The transfer equations for normal waves and radiation polarization in an anisotropic medium

Yu. N. Gnedin and G. G. Pavlov

A. F. Ioffe Physico-technical Institute, USSR Academy of Sciences
(Submitted May 10, 1973)
Zh. Eksp. Teor. Fiz. **65**, 1806–1817 (November 1973)

The conditions of applicability of the transfer equations for normal waves are found: the orthogonality of the waves, or a large phase shift between them over a mean free path. Expressions are obtained for the degree of orthogonality of normal waves in an axially symmetric medium (e.g., a magnetoactive plasma or a gas consisting of atoms experiencing the normal Zeeman effect). Simple formulas for the degree of circular polarization of the radiation emerging from the atmosphere of a star with a strong dipole-type magnetic field are derived on the basis of the equations for the normal-wave intensities. The question of radiation polarization in layers located deep inside an anisotropic medium is considered, and the cases when such polarization occurs are indicated.

1. INTRODUCTION

The problem of radiation transfer in an anisotropic medium is of great practical importance both for the laboratory plasma and in astrophysics.

Magnetic fields $H \sim 10^7$ G have now been attained in laboratories. Therefore, the possibility is presently discussed of measuring the polarization of the short-wave radiation (in the optical and x-ray bands) of a plasma in a magnetic field[1]. Polarization measurements yield information about the magnetic-field and temperature distributions inside the plasma.

The problem of the polarization of the radiation of an anisotropic medium has become especially pressing in astrophysics in connection with the recently discovered magnetic white dwarfs (for references, see [2]) with magnetic fields of 10^6–10^8 G and pulsars, which, apparently, have fields of up to 10^{12}–10^{13} G. The radiation of an anisotropic medium is, as a rule, polarized, and the measurement of this polarization allows us to obtain important data on the characteristics of the emitting object. Two methods are available for computing the polarization of radiation. The first consists in solving a system of transport equations for four Stokes parameters[3], or in solving a matrix equation for the radiation-density polarization matrix[4]. This method is the one most frequently used in astrophysics, especially in the optical region. The second method is based on the concept of the two normal (natural) waves—the ordinary and extraordinary—which propagate in an anisotropic medium independently of each other if the radiation and scattering processes are unimportant. It consists in solving a system of two transport equations for the intensities of the normal waves, and it is usually used in the problem of radio-wave propagation in a plasma[5].

The first method is more general, but considerably more laborious. The second method is considerably simpler, since it allows us to directly use the well-developed apparatus of the theory of unpolarized-radiation transfer, but it is far from being always applicable.

In the present paper we show the connection between these two methods and establish, using the equations for the intensities of the normal waves, the criteria for the applicability of the radiation-transfer description. It is found that such a description is applicable in an optically thick medium if at distances corresponding to a unit optical thickness the phase shift of one normal wave relative to the other is large, i.e., $\omega|n_1 - n_2|/c \gg (\mu_1 + \mu_2)/2$, where ω is the angular frequency of the radiation, n_1 and n_2 are the refractive indices of the normal waves, and μ_1 and μ_2 are their absorption coefficients. If even this condition is not satisfied, for the important particular case of local thermodynamic equilibrium (LTE), the simplified description is applicable when the normal waves are orthogonal to each other, i.e., when the principal axes of their polarization ellipses are perpendicular to each other and the ellipses themselves are similar. In a magnetoactive plasma the normal waves are orthogonal for any angle of propagation of the waves and for any magnitude of the magnetic field if the radiation frequency ω is much higher than the electron-ion collision rate ν_e. In a gas consisting of atoms experiencing the Zeeman effect, the normal waves can be far from being orthogonal near the center of the spectral line.

Equations for the intensities of the normal waves, obtained from the general transport equations, yield formulas for the degree of circular polarization of the radiation emerging from the atmosphere of a star with a strong dipole-type magnetic field. In particular, for the Rayleigh-Jeans region of the spectrum and for a hot atmosphere, when the bremsstrahlung absorption is dominant, the degree of circular polarization is described by the approximate formula $P_V = 0.024[\hbar^2\omega\omega_H/(kT)^2]\cos\alpha$, where $\omega_H = eH/m_ec$ is the electron cyclotron frequency corresponding to the field at the magnetic pole of the star, while α is the angle between the normal to the surface and the direction of the axis of the magnetic dipole.

Analysis of the equations obtained allows us to determine the polarization of the radiation in layers located deep inside the medium. It is indicated that radiation can be polarized in layers located deep inside a dense plasma with a magnetic field when the plasma frequency is much greater than the Doppler broadening. In such a plasma the generation of the ordinary wave is inhibited. In the case of a plasma with a very strong magnetic field ($\omega_H \gg \omega$), when $\tau(\omega/\omega_H)^2 \ll 1$ (τ is the total optical width of the plasma), only the ordinary wave will be effectively generated and propagated, even if $\tau \gg 1$.

