnitude, as is required for reconciliation with the experimental data.[5] Thus, the effect observed in[5] cannot be ascribed with assurance to any of the aforementioned mechanisms.

Experimental observation of the low-frequency plasma waves in superconductors, predicted in the present paper, can be of definite physical and possibly also practical interest. It appears that this phenomenon is of great significance in the analysis of the problem of "high-temperature" superconductivity in one-dimensional and two-dimensional systems. In accordance with the results of[17], on going to the three-dimensional case (a system of parallel films or a two-dimensional conglomerate of "filaments") the thresholdless mode of the plasma oscillations vanishes. At the same time, the possibility is restored of long-range order—phase correlation at large distances ($|x-x'|\to\infty$) and for long time intervals ($|t-t'|\to\infty$).

In conclusion, I take the opportunity to thank A. F. Andreev, V. M. Dmitriev, and I. K. Yanson for discussing the work and for a number of useful remarks.

[1)]Little [4] cites an unpublished remark by Onsager on the possibility of the existence of "baby plasmons" in one-dimensional superconductors. It appears that the density-oscillation modes considered by us are indeed such plasmons. We present below a detailed theory of the corresponding phenomenon and its connection with recent experiments [5].

[2)]At first glance it seems paradoxical that $\mathscr{L}$ becomes infinite as $N_s \to 0$. It must be recognized, however, that at small N_s (sufficiently close to T_c) nonlinear effects manifest themselves quite early and change the picture in question.

[3)]In addition to the "dissipative" damping described by formula (15) it is necessary to take into account also radiative damping due to radiation of waves into the surrounding space. By regarding the radiation of a superconducting-filament segment of length L as dipole radiation, we easily obtain $\gamma_{rad} \sim N_s e^2 S\omega^2/mc^3k$, where $k \sim L^{-1}$. Generally speaking, the radiative damping decrement (γ_{rad}) may not be small in comparison with the damping coefficient $\gamma = \gamma_{dis}$ calculated above. Nonetheless, the ratio γ_{rad}/γ, being proportional to $(v/c)^3$, is always small in comparison with unity.

[4)]In the case of a filament the velocity v(k) has increased with decreasing k only logarithmically, and since the coefficient preceding the logarithm is much smaller than c, it has in fact always been smaller than the speed of light.

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Translated by J. G. Adashko
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