2. THE NORMAL WAVES AND THE DIFFERENT FORMS OF THE TRANSFER EQUATION

If the medium is such that the phenomena of electromagnetic-energy emission and scattering are important, then to determine the characteristics of the radiation of the medium we must solve the radiation-transfer equation. For an arbitrary anisotropic medium it is convenient to use the transfer equation for the polarization density matrix, whose elements are linear combinations of the Stokes parameters. If the refractive index of the medium differs little from unity, then the transfer equation is of the form[4]

$$(\mathbf{n}\nabla)\rho_{\alpha\beta}=-{}^1/_2(T_{\alpha\gamma}\rho_{\gamma\beta}+\rho_{\alpha\gamma}T_{\gamma\beta}{}^+)+S_{\alpha\beta}. \tag{1}$$

Here $\rho_{\alpha\beta}(\mathbf{n}\omega)$ is the density matrix of radiation with frequency ω and direction of propagation $\mathbf{n}$. If α, $\beta = \pm 1$ are the indices of the right and left circular polarizations, then

$$\rho_{\alpha\alpha}=(I+\alpha V)/2, \quad \rho_{\alpha-\alpha}=-(Q+i\alpha U)/2, \tag{2}$$

where I, Q, U, and V are the Stokes parameters. The quantity $T_{\alpha\beta}(\mathbf{n}\omega)$ is the transfer matrix, the Hermitian part of which describes the absorption of the radiaton, while the anti-Hermitian part describes the transition from one polarization to the other. The quantity $S_{\alpha\beta}(\mathbf{n}\omega)$ is the source matrix, which, generally speaking, includes both an integral term depending on ρ and describing the scattering and an additional radiation source (e.g., a thermal radiation source). In Eq. (1), as below, summation over repeated polarization indices is implied.

Let

$$T_{\alpha\beta}u_{\beta j}=T_j u_{\alpha j}. \tag{3}$$

In expression (3), the T_j's (j = 1, 2) are the eigenvalues of the transfer matrix, while $u_{\alpha j} \equiv f_\alpha^{(j)}$ are the cyclic components of the eigenvectors of the matrix T. The vectors $\mathbf{f}^{(j)} = \mathbf{e}_\alpha u_{\alpha j}$ are the basis vectors of the polarization of two waves, which, in an anisotropic medium, are called normal waves (the j = 1 wave is called the extraordinary wave, the j = 2 the ordinary wave). The real and imaginary parts of T_j are the coefficients of absorption μ_j and refraction $\omega n_j/c$ for the j-th normal wave. The quantities

$$\alpha_j=f_1^{(j)}/f_{-1}^{(j)}=(T_j-T_{-1-1})/T_{-11} \tag{4}$$

determine the j-th wave's polarization, which in the general case is elliptical. The parameters of the polarization ellipse (see the figure) can be expressed in terms of the modulus r_j and phase δ_j of the complex coefficients α_j as follows:

$$a_j=(r_j+1)/[2(r_j^2+1)]^{1/2}, \quad b_j=|r_j-1|/[2(r_j^2+1)]^{1/2},$$
$$\chi_j=(\delta_j\pm\pi)/2. \tag{5}$$

Let us introduce the density matrix in the normal-wave representation

$$R_{jk}=u_{j\alpha}{}^{-1}\rho_{\alpha\beta}(u^+)_{\beta k}{}^{-1}. \tag{6}$$

From Eq. (1) we obtain for R_{jk} the equation

$$(\mathbf{n}\nabla)R_{jk}=-g_{jk}R_{jk}+S_{jk}, \tag{7}$$

where

$$g_{jk}=(T_j+T_k{}^*)/2=(\mu_j+\mu_k)/2+i\omega(n_j-n_k)/c. \tag{8}$$

The quantity $S_{jk} = u_{j\alpha}^{-1}S_{\alpha\beta}(u^+)_{\beta k}^{-1}$ is the source matrix in the normal-wave representation; we do not sum over j in (7). Equation (7) is simpler in form than (1), but, as before, it is a system of four coupled equations for the quantities R_{jk}, since the source matrix S, generally speaking, depends on R.

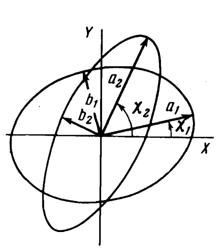

FIG. 1. The polarization ellipses of the normal waves.

The diagonal elements of the matrix R have the meaning of normal-wave intensities $R_{jj} \equiv I_j$. There are practically important cases when, instead of (7), one can write a system of equations for the I_j's. Let us rewrite (7) in the form

$$R_{jk}(z)=R_{jk}(z_0)\exp\left[-\int_{z_0}^{z}g_{jk}(z')dz'\right]+\int_{z_0}^{z}S_{jk}(z_1)\exp\left[-\int_{z_1}^{z}g_{jk}(z')dz'\right]dz_1, \tag{9}$$

where z is the coordinate in the direction n and z_0 is some fixed value of z. The equality (9) is an integral equation for R_{jk}. The second term on the right-hand side of (9) contains integration of the source matrix with the function $\exp\left(-\int_{z_1}^{z} g_{jk}dz'\right)$. Only those z_1 which satisfy the condition $\mathrm{Re}\int_{z_1}^{z} g_{jk}dz' \leq 1$ contribute to the integral. If

$$\mathrm{Im}\int_{z_1}^{z} g_{jk}\,dz'\gg \mathrm{Re}\int_{z_1}^{z} g_{jk}\,dz' \tag{10a}$$

or (for a homogeneous medium)

$$\omega(n_j-n_k)/c\gg(\mu_j+\mu_k)/2, \tag{10b}$$

then the integral over z_1 for $j \neq k$ is much smaller than that for j = k, because of the presence of the rapidly oscillating factor $\exp\left[i\mathrm{Im}\int_{z_1}^{z} g_{jk}dz'\right]$. The conditions (10) imply that the phase shift between the two normal waves, which is due to the difference in the phase velocities, is large at distances corresponding to one optical thickness.

For the particular case of circularly polarized normal waves, the condition (10) is called the condition for large Faraday depolarization. Under this condition

$$(\mathbf{n}\nabla)R_{jk}=-g_{jk}R_{jk}+S_{jj}\delta_{jk}. \tag{11}$$

It can be seen from Eq. (11) that the off-diagonal elements are damped out over a distance of $\sim[(\mu_j+\mu_k)/2]^{-1}$ from the radiation sources, something that does not happen to the diagonal elements because of the presence of the term $S_{jj}\delta_{jk}$. Consequently, if we are interested in the propagation of radiation at sufficiently large distances from the radiation sources (e.g., from the lower limit of a flat atmosphere) and the condition (10) is fulfilled, then instead of (7) we have

$$(\mathbf{n}\nabla)I_j=-\mu_jI_j+S_{jj}, \tag{12}$$

where S_{jj} depends only on the diagonal elements of the matrix (i.e., only on the intensities of the normal waves).

3. CONDITIONS FOR ORTHOGONALITY OF THE NORMAL WAVES

The connection between the Stokes parameters and the elements of the matrix R, as well as the explicit form of the transfer equations in the normal-wave representation, essentially depends on whether the matrix u determined by Eq. (3) is unitary and the polarization basis vectors $\mathbf{f}^{(j)}$ of the normal waves are mutually orthogonal. For example, if u is unitary ($u^+u = 1$), then the radiation intensity $I = I_1 + I_2$/ in the opposite case $I = u_{\alpha j}R_{jk}u_{k\alpha}^+$ is a linear combination of all the elements of the matrix R. The effect of the unitarity of u on the explicit form of the transfer equations is especially clearly manifested in the LTE approximation. In this case[6] $S_{\alpha\beta} = B(T_{\alpha\beta} + T^+_{\alpha\beta})/2$, where B is the Planck function. If the matrix u is unitary, then the eigenvectors of the matrix $T + T^+$ coincide with those of T, and $S_{jk} = B\mu_j\delta_{jk}/2$. Therefore, if the normal waves are orthogonal, then even when the condition (10) is not fulfilled, a system of two equations can be derived in the LTE approximation for the intensities of the normal waves; what is more the equations are not coupled, which implies independence of propagation of the normal waves in this case. If, however, $u^+ \neq u^{-1}$, then even when the condition (10) is fulfilled the u-dependent term in (12) has a more complicated form: $S_{jj} = B\mu_j[u^{-1}(u^+)^{-1}]_{jj}/2$.

Thus, it is of interest to investigate the question of the orthogonality of the polarization vectors of the normal waves. It follows from (3) that

$$(u^+u)_{11}=(u^+u)_{-1-1}=1, \quad (u^+u)_{1-1}=x, \quad (u^+u)_{-11}=x^*, \tag{13}$$

where $x = (\alpha_1^*\alpha_2 + 1)/(1 + |\alpha_1|^2)^{1/2}(1 + |\alpha_2|^2)^{1/2}$. It is clear that as a natural measure of the nonorthogonality of the polarization vectors we can choose the quantity $|x|$, which satisfies the inequality $0 \le |x| \le 1$. For $|x| = 0$ ($\alpha_2^{-1} = \alpha_1^*$), the polarization vectors are orthogonal, i.e., the polarization ellipses are similar to each other ($a_1b_2 = a_2b_1$) and the principal axes of the ellipses are perpendicular to each other ($|\chi_2 - \chi_1| = \pi/2$). For $|x| = 1$ ($\alpha_1 = \alpha_2$) we have maximum nonorthogonality: the polarization ellipses coincide with each other.

Let us express $|x|$ in terms of the parameters of the medium. For this purpose let us write the elements of the transfer matrix in the form

$$T_{\alpha\alpha}=t_I+\alpha t_V, \quad T_{\alpha-\alpha}=-(t_Q+i\alpha t_U). \tag{14}$$

The quantities t_I, t_Q, t_U, and t_V have the meaning of complex transfer coefficients for the corresponding Stokes parameters. Let us also introduce $t_l = (t_Q^2 + t_U^2)^{1/2}$. Then

$$T_{1,2}=t_I\pm(t_V^2+t_l^2)^{1/2}, \quad \alpha_{1,2}=-[t_V\pm(t_V^2+t_l^2)^{1/2}]/t_l. \tag{15}$$

The quantities α_j, and, consequently, x, are completely determined by the dimensionless parameter

$$\gamma=t_l/t_V.$$

Let a and b be the real and imaginary parts of this parameter. Then

$$|x|^2=(1-\sqrt{1-y^2})/(1+\sqrt{1-y^2}), \quad y=2b/(a^2+b^2+1). \tag{16}$$

Thus, we have total orthogonality if $b = 0$ or $a^2 + b^2 = \infty$, and total nonorthogonality if $a = 0$ and $b = \pm1$ at the same time.

The formulas (15) and (16) are general formulas and are applicable to an arbitrary anisotropic medium. In the important particular case of an anisotropic medium with one distinct direction (a tenuous magnetoactive plasma; an elemental gas experiencing the normal Zeeman effect), the elements of the matrix T can be written in the form

$$T_{\alpha\beta}(\mathbf{n}\omega)=e^{i(\beta-\alpha)\varphi}\sum_{m=-1}^{1} d_{m\alpha}(\vartheta)t_m(\omega)d_{m\beta}(\vartheta), \tag{17}$$

where ϑ and φ are the polar and azimuthal angles of the vector n in the coordinate system whose polar axis coincides with the distinct direction, and $d_{m\alpha}(\vartheta)$ are the elements of the finite-rotation matrix[7]. They are given by the formulas

$$d_{\alpha\beta}=(1+\alpha\beta\cos\vartheta)/2, \quad d_{0\alpha}=-d_{\alpha 0}=\alpha\sin\vartheta/\sqrt{2}, \quad d_{00}=\cos\vartheta. \tag{18}$$

The quantities $t_m(\omega)$ are proportional to the diagonal components of the complex polarizability tensor in cyclic coordinates.

From (17) and (18) we obtain

$$t_I=(t_1+t_{-1})/2+\sin^2\vartheta(2t_0-t_1-t_{-1})/4, \quad t_V=\cos\vartheta(t_1-t_{-1})/2,$$
$$t_Q=\sin^2\vartheta\cos 2\varphi(2t_0-t_1-t_{-1})/4, \quad t_U=-t_Q \operatorname{tg} 2\varphi. \tag{19}$$

The formulas (19) were used in the solution of the problem of radiation transfer in a spectral line in the presence of a magnetic field[8]. They allow us to separate out the angular dependence in its explicit form.

Using (13), we obtain

$$\gamma=\frac{2t_0-t_1-t_{-1}}{2(t_1-t_{-1})}\sin\vartheta \operatorname{tg}\vartheta. \tag{20}$$

For a cold tenuous ($|n_j - 1| \ll 1$) magnetoactive plasma

$$t_m=\omega_p^2/c[\nu_e-i(\omega-m\omega_H)], \tag{21}$$

where ω_p and $\omega = eH/m_ec$ are the plasma and cyclotron frequencies and ν_e is the effective collision rate. In this case

$$\gamma=\frac{1}{2}\frac{\omega_H}{\omega+i\nu_e}\sin\vartheta \operatorname{tg}\vartheta, \quad y=\frac{\omega_H\nu_e\sin\vartheta \operatorname{tg}\vartheta}{\omega^2+\nu_e^2+{}^1/_4\omega_H^2\sin^2\vartheta \operatorname{tg}^2\vartheta}. \tag{22}$$

We see from (22) that total orthogonality occurs in a cold magnetoactive plasma only for longitudinal and transverse propagations of the waves ($\vartheta = 0$ and $\vartheta = \pi/2$), or under the condition that $\nu_e = 0$, i.e., when collisions are unimportant. If $\omega \gg \nu_e$ (or more precisely when $\omega^2 + 1/4\omega_H^2\sin^2\vartheta\tan^2\vartheta \gg \nu_e^2$), then the nonorthogonality parameter

$$|x|=\frac{2\nu_e\omega_H|\sin\vartheta \operatorname{tg}\vartheta|}{4\omega^2+\omega_H^2\sin^2\vartheta \operatorname{tg}^2\vartheta}\ll 1, \tag{23}$$

so that the normal waves are almost orthogonal. If the two conditions

$$\omega_H|\sin\vartheta \operatorname{tg}\vartheta|=2\nu_e, \quad \omega=0, \tag{24}$$

are fulfilled at the same time, then we have total nonorthogonality. In the case of total nonorthogonality, $\gamma = \pm i$, $\alpha_1 = \alpha_2 = \mp i$, and $T_1 = T_2 = t_I$, i.e., the eigenvalues and eigenvectors of the matrix T coincide. In this case the parameters of the ellipses of polarization also coincide, and are equal to: $a_{1,2} = 1$, $b_{1,2} = 0$, and $\chi_{1,2} = \pi \pm \pi/4$ (the upper sign corresponds to $\vartheta > \pi/2$, the lower sign to $\vartheta < \pi/2$), i.e., instead of two normal waves we obtain one linearly polarized in a direction inclined at an angle of $\pi/4$ to the direction of the magnetic field. Thus, total nonorthogonality obtains in the case when the dispersion equation has essentially multiple roots[9]. The conditions given in[9] for the appearance of such roots differ from (24) in that instead of the condition that $\omega = 0$, it is required that $\omega = \omega_p$, but it is actually the same thing, since we considered the case of only a tenuous plasma ($\omega \gg \omega_p$).

The formulas obtained for the cold plasma can be easily extended to the case of an atomic gas in a magnetic field with a Lorentzian absorption-line contour. For such a gas[6] (if the normal Zeeman effect obtains),

$$t_m=N_J\lambda^2 g_{J'}\Gamma_{J'J}/4\pi[\nu_{J'}-i(\omega-m\omega_H/2-\omega_{J'J})], \tag{25}$$

where N_J is the concentration of atoms in the lower state, $\Gamma_{J'J}$ is the partial width of the transition from the upper state J′ to the lower state J, $\omega_{J'J}$ is the frequency of the transition, $g_{J'}$ is the multiplicity of the degeneracy of the level J′, and $\nu_{J'}$ is the total width of the level J′, which is made up of the natural and collision widths. Since the formula (21) is obtained from (25) by making the substitutions $N_J\lambda^2\Gamma_{J'J}g_{J'} \to 4\pi\omega_p^2/c$, $\nu_{J'} \to \nu_e$, $\omega - \omega_{J'J} \to \omega$, and $\omega_H \to \omega_H/2$, all the formulas for the cold plasma are applicable, after the appropriate substitutions, to the case of radiation transfer in a line in the presence of a magnetic field. In contrast to the plasma, for which the orthogonality conditions are fulfilled in the majority of practically important cases, the conditions for a line are fulfilled only at the wings. At the line center

$$|x|=\begin{cases} \omega_H|\sin\vartheta\,\mathrm{tg}\,\vartheta|/\nu_{J'}, & \omega_H|\sin\vartheta\,\mathrm{tg}\,\vartheta|>\nu_{J'},\\ \nu_{J'}/\omega_H|\sin\vartheta\,\mathrm{tg}\,\vartheta|, & \omega_H|\sin\vartheta\,\mathrm{tg}\,\vartheta|<\nu_{J'}.\end{cases}$$

Let us emphasize that for any relation between the magnitude of the Zeeman splitting ω_H and the line width ω_J, we can find such a direction $|\sin\vartheta\tan\vartheta| = \nu_{J'}/\omega_H$ in which, for $\omega = \omega_{J'J}$, the normal waves will be totally nonorthogonal.

4. THE TRANSFER EQUATIONS FOR THE NORMAL-WAVE AMPLITUDES IN DIFFERENT APPROXIMATIONS

Let us write out the explicit form of the transfer equations for the intensities of the normal waves in two practically important cases.

A. The LTE Approximation. This approximation is applicable if the medium is so dense that the populations of the electron states are largely determined by the collisions and are described by the Boltzmann distribution function with a temperature that can change from point to point. If the conditions (10) are fulfilled, then the transfer equation (12) assumes the form

$$(\mathbf{n}\nabla)I_j=-\mu_j[I_j-B(u^+u)_{jj}{}^{-1}/2], \tag{26}$$

where B is the Planck function, which depends on the local temperature, μ_j is the absorption coefficient for the j-th wave, and the matrices u are given by the formula (3). If the normal waves are orthogonal ($u^+u = 1$), then irrespective of whether the conditions (10) are fulfilled or not

$$(\mathbf{n}\nabla)I_j=-\mu_j(I_j-B/2). \tag{27}$$

If the refractive indices n_j for the normal waves are very different from unity, then instead of (27) we have[5]

$$\frac{n_j^2}{|\cos\Theta_j|}\frac{d}{dz}\left(\frac{I_j|\cos\Theta_j|}{n_j^2}\right)=-\mu_j\left(I_j-\frac{n_j^2}{|\cos\Theta_j|}\frac{B}{2}\right), \tag{28}$$

where Θ_j is the angle between the group velocity and the wave vector of the j-th wave. Let us emphasize that if the normal waves are nonorthogonal, then Eqs. (27) and (28) are inapplicable, since in this case the normal waves cannot propagate independently of each other.

B. The Pure-Scattering Approximation. In this case the medium is so tenuous that an absorbed radiation is not converted into thermal radiation as in the case of the LTE, but is re-emitted directly following the absorption process. In this case the transfer equations for the normal-wave intensities can be written down if the conditions (10) are fulfilled. Transforming the general transfer equation[4] under the conditions (10), we can obtain

$$(\mathbf{n}\nabla)I_j(\mathbf{n}\omega)=-\mu_j(\mathbf{n}\omega)I_j(\mathbf{n}\omega)+N\int\sigma_{jk}(\mathbf{n}\omega,\mathbf{n}'\omega')I_k(\mathbf{n}'\omega')\,d\mathbf{n}'\,d\omega'. \tag{29}$$

The quantities $\sigma_{jk}(\mathbf{n}\omega, \mathbf{n}'\omega')$ are the cross sections for scattering of the k-th normal wave with the characteristics $\mathbf{n}'\omega'$ into the j-th normal wave with the characteristics $\mathbf{n}\omega$, and N is the concentration of the scatterers. In contrast to the LTE case, we have two coupled integro-differential equations for $I_{1,2}$. The quantities $N\sigma_{jk}$ ($j \neq k$) may be called the coefficients of transfoı mation of the k-th wave into the j-th wave, and the quantity $N\sigma_{jj}$ the scattering coefficient for the j-th wave. Notice that in the approximation under consideration

$$\mu_j(\mathbf{n}\omega)=N\sum_k\int\sigma_{jk}(\mathbf{n}\omega,\mathbf{n}'\omega')\,d\mathbf{n}'\,d\omega'.$$

The condition (10), which must hold for Eq. (29) to be valid, is fulfilled in many practically important cases. It can be rewritten in the form

$$\mathrm{Im}[(t_V^2+t_l^2)^{1/2}]\gg\mathrm{Re}\,t_I. \tag{30}$$

For example, for the longitudinal propagation in a magnetoactive plasma, the condition (30) has the form

$$(\omega^2-\omega_H^2-\nu_e^2)\,\omega_H\gg(\omega^2+\omega_H^2+\nu_e^2)\,\nu_e.$$

For $\omega \gg \omega_H, \nu_e$, it reduces to the requirement that the cyclotron frequency be high in comparison with the collision rate ($\omega_H \gg \nu_e$). Knowing I_j, we can find all the Stokes parameters:

$$I=\sum_j I_j,\quad V=\sum_j p_V{}^jI_j,\quad Q=\sum_j p_Q{}^jI_j,\quad U=\sum_j p_U{}^jI_j, \tag{31}$$

where from (2), (4), and (6), we have

$$p_V{}^j=\frac{|\alpha_j|^2-1}{|\alpha_j|^2+1},\quad p_Q{}^j=-\frac{2\,\mathrm{Re}\,\alpha_j}{|\alpha_j|^2+1},\quad p_U{}^j=-\frac{2\,\mathrm{Im}\,\alpha_j}{|\alpha_j|^2+1}. \tag{32}$$

Taking (5) into account, we can express the formulas (32) in terms of the parameters of the polarization ellipses

$$p_V{}^j=\pm\frac{2K_j}{1+K_j^2},\quad p_Q{}^j=\frac{1-K_j^2}{1+K_j^2}\cos 2\chi_j,\quad p_U{}^j=\frac{1-K_j^2}{1+K_j^2}\sin 2\chi_j, \tag{33}$$

where $K_j = a_j/b_j$ is the ratio of the axes of the ellipse, and the upper sign in the formula for p_V^j corresponds to $|\alpha_j| > 1$ (counterclockwise rotation).

5. CIRCULAR POLARIZATION OF THE RADIATION OF A STAR WITH A STRONG MAGNETIC FIELD

Let us use the equations obtained for the intensities of the normal waves to compute the polarization of the radiation emitted in the continuous spectrum by a star possessing a strong magnetic field. The magnetic fields of the majority of these stars (excluding neutron stars) are such that $\omega_H \ll \omega$ in the optical band. In this case the polarization of the normal waves can be assumed to be circular for all angles of propagation except in the narrow interval near $\vartheta = \pi/2$; $|\cos\vartheta| \ll \omega_H\sin^2\vartheta/2\omega$. As a result, the polarization of the radiation of magnetic stars should be predominantly circular. The degree of linear polarization contains (as compared to that of the circular polarization) an additional small parameter ω_H/ω. We shall find the degree of circular polarization of the radiation of magnetic stars in two limiting cases: in the LTE and pure-scattering approximations. For the degree of circular polarization we have the formula

$$P_V=F_V/F_I, \tag{34}$$

where F_V and F_I are the fluxes of the Stokes parameters V and I.

A. The Circular Polarization of the Radiation in a

Stellar Atmoshpere in the Presence of LTE. As a rule, stellar atmospheres are dense enough for the LTE approximation to be applicable. Moreover, in the optical band the effective collision rate is usually much lower than the radiation frequency. Consequently, the normal waves are mutually orthogonal (see (23)), and we can use Eq. (27). Assuming, as usual, that the temperature is a linear function of the optical depth, we find from (27) that

$$I_j=\frac{1}{2}B_\omega(T_0)\left(1+\frac{\bar\mu}{\mu_j}\beta\cos\Theta\right), \tag{35}$$

where $B_\omega(T_0)$ is the Planck function corresponding to the temperature on the stellar surface, μ_j is the absorption coefficient for the j-th wave, and $\bar\mu$ is the frequency-averaged absorption coefficient[10]. The coefficient β in (35) is, in the Eddington approximation, equal to[10]

$$\beta=3\hbar\omega/8kT_0[1-\exp(-\hbar\omega/kT_0)]. \tag{36}$$

The angle Θ is the angle between the normal to the stellar surface at the point in question and the line of sight. In (35) μ_j depends on the frequency and the angle ϑ between the magnetic field at the given point and the line of sight. Since the magnetic field of a real star cannot be the same at all points on the surface, as was assumed, for example, in[11], μ_j is a position function on the stellar surface. In computing $\bar\mu$, we can assume that $\mu_1=\mu_2=\mu_0$ (μ_0 is the absorption coefficient in the absence of a magnetic field), since for $\omega_H \ll \omega$, allowance for the difference between μ_j and μ_0 would only give a small correction $\sim(\omega_H/\omega)^2$.

Substituting (35) in (31) and recognizing that for mutually orthogonal waves $p_V^1=-p_V^2$, we obtain

$$I=B\left(1+\frac{\bar\mu}{\mu_0}\beta\cos\Theta\right),\quad V=\frac{B}{2}p_V{}^1\frac{\bar\mu}{\mu_0}\frac{\mu_2-\mu_1}{\mu_0}\beta\cos\Theta. \tag{37}$$

Assuming that only the magnetic field is inhomogeneous over the stellar surface, we find from (37) and (31) that

$$P_V=\beta\bar\mu\int d\Omega\cos^2\Theta p_V{}^1(\mu_2-\mu_1)/2\pi\mu_0{}^2(1+2\bar\mu\beta/3\mu_0), \tag{38}$$

where $\int d\Omega$ denotes integration over the apparent surface of the star. The subsequent calculations depend on which absorption mechanism is dominant in the atmosphere of star in question and on the nature of the magnetic-field geometry. In general, $\mu_j=\mu_j^e+\mu_j^a$, where μ_j^e and μ_j^a are the coefficients of absorption on the electrons (bremsstrahlung and synchrotron-radiation absorption) and atoms (bound-bound transitions and the photoelectric effect). In a cold magnetoactive plasma for which $\Delta\omega_D\equiv(kT/mc^2)^{1/2}\omega\ll\nu_e$ provided $|\cos\vartheta|\gg\omega_H\sin^2\chi/2\omega$, the coefficient of absorption on electrons[12] is given by

$$\mu_j{}^e=\mu_0\omega^2/(\omega\mp\omega_H|\cos\vartheta|)^2\approx\mu_0(1\pm2\omega_H|\cos\vartheta|/\omega). \tag{39}$$

For the atomic coefficient of continuous absorption in a strong magnetic field, there are at present no detailed calculations. The cross section obtained in[13] by one of the present authors for the photoelectric effect in a magnetic field in the frequency region of interest to us is applicable only for negative ions. Kemp's calculations[14] show that for $\omega_H\ll\omega$ we can obtain for the various continuous-absorption models the relation

$$p_V{}^1(\mu_2-\mu_1)=-\mu_0 a\omega_H\cos\vartheta/\omega, \tag{40}$$

where the value of a depends on the chosen model. For bremsstrahlung absorption in a magnetoactive plasma, a = 4.

Let us assume that a magnetic star can be represented by a magnetic dipole whose center coincides with the center of the star. In that case

$$\int d\Omega\cos^2\Theta H\cos\vartheta=4\pi H_p\cos\alpha/15, \tag{41}$$

where H_p is the field intensity at the pole and α is the angle between the axis of the magnetic dipole and the line of sight. Substituting (41) in (38), we obtain

$$P_V=-\frac{2a}{15}\frac{\omega_{H_p}\cos\alpha}{\omega}\frac{\beta\bar\mu}{\mu_0}\left(1+\frac{2}{3}\frac{\beta\bar\mu}{\mu_0}\right)^{-1}. \tag{42}$$

The ω and T dependence of P_V is determined by the ratio $\beta\bar\mu/\mu_0$. If the star is so hot that the dominant absorption mechanism is the bremsstrahlung-absorption mechanism, then computing $\bar\mu$ by the Rosseland method[10], we have

$$\bar\mu/\mu_0=0.137(\hbar\omega/kT)^3[1-\exp(-\hbar\omega/kT)]^{-1} \tag{43}$$

and

$$P_V=-\frac{0.027(\hbar\omega/kT_0)^4}{[1-\exp(-\hbar\omega/kT_0)]^2+0{,}034(\hbar\omega/kT_0)^4}\frac{\omega_{H_p}\cos\alpha}{\omega}. \tag{44}$$

For $\hbar\omega\ll kT$

$$P_V=-0.027(\hbar\omega/kT_0)^2(\omega_{H_p}/\omega)\cos\alpha \tag{45a}$$

while for $\hbar\omega\gg kT$

$$P_V=-0.805(\omega_{H_p}/\omega)\cos\alpha. \tag{45b}$$

Notice that the value of P_V for the homogeneous-field model would be greater by a factor of 2.5 when $H=H_p$. Therefore, the formula (42) (after it has been multiplied by 2.5) is valid for the laboratory plasma also:

$$P_V=-\frac{a}{3}\frac{\omega_{H_p}\cos\alpha}{\omega}\frac{\beta\bar\mu}{\mu_0}\left(1+\frac{2}{3}\frac{\beta\bar\mu}{\mu_0}\right)^{-1}.$$

B. The Circular Polarization of Radiation Scattered by Electrons. In certain cosmic sources (x-ray stars) the scattering by electrons predominates over the free-free transitions: $\mu^s=N_e\sigma_T\gg\mu^{ff}$. In this case Eqs. (29) can be used when the condition (10) is fulfilled. For an approximate solution of the problem, let us represent the integral term in (29) in the form $\mu_jS_j/2$, and for S_j use its value for a dispersive atmosphere without a magnetic field. In the Eddington approximation this value is $S_j=F(1+3\tau/2)/2\pi$, where F is the total radiation flux and τ is the optical depth in the absence of a magnetic field. Then

$$I_j=\frac{F}{4\pi}\left(1+\frac{3}{2}\frac{\mu^s}{\mu_j}\cos\Theta\right). \tag{46}$$

Comparing (46) with (35) and taking account of the fact that for scattering by electrons in a magnetic field the formula (39) is valid, we see that the degree of circular polarization is given by the formula (42) if we set in it a = 4 and $\beta\bar\mu/\mu_0=3/2$:

$$P_V=-0.4(\omega_{H_p}/\omega)\cos\alpha. \tag{47}$$

The formula (47) practically completely coincides with the result obtained in[15] with the aid of a more rigorous solution of the transfer equations.

6. THE POLARIZATION OF RADIATION DEEP INSIDE A MEDIUM

On the face of it, it seems that in layers located deep inside an anisotropic medium, the radiation can possess appreciable polarization; for there is a distinct direction in the medium. It is, however, not difficult to show that this is not the case if the medium is homogeneous. Deep inside a homogeneous medium the characteristics of the radiation do not depend on the coordinates: $(n\nabla)\rho_{\alpha\beta} = 0$, and from (7) we have for the density matrix deep inside the medium the equation

$$g_{jk}R^{(0)}_{jk}=S_{jk}. \quad (48)$$

In the LTE approximation, this immediately yields $R^{(0)}_{jk} = \delta_{jk}B/2$. In the pure-scattering approximation, the integral equation to which (31) reduces is identically satisfied upon substitution of the matrix $R^{0}_{jk} = R^{0}\delta_{jk}$, in virtue of the optical theorem[16]. Thus, radiation deep inside a homogeneous anisotropic medium is isotropic and unpolarized.

It is possible, however, to indicate a number of cases when the radiation can possess appreciable polarization at large distances from the boundaries of the medium and sources. Such a situation obtains when the transfer coefficients for one normal wave are much larger than those for the other wave, and the coefficients of transformation of one wave into the other are small[1)].

As an example, we can indicate the problem of the generation and scattering by electrons of radiation in a plasma with a very strong magnetic field, such that $\omega_H \gg \omega$. Then[12] the cross section for scattering of the ordinary wave by an isolated electron differs little from the Thomson cross section $\sigma_2 \approx \sigma_T \sin^2\vartheta$ (with the exception of the narrow angle range $\vartheta \lesssim (\omega/\omega_H)^{1/2}$). As to the extraordinary wave, for it $\sigma_1 \sim (\omega/\omega_H)^2\sigma_T \ll \sigma_2$ in the frequency and angle regions under consideration. It is significant that the cross section for transformation of the normal waves into each other is also less than the Thomson cross section by a factor of $(\omega/\omega_H)^2$[17]. A similar situation obtains for the bremsstrahlung absorption coefficient. Therefore, a thermal plasma will be optically thick for the extraordinary wave provided $\tau_0 \ll (\omega_H/\omega)^2$, where τ_0 is the total optical thickness of the plasma without a magnetic field. In such a plasma, only the ordinary wave will be effectively generated and scattered. The radiation will be concentrated in the plane perpendicular to the magnetic field, and will possess a large linear polarization right up to depths $\tau \sim (\omega_H/\omega)^2\tau_0$. The direction of the preferred oscillations of the electric vector lies in the plane containing the direction of propagation of the radiation and the magnetic field.

Another example when radiation can be polarized at large distances from the boundaries and sources is the radiation of a dense hot plasma, for which $(\omega_p/\omega)^2 \gg n_j(\omega)(kT_e/m_ec^2)^{1/2}$ and $\omega_p^2 \gtrsim \omega^2(1-\omega_H/\omega)$. In such a plasma the dominant role is played by the magnetobremsstrahlung emission and absorption of photons at higher harmonics of the gyrofrequency[5], the absorption coefficient for the ordinary wave being, as a rule, two-three orders of magnitude smaller than for the extraordinary wave. As a result, the radiation can possess an appreciable linear polarization far from the boundaries of the medium. In this case the direction of the preferred oscillations of the electric vector lies in the plane perpendicular to the magnetic field.

The authors are grateful to A. Z. Dolginov and N. A. Silant'ev for useful discussions.

[1)]The assertion made in[15] that radiation in layers located deep inside a medium is polarized is valid only in these cases.

[1]A. C. England, R. A. Dandle, H. O. Eason, and J. C. Sprott, Annual Meeting of the Division of Plasma Physics, November 13–16, 1972, Monterey, California, USA.

[2]V. N. Sazonov, Usp. Fiz. Nauk 108, 583 (1972) [Sov. Phys.-Uspekhi 15, 822 (1973)].

[3]S. Chandrasekhar, Radiative Transfer, Dover Publications, New York, 1960 (Russ. Transl., IIL, 1953).

[4]A. Z. Dolginov, Yu. N. Gnedin, and N. A. Silant'ev, JQSRT 10, 707 (1970).

[5]V. V. Zhelezhyakov, Radioizluchenie Solntsa i planet (Radio Emission of the Sun and Planets), Nauka, 1964 (Eng. Transl., Pergamon Press, New York, 1970).

[6]A. Z. Dolginov and G. G. Pavlov, Astron. Zh. 50, 762 (1973) [Sov. Astron. AJ 17, No. 4 (1974)].

[7]A. R. Edmonds, CERN 55-26, Geneva, 1955.

[8]V. E. Stepanov, Astron. Zh. 37, 631 (1960) [Sov. Astron. AJ 4, 603 (1960)].

[9]V. L. Ginzburg, Rasprostranenie élektromagnitnykh voln v plazme (The Propagation of Electromagnetic Waves in Plasmas), Nauka, 1967 (Eng. Transl., Gordon and Breach, New York, 1962).

[10]V. V. Sobolev, Kurs teoreticheskoĭ astrofiziki (A Course in Theoretical Astrophysics), Nauka, 1967.

[11]H. L. Shippman, Astrophys. J. 167, 165 (1971).

[12]V. Canuto, J. Lodenquai, and M. Ruderman, Phys. Rev. D3, 2303 (1971).

[13]G. G. Pavlov, Opt. Spektrosk. 33, 1006 (1972) [Opt. Spectrosc. 33, 554 (1972)].

[14]J. C. Kemp, Astrophys. J. 162, 169 (1970).

[15]Yu. N. Gnedin, A. Z. Dolginov, and N. A. Silant'ev, Zh. Eksp. Teor. Fiz. 59, 865 (1970) [Sov. Phys.-JETP 32, 472 (1971)].

[16]E. M. Lifshitz, V. B. Berestetskiĭ, and L. P. Pitaevskiĭ, Relyativistskaya kvantovaya teoriya, (Relativistic Quantum Theory), Vol. 1, Nauka, 1968 (Eng. Transl., Pergamon Press, New York, 1971).

[17]Yu. N. Gnedin and R. A. Syunyaev, Zh. Eksp. Teor. Fiz. 65, 102 (1973) [Sov. Phys.-JETP 38, No. 1 (1974)].

Translated by A. K. Agyei
